Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 1238–1257
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

L^p-uniqueness for elliptic operators with unbounded coefficients in $\mathbb{R}^N$ ☆

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Received 29 April 2008; accepted 21 July 2008

Available online 12 September 2008

Communicated by Paul Malliavin

Abstract

Let $\mathcal{A}$ be an elliptic operator with unbounded and sufficiently smooth coefficients and let μ be a (sub)-invariant measure of the operator $\mathcal{A}$. In this paper we give sufficient conditions guaranteeing that the closure of the operator $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ generates a sub-Markovian strongly continuous semigroup of contractions in $L^p(\mathbb{R}^N, \mu)$. Applications are given in the case when $\mathcal{A}$ is a generalized Schrödinger operator.

Keywords: Elliptic operators with unbounded coefficients; (Sub-)invariant measures; Cores

1. Introduction

The qualitative properties of elliptic operators $\mathcal{A}$ with unbounded coefficients in $\mathbb{R}^N$ have been investigated intensively in recent years, after the seminal papers [3,4,12,18], motivated by the important impact of these operators on stochastic processes, and their application to branches of applied sciences such as mathematical finance.

☆ Work partially supported by the research project "Equazioni di Kolmogorov" of the Ministero dell'Istruzione, dell'Università e della Ricerca (M.I.U.R.).

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.07.022

The natural settings where to study these operators are the space of bounded and continuous functions (say $C_b(\mathbb{R}^N)$) and L^p-spaces. In $C_b(\mathbb{R}^N)$ one can associate a semigroup of bounded operators with such operators under rather weak assumptions on their coefficients. Indeed, if

$$\mathcal{A}\phi = \sum_{i,j=1}^{N} q_{ij} D_{ij}\phi + \sum_{j=1}^{N} b_j D_j\phi + c\phi,$$

on smooth functions ϕ, requiring that the matrix $Q(x) = (q_{ij}(x))$ is definite positive at any $x \in \mathbb{R}^N$, that the coefficients q_{ij}, b_j $(i, j = 1, \dots, N)$ and c are locally α-Hölder continuous for some $\alpha \in (0, 1)$ and c is bounded from above, is all what one needs to show that, for any $f \in C_b(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = (\mathcal{A}u)(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = f, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

admits (at least) a classical solution, bounded with respect to the sup-norm in $[0, T] \times \mathbb{R}^N$, for any $T > 0$. Here, by classical solution we mean any function u which (i) is continuous up to $t = 0$, (ii) is continuously differentiable in $(0, +\infty) \times \mathbb{R}^N$, once with respect to time and twice with respect to the spatial variables, (iii) solves (1.1).

In general, the Cauchy problem (1.1) admits more than a unique bounded classical solution (but it turns out to be uniquely solvable if an additional algebraic condition on its coefficients is prescribed). Nevertheless, it is possible to associate a semigroup of bounded operators with the operator $\mathcal{A}$, setting $T(t)f = u(t, \cdot)$ for any $t \geqslant 0$, where for any positive f, $u(t, \cdot)$ is the value at t of the *minimal positive* solution to (1.1). This semigroup is, in general, neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$. In fact, $T(t)f$ tends to f in the weak topology of $\mathbb{R}^N$, i.e., the sequence $\{T(t)f\}$ is bounded and converges to f locally uniformly in $\mathbb{R}^N$ as $t \to 0^+$.

A rather complete analysis of the semigroup $\{T(t)\}$ and its main smoothing properties is available nowadays and we refer the reader to [5].

The analysis of the operator $\mathcal{A}$ in the usual L^p-spaces related to the Lebesgue measure is much more difficult than in $C_b(\mathbb{R}^N)$. As a one-dimensional example in [30, Section 2] shows, whatever $\varepsilon > 0$ is fixed, the operator $\mathcal{A}$ defined by

$$\mathcal{A}\phi(x) = \phi''(x) - \operatorname{sign}(x)|x|^{1+\varepsilon}\phi'(x), \quad x \in \mathbb{R}, \tag{1.2}$$

does not generate a strongly continuous semigroup in $L^p(\mathbb{R})$ for any $p \in [1, +\infty)$. Hence, assumptions on the growth at infinity of the coefficients of the operator $\mathcal{A}$ more restrictive than in the C_b-case are to be prescribed. Typically, the diffusion coefficients are supposed to be bounded or to grow at most slightly more than quadratically at infinity, and some suitable compensation conditions on the coefficients are prescribed (see, e.g., the papers [9–11,16,17,23,25,26,28,30,31]).

The suitable L^p-spaces where to analyze elliptic operators with unbounded coefficients are L^p-spaces related to particular measures, the so-called *invariant measures* and *infinitesimally invariant measures*. Whenever existing, an invariant measure is any (probability) measures μ such that

$$\int_{\mathbb{R}^N} T(t)f\,\mu(dx) = \int_{\mathbb{R}^N} f\,\mu(dx), \quad t > 0, \tag{1.3}$$

for any $f \in C_{\mathrm{b}}(\mathbb{R}^N)$. Here, $\{T(t)\}$ is the semigroup introduced above. Condition (1.3) may be rephrased requiring that

$$\int_{\mathbb{R}^N} \mathcal{A}\phi\,\mu(dx) = 0, \tag{1.4}$$

for any $\phi \in C_{\mathrm{b}}(\mathbb{R}^N)$ such that $\sup_{t\in(0,1)} t^{-1}\|T(t)\phi - \phi\|_\infty < +\infty$ and $t^{-1}(T(t)\phi - \phi)$ converges locally uniformly in $\mathbb{R}^N$ to a bounded and continuous function. In particular, (1.4) should be satisfied by any smooth and compactly supported function ϕ.

Any measure μ satisfying (1.4) for any $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ is usually called *infinitesimally invariant*. Note that whenever an infinitesimally invariant measure of $\mathcal{A}$ exists, the operator $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N, \mu)$ for any $p \in [1, +\infty)$.

In the case when μ is an invariant measure of $\mathcal{A}$, the semigroup $\{T(t)\}$ extends in a rather straightforward way by a strongly continuous semigroup in $L^p(\mathbb{R}^N, \mu)$ for any $p \in [1, +\infty)$. Its infinitesimal generator A_p turns out to be an extension of the operator $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$. For instance, in the case of the operator in (1.2) with $\varepsilon = 2$ the measure $\mu(dx) = Ke^{-|x|^4/4}\,dx$ (where K is a suitable normalizing constant) satisfies (1.3), so that the realization of $\mathcal{A}$ in $L^p(\mathbb{R}^N, \mu)$ generates a strongly continuous semigroup of contractions for any $p \in [1, +\infty)$.

In general it is not known whether there exists only one strongly continuous semigroup on $L^p(\mathbb{R}^N, \mu)$ whose generator extends $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$. If the answer is positive, then $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$ is said to be $L^p(\mathbb{R}^N, \mu)$-unique. Standard results in semigroup theory show that the $L^p(\mathbb{R}^N, \mu)$-uniqueness is equivalent to the condition that the closure of $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N, \mu)$. If this is the case, then $C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ is a core for A_p and this is of great importance since, in general, characterizing the domain of A_p is a hard task. Indeed, it is rather easy to show that it contains the set of all compactly functions $f \in L^p(\mathbb{R}^N, \mu)$ such that their first- and second-order derivatives are in $L^p(\mathbb{R}^N, \mu)$ as well. Moreover, whenever pointwise gradient estimates for the function $T(t)f$ are available for any $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$, one can partially characterize $D(A_p)$, showing that it is continuously embedded in the Sobolev space $W^{1,p}(\mathbb{R}^N, \mu)$ (the set of functions $f \in L^p(\mathbb{R}^N, \mu)$ such that the distributional gradient of f is in $L^p(\mathbb{R}^N, \mu)^N$) for any $p \in (1, +\infty)$. Anyway, a complete characterization of $D(A_p)$ is known, to the best of our knowledge only in very few cases (see, e.g., [13,22,24,25]).

In this paper, under suitable assumptions on the coefficients of the operator $\mathcal{A}$ and assuming that $c \equiv 0$, we prove the $L^p(\mathbb{R}^N, \mu)$-uniqueness of $(\mathcal{A}, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$, thus generalizing the results in [1,2].

For the sake of generality, we assume that μ is just a *sub-invariant* measure of the operator $\mathcal{A} - \alpha$ for some $\alpha \in \mathbb{R}$, in the sense that

$$\int_{\mathbb{R}^N} \mathcal{A}\phi\,\mu(dx) \leqslant \alpha \int_{\mathbb{R}^N} \phi\,\mu(dx),$$

for any positive function $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$. Whenever it admits a sub-invariant measure, the operator $(\mathcal{A} - \frac{\alpha}{p}I, C_{\mathrm{c}}^\infty(\mathbb{R}^N))$ turns out to be dissipative (see e.g. [14, Appendix B, Lemma 1.8]), hence closable, in $L^p(\mathbb{R}^N,\mu)$ for any $p\in[1,+\infty)$. Under some integrability conditions involving a Lyapunov function and the coefficients of the operator $\mathcal{A}$, we prove that the closure of $(\mathcal{A},C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$ for any $p\in[1,+\infty)$. Our results cover also some situations in which the coefficients of the operator $\mathcal{A}$ may grow at infinity with some exponential rate. In particular, we partly generalize some results in [14].

In the second part of the paper, we give another criterion ensuring that the closure of the operator $(\mathcal{A},C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N,\mu)$. In particular, this criterion is useful when the measure μ is symmetrizing for the operator $\mathcal{A}$ (i.e., in the case when $\int_{\mathbb{R}^N}\phi\mathcal{A}\psi\,\mu(dx)=\int_{\mathbb{R}^N}\psi\mathcal{A}\phi\,\mu(dx)$ for any $\phi,\psi\in C_c^\infty(\mathbb{R}^N)$). For instance, this is the case when $\mathcal{A}$ is a generalized Schrödinger operator. In this setting, our results generalize similar results obtained by Eberle [14], Liskevič [20], Liskevič and Semenov [21].

2. A first criterion for $L^p(\mathbb{R}^N,\mu)$-uniqueness

Let $\mathcal{A}$ be the second order elliptic partial differential operator defined on smooth functions by

$$(\mathcal{A}\psi)(x)=\sum_{i,j=1}^N q_{ij}(x)D_{ij}\psi(x)+\sum_{i=1}^N b_i(x)D_i\psi(x),\quad x\in\mathbb{R}^N. \tag{2.1}$$

We make the following assumptions.

Hypothesis 2.1.

(i) The coefficients $q_{ij}=q_{ji}$ and b_j $(i,j=1,\dots,N)$ belong to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N,dx)$ and $L^r_{\mathrm{loc}}(\mathbb{R}^N,dx)$, respectively, for some $r>N\geqslant 2$.

(ii) The matrix $Q:=(q_{ij})_{i,j=1}^N$ satisfies the ellipticity condition

$$\big(Q(x)\xi\big)\cdot\xi\geqslant\eta(x)|\xi|^2,\quad \xi\in\mathbb{R}^N,\ x\in\mathbb{R}^N,$$

for some positive function η such that $\inf_K\eta>0$ for every compact $K\subset\mathbb{R}^N$.

(iii) There exists a positive locally finite Borel measure μ on $\mathbb{R}^N$ such that:
  (a) μ is absolutely continuous with respect to the Lebesgue measure and its density ϱ is everywhere positive in $\mathbb{R}^N$. Moreover, $\inf_K\varrho>0$ for any compact set $K\subset\mathbb{R}^N$.
  (b) μ is sub-invariant for the operator $(\mathcal{A}-\alpha,C_c^\infty(\mathbb{R}^N))$ for some $\alpha\geqslant 0$, i.e.,

$$\int_{\mathbb{R}^N}\mathcal{A}f\,\mu(dx)\leqslant\alpha\int_{\mathbb{R}^N}f\,\mu(dx)$$

for all $0\leqslant f\in C_c^\infty(\mathbb{R}^N)$.

We remark that condition (i) in Hypothesis 2.1 guarantees that the functions q_{ij} $(i,j=1,\dots,N)$ are locally Hölder continuous and, hence, locally bounded in $\mathbb{R}^N$.

In this setting we are able to prove the following result.

Theorem 2.2. *Assume that Hypothesis* 2.1 *holds. Assume further that there exists a positive function $V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and*

$$\frac{\mathcal{A}V}{V\log V} \in L^p\big(\mathbb{R}^N,\mu\big) \quad \textit{and} \quad \frac{(Q\nabla V)\cdot\nabla V}{V^2\log V} \in L^p\big(\mathbb{R}^N,\mu\big), \tag{2.2}$$

for some $1 \leqslant p < +\infty$ and $b_i \in L^p_{\mathrm{loc}}(\mathbb{R}^N,\mu)$ $(i = 1,\dots,N)$. Then, the closure of the operator $(\mathcal{A}, C^\infty_{\mathrm{c}}(\mathbb{R}^N))$ on $L^p(\mathbb{R}^N,\mu)$ generates a sub-Markovian strongly continuous semigroup. In particular, $(\mathcal{A}, C^\infty_{\mathrm{c}}(\mathbb{R}^N))$ is $L^p(\mathbb{R}^N,\mu)$ unique.

Proof. To begin with we observe that, up to replacing V with $V + e$, we can assume, without any loss of generality, that $V \geqslant e$ in $\mathbb{R}^N$.

Let us prove that the space $(\lambda I - \mathcal{A})(C^\infty_{\mathrm{c}}(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N,\mu)$ for all $\lambda > 0$. For this purpose, let us denote by p' the conjugate exponent of p and suppose that

$$\big\langle(\lambda I - \mathcal{A})\varphi, \psi\big\rangle_{L^p(\mathbb{R}^N,\mu)} = 0, \tag{2.3}$$

for some $\psi \in L^{p'}(\mathbb{R}^N,\mu)$, $\lambda > 0$, and all $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$. Setting $\nu(dx) = \psi\varrho\,dx$, condition (2.3) can be rewritten equivalently as

$$\int_{\mathbb{R}^N} (\lambda - \mathcal{A})\varphi\,\nu(dx) = 0, \quad \varphi \in C^\infty_{\mathrm{c}}\big(\mathbb{R}^N\big). \tag{2.4}$$

By [7, Corollary 2.10] the density $\psi\varrho$ of the measure ν belongs to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N,dx)$. Hence, it is continuous and locally bounded in $\mathbb{R}^N$.

For all $i = 1,\dots,N$, set $\hat{b}_i = b_i - \sum_{j=1}^N D_j q_{ij}$. Since $\hat{b}_i \in L^r_{\mathrm{loc}}(\mathbb{R}^N,dx)$ for any i, integrating by parts, from (2.4) we obtain that

$$\lambda\int_{\mathbb{R}^N} \varphi\psi\varrho\,dx = \int_{\mathbb{R}^N} \mathcal{A}\varphi\psi\varrho\,dx = \int_{\mathbb{R}^N} \big[-\nabla\varphi\cdot\big(Q\nabla(\psi\varrho)\big) + \psi\varrho\hat{b}\cdot\nabla\varphi\big]\,dx, \tag{2.5}$$

for every $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$. By density, (2.5) can be extended to every $\varphi \in C^1_{\mathrm{c}}(\mathbb{R}^N)$.

Fix now $v \in C^1(\mathbb{R}^N)$ and $\zeta \in C^1_{\mathrm{c}}(\mathbb{R}^N)$ with $\zeta \geqslant 0$. Replacing φ with $v\zeta$ in (2.5), we get

$$\lambda\int_{\mathbb{R}^N} v\zeta\psi\varrho\,dx = \int_{\mathbb{R}^N} \big[-v\nabla\zeta\cdot\big(Q\nabla(\psi\varrho)\big) - \zeta\nabla v\cdot\big(Q\nabla(\psi\varrho)\big) + \psi\varrho\hat{b}\cdot\nabla(v\zeta)\big]\,dx. \tag{2.6}$$

Since $\psi\varrho$ is locally Hölder continuous and $(Q\nabla(\psi\varrho))_i, \hat{b}_i \in L^r_{\mathrm{loc}}(\mathbb{R}^N,dx)$ for any $i = 1,\dots,N$, we can extend equality (2.6) by density to every $v \in W^{1,r'}_{\mathrm{loc}}(\mathbb{R}^N,dx)$, where r' denotes the conjugate exponent of r.

Let $F : \mathbb{R} \to [-1,1]$ be an increasing C^1-function such that $F(s) = 0$ if $|s| \leqslant 1$, $F(s) = -1$ if $s \leqslant -2$ and $F(s) = 1$ if $s \geqslant 2$. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$, set $u_n(x) := F(n\psi(x)\varrho(x))$. Then, $u_n \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N,dx) \subset W^{1,r'}_{\mathrm{loc}}(\mathbb{R}^N,dx)$, since $r' < 2 < r$. Moreover, $|u_n| \leqslant 1$ and u_n pointwise tends to $\mathrm{sign}(\psi\varrho)$ in $\mathbb{R}^N$, as $n \to +\infty$. Replacing v with u_n in (2.6) and observing that

$$\nabla u_n \cdot Q\nabla(\psi\varrho) = nF'(n\psi\varrho)\nabla(\psi\varrho)\cdot Q\nabla(\psi\varrho) \geqslant 0, \quad n\in\mathbb{N},$$

we obtain

$$\begin{aligned}
\lambda\int_{\mathbb{R}^N} u_n\zeta\psi\varrho\,dx &= \int_{\mathbb{R}^N}\bigl[-u_n\nabla\zeta\cdot\bigl(Q\nabla(\psi\varrho)\bigr) - n\zeta F'(n\psi\varrho)\nabla(\psi\varrho)\cdot\bigl(Q\nabla(\psi\varrho)\bigr)\bigr]\,dx\\
&\quad + \int_{\mathbb{R}^N}\bigl[u_n\psi\varrho\hat{b}\cdot\nabla\zeta + n\zeta F'(n\psi\varrho)\psi\varrho\hat{b}\cdot\nabla(\psi\varrho)\bigr]\,dx\\
&\leqslant \int_{\mathbb{R}^N}\bigl[-\nabla\zeta\cdot\bigl(Q\nabla(u_n\psi\varrho)\bigr) + \psi\varrho\nabla\zeta\cdot(Q\nabla u_n)\bigr]\,dx\\
&\quad + \int_{\mathbb{R}^N}\bigl[u_n\psi\varrho\hat{b}\cdot\nabla\zeta + n\zeta F'(n\psi\varrho)\psi\varrho\hat{b}\cdot\nabla(\psi\varrho)\bigr]\,dx\\
&\leqslant \int_{\mathbb{R}^N}\bigl[u_n\psi\varrho\,\mathrm{div}(Q\nabla\zeta) + \psi\varrho nF'(n\psi\varrho)\nabla\zeta\cdot\bigl(Q\nabla(\psi\varrho)\bigr)\bigr]\,dx\\
&\quad + \int_{\mathbb{R}^N}\bigl[u_n\psi\varrho\hat{b}\cdot\nabla\zeta + n\zeta F'(n\psi\varrho)\psi\varrho\hat{b}\cdot\nabla(\psi\varrho)\bigr]\,dx,
\end{aligned}\tag{2.7}$$

for each $n\in\mathbb{N}$. Since $F'(y)=0$ if $|y|\leqslant 1$ or $|y|\geqslant 2$, it holds that $|nF'(n\psi\varrho)||\psi\varrho|\leqslant 2\|F'\|_\infty$ in $\mathbb{R}^N$ for every $n\in\mathbb{N}$. On the other hand, if $\psi(x)\varrho(x)\neq 0$, then there exists $n_0\in\mathbb{N}$ such that, for every $n\geqslant n_0$, $|\psi(x)\varrho(x)|>2n^{-1}$. Thus, $nF'(n\psi(x)\varrho(x))=0$ for all $n\geqslant n_0$. It follows that $nF'(n\psi\varrho)\psi\varrho$ tends to 0 in a dominated way as $n\to+\infty$. Hence, passing to the limit in the first and last sides of (2.7), and taking into account that the supports of all the involved functions are contained in the support of ζ, we get

$$\lambda\int_{\mathbb{R}^N}\zeta|\psi\varrho|\,dx \leqslant \int_{\mathbb{R}^N}|\psi\varrho|\,\mathrm{div}(Q\nabla\zeta)\,dx + \int_{\mathbb{R}^N}|\psi\varrho|\hat{b}\cdot\nabla\zeta\,dx = \int_{\mathbb{R}^N}|\psi\varrho|\mathcal{A}\zeta\,dx. \tag{2.8}$$

Next, let $\zeta:\mathbb{R}\to[0,1]$ be a decreasing C^1-function such that $\zeta(s)=1$ if $s\leqslant 1$ and $\zeta(s)=0$ if $s\geqslant 2$, and define $\zeta_n := H(\frac{\log V}{n})$ for every $n\in\mathbb{N}$. Since V blows up as $|x|\to+\infty$, each function ζ_n belongs to $C^1_c(\mathbb{R}^N)$. Moreover, $\zeta_n\leqslant 1$, $\lim_{n\to+\infty}\zeta_n=1$ pointwise in $\mathbb{R}^N$.

A straightforward computation shows that

$$\begin{aligned}
\mathcal{A}\zeta_n &= \frac{\log V}{n}\zeta'\biggl(\frac{\log V}{n}\biggr)\frac{\mathcal{A}V}{V\log V} + \frac{(\log V)^2}{n^2}\zeta''\biggl(\frac{\log V}{n}\biggr)\frac{(Q\nabla V)\cdot\nabla V}{V^2(\log V)^2}\\
&\quad - \frac{\log V}{n}\zeta'\biggl(\frac{\log V}{n}\biggr)\frac{(Q\nabla V)\cdot\nabla V}{V^2\log V},
\end{aligned}$$

for every $n\in\mathbb{N}$. Hence, $\mathcal{A}\zeta_n$ pointwise tends to 0 in $\mathbb{R}^N$ as $n\to+\infty$. Moreover, observing that $\zeta'(\frac{V}{n})$, $\zeta''(\frac{V}{n})$ vanish if $\frac{\log V}{n}\notin[1,2]$, and $V\geqslant e$, we get

$$|\mathcal{A}\zeta_n| \leqslant 2\|\zeta'\|_\infty \left|\frac{\mathcal{A}V}{V\log V}\right| + \left(4\|\zeta''\|_\infty + 2\|\zeta'\|_\infty\right)\frac{(Q\nabla V)\cdot\nabla V}{V^2\log V},$$

for every $n \in \mathbb{N}$. Then, by replacing ζ with ζ_n in (2.8), recalling that the functions $(V\log V)^{-1}\mathcal{A}V$ and $(V^2\log V)^{-1}(Q\nabla V)\cdot\nabla V$ belong to $L^p(\mathbb{R}^N,\mu)$ and applying the dominated convergence theorem, we obtain that

$$\lambda \int_{\mathbb{R}^N} |\psi|\varrho\, dx \leqslant 0,$$

which, of course, implies that $\psi = 0$ μ-a.e.

Since Hypothesis 2.1(iii)-(b) implies that the operator $(\mathcal{A} - \frac{\alpha}{p}, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N,\mu)$ (also in the case $p = 1$) (see, e.g., [14, Lemma 1.8 in Appendix B]), and the space $(\frac{\alpha}{p} + 1 - \mathcal{A})(C_c^\infty(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N,\mu)$, the closure of the operator $(\mathcal{A} - \frac{\alpha}{p}, C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup of contractions in $L^p(\mathbb{R}^N,\mu)$ by the well-known Lumer–Phillips' generation theorem (see, e.g., [15, Chapter 2, Theorem 3.15]). Therefore, the closure of the operator $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup on $L^p(\mathbb{R}^N,\mu)$. Finally, by [14, Lemma 1.9 in Appendix B] such a semigroup is also sub-Markov. This completes the proof. □

Remark 2.3. Theorem 2.2 to be applied requires to prove the integrability of two suitable functions with respect to the measure μ. The fact that in most the cases the measure μ is not explicit makes things difficult. A strategy to prove the integrability of the functions in (2.2) consists in comparing them with functions which we know *a priori* that are in some L^p space related to the measure μ. For instance this is the case when the operator $\mathcal{A}$ admits a Lyapunov function, i.e., when there exists a positive smooth function φ diverging to $+\infty$ as $|x| \to +\infty$, such that $\mathcal{A}\varphi$ tends to $-\infty$ as $|x| \to +\infty$. Indeed, in this situation, the functions φ and $\mathcal{A}\varphi$ are integrable with respect to the measure μ.

2.1. An example

In this subsection we provide the reader with a class of elliptic operators with unbounded coefficients to which Theorem 2.2 applies. We assume that the coefficients of the operator $\mathcal{A}$ satisfy conditions (i) and (ii) in Hypothesis 2.1.

Let $V \in C^1(\mathbb{R}^N)$ be any function such that $V(x) \geqslant 2$ for any $x \in \mathbb{R}^N$ and $V(x) := 2\exp(\delta|x|^\beta)$ for any $x \in \mathbb{R}^N$ with $|x| \geqslant 1$, where β and δ are positive constants. Further, assume that

$$\limsup_{|x|\to+\infty}\left(C\Lambda(x) + \frac{b(x)\cdot x}{|x|^\beta}\right) < 0, \tag{2.9}$$

for some $C > 0$, where $\Lambda(x)$ denotes the maximum eigenvalue of the matrix $Q(x)$. By [27, Proposition 2.4], V is a Lyapunov function for the operator $\mathcal{A}$ defined in (2.1) if $\delta < \beta^{-1}C$, i.e., $0 < V \in C^2(\mathbb{R}^N)$, $V(x) \to +\infty$ and $\mathcal{A}V(x) \to -\infty$ as $|x| \to +\infty$. Therefore, by Khas'minskii theorem (see [19, Chapter 3, Theorem 5.1], see also [29, Theorem 6.3] or [5, Section 8.1.2]) $\mathcal{A}$ admits a unique invariant measure μ whose density ϱ is a positive and continuous function by

[8, Lemma 1.1], and the functions V and $\mathcal{A}V$ are integrable with respect to μ for $\delta < \beta^{-1}C$ by [27, Proposition 2.4].

Lemma 2.4. *Let $1 \leqslant p < +\infty$ and suppose that $\beta\delta < C$. If*

$$\limsup_{|x|\to+\infty} |x|^{\beta-2}\Lambda(x)e^{-\frac{\delta}{p}|x|^\beta} < +\infty, \tag{2.10}$$

then, the function $(V^2 \log V)^{-1}((Q\nabla V)\cdot\nabla V)$ belongs to $L^p(\mathbb{R}^N, \mu)$.

Proof. Since $V(x) = 2e^{\delta|x|^\beta}$ for $|x| \geqslant 1$, then $\nabla V(x) = \delta\beta|x|^{\beta-2}xV(x)$ for such x's. Hence,

$$\begin{aligned}\big(Q(x)\nabla V(x)\big)\cdot\nabla V(x) &= \delta^2\beta^2|x|^{2\beta-4}\big(\big(Q(x)x\big)\cdot x\big)\big(V(x)\big)^2\\ &\leqslant \delta^2\beta^2|x|^{2\beta-2}\Lambda(x)\big(V(x)\big)^2,\end{aligned}$$

for any $|x| \geqslant 1$. Consequently,

$$\frac{(Q(x)\nabla V(x))\cdot\nabla V(x)}{(V(x))^2\log V(x)} \leqslant \delta\beta^2|x|^{\beta-2}\Lambda(x), \quad |x| \geqslant 1. \tag{2.11}$$

Using condition (2.10) it follows easily that the right-hand side of (2.11) can be estimated from above by $K(V(x))^{1/p}$ for some positive constant K. Since $V \in L^1(\mathbb{R}^N, \mu)$, the assertion follows. $\square$

Lemma 2.5. *Suppose that $\beta\delta < C$. Then, the function $(V\log V)^{-1}\mathcal{A}V$ belongs to $L^1(\mathbb{R}^N, \mu)$. Moreover, let $1 < p < +\infty$ and assume that condition* (2.10) *is satisfied and*

$$\limsup_{|x|\to+\infty} \frac{|b(x)\cdot x|}{|x|^{2+\beta(p'-1)}\exp(\delta(p'-1)|x|^\beta)} < +\infty, \tag{2.12}$$

where p' is the conjugate exponent of p. Then the function $(V\log V)^{-1}\mathcal{A}V$ belongs to $L^p(\mathbb{R}^N, \mu)$.

Proof. Since, by assumptions, $V \geqslant 2$, we can estimate

$$\big|(V\log V)^{-1}\mathcal{A}V\big| \leqslant \big(\log(4)\big)^{-1}|\mathcal{A}V|.$$

Since $\mathcal{A}V \in L^1(\mathbb{R}^N, \mu)$, it follows immediately that the function $(V\log V)^{-1}\mathcal{A}V$ is in $L^1(\mathbb{R}^N, \mu)$ as well.

Now, let us consider the case when $p > 1$. To prove that $(V\log V)^{-1}\mathcal{A}V$ belongs to $L^p(\mathbb{R}^N, \mu)$, we will show that such a function can be estimated from above by $M|\mathcal{A}V|^{1/p}$ for some positive constant M or, equivalently, that the function $(V\log V)^{-p'}|\mathcal{A}V|$ is bounded. For this purpose, we observe that

$$
\begin{aligned}
(\mathcal{A}V)(x) = \beta\delta|x|^{\beta-1}V(x)\biggl(&\frac{\mathrm{Tr}(Q(x))}{|x|} + \frac{\delta\beta(\beta-2)(Q(x)x)\cdot x}{|x|^3} \\
&+ \frac{b(x)\cdot x}{|x|} + \delta\beta|x|^{\beta-3}\bigl(Q(x)x\bigr)\cdot x\biggr),
\end{aligned}
$$

for any $|x| \geqslant 1$. Since $\mathcal{A}V < 0$ for large $|x|$, it follows

$$
\begin{aligned}
\frac{|(\mathcal{A}V)(x)|}{(V(x)\log V(x))^{p'}} \leqslant \frac{\beta}{\delta^{p'-1}|x|^{1+\beta(p'-1)}\exp(\delta(p'-1)|x|^{\beta})}\biggl(&-\frac{\mathrm{Tr}(Q(x))}{|x|} - \frac{b(x)\cdot x}{|x|} \\
&- \frac{(\beta-2)(Q(x)x)\cdot x}{|x|^3} - \delta\beta|x|^{\beta-3}\bigl(Q(x)x\bigr)\cdot x\biggr).
\end{aligned} \tag{2.13}
$$

We now consider the cases $0 < \beta < 2$ and $\beta \geqslant 2$ separately.

Case $\beta \geqslant 2$. Since all the terms in (2.13) are negative but the second one, we can estimate

$$
\frac{|(\mathcal{A}V)(x)|}{(V(x)\log V(x))^{p'}} \leqslant \frac{\beta}{\delta^{p'-1}}\frac{|b(x)\cdot x|}{|x|^{2+\beta(p'-1)}\exp(\delta(p'-1)|x|^{\beta})}, \tag{2.14}
$$

for $|x|$ sufficiently large. From condition (2.12), it now follows immediately that the function $(V\log V)^{-p'}|\mathcal{A}V|$ is bounded.

Case $0 < \beta < 2$. By (2.9), (2.10) and (2.13) we obtain, for a suitable $\kappa > 0$ and large $|x|$,

$$
\begin{aligned}
&\frac{|(\mathcal{A}V)(x)|}{(V(x)\log V(x))^{p'}} \\
&\quad \leqslant \frac{\beta}{\delta^{p'-1}}\frac{1}{|x|^{1+\beta(p'-1)}\exp(\delta(p'-1)|x|^{\beta})}\biggl(\frac{|b(x)\cdot x|}{|x|} + (2-\beta)\frac{(Q(x)x)\cdot x}{|x|^3}\biggr) \\
&\quad \leqslant \frac{\beta}{\delta^{p'-1}}\frac{1}{|x|^{2+\beta(p'-1)}\exp(\delta(p'-1)|x|^{\beta})}\bigl(\bigl|b(x)\cdot x\bigr| + (2-\beta)\Lambda(x)\bigr) \\
&\quad \leqslant \frac{\beta}{\delta^{p'-1}}\frac{1}{|x|^{2+\beta(p'-1)}\exp(\delta(p'-1)|x|^{\beta})}\biggl(\bigl|b(x)\cdot x\bigr| + (2-\beta)\kappa\frac{\exp(\frac{\delta}{p}|x|^{\beta})}{|x|^{\beta-2}}\biggr).
\end{aligned}
$$

As $p'-1 > \frac{1}{p}$, it follows that the function $(V\log V)^{-p'}|\mathcal{A}V|$ is bounded by using again condition (2.12). $\square$

In view of Theorem 2.2, we have proved the following result.

Proposition 2.6. *Let $p \geqslant 1$ and let us assume that the diffusion and the drift coefficients of the operator $\mathcal{A}$ satisfy conditions* (i) *and* (ii) *in Hypothesis* 2.1 *and $b_i \in L^p_{\mathrm{loc}}(\mathbb{R}^N, dx)$ $(i = 1, \ldots, N)$. Further, assume that conditions* (2.9)*,* (2.10) *and* (2.12) *are satisfied* (*the latter one only in the case when $p > 1$*)*. Then, the closure of the operator $(\mathcal{A}, C_c^{\infty}(\mathbb{R}^N))$ generates a strongly continuous semigroup of contractions in $L^p(\mathbb{R}^N, \mu)$.*

Remark 2.7. It is worth stressing that Proposition 2.6 covers also some situation in which the diffusion and the drift coefficients of $\mathcal{A}$ have some exponential growth at infinity. This is an improvement of [14, Theorem 2.3, p. 67] where it is required that

$$\limsup_{|x|\to+\infty} \frac{(Q(x)x)\cdot x}{|x|^4(\log|x|)^2} < +\infty. \tag{2.15}$$

For instance, Theorem 2.2 applies to the operator $\mathcal{A}$ defined on smooth functions ϕ by

$$(\mathcal{A}\phi)(x) = e^{\frac{1}{2}|x|^2}\Delta\phi - e^{|x|^2}x\cdot\nabla\phi(x), \quad x\in\mathbb{R}^N,$$

which, of course, does not satisfy condition (2.15).

3. A second criterion for $L^p(\mathbb{R}^N,\mu)$ uniqueness

In this section we give a second criterion which guarantees that the operator $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ uniquely extends to $L^p(\mathbb{R}^N,\mu)$ by a strongly continuous semigroup. As we will see in Section 3.1 this criterion is particularly useful in the case when the measure μ is symmetrizing for the operator $\mathcal{A}$.

Hypothesis 3.1.

(i) Hypothesis 2.1 is satisfied.
(ii) The density ϱ of μ belongs to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N,dx)$, where r is the same exponent as in Hypothesis 2.1(i).

Remark 3.2. Observe that if the measure μ is infinitesimally invariant for the operator $\mathcal{A}$, i.e.,

$$\int_{\mathbb{R}^N} \mathcal{A}\phi\,\mu(dx) = 0, \quad \phi\in C_c^\infty\big(\mathbb{R}^N\big),$$

by [7, Corollary 2.10], the density ϱ of the measure μ with respect to the Lebesgue measure belongs to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N,dx)$. Hence Hypothesis 3.1(ii) is satisfied.

For our purposes, it is much more convenient to deal with operators whose principal part is in divergence form. Hence, we write $\mathcal{A}$ in the following equivalent way:

$$\mathcal{A}\psi = \frac{1}{\varrho}\sum_{i,j=1}^N D_i(\varrho q_{ij}D_j\psi) + \beta\cdot\nabla\psi = \frac{1}{\varrho}\mathrm{div}\big(\varrho(Q\nabla\psi)\big) + \beta\cdot\nabla\psi = \mathcal{A}_0\psi + \beta\cdot\nabla\psi,$$

for any smooth function ψ, where $\beta_j = b_j - \frac{1}{\varrho}\sum_{i=1}^N D_i(\varrho q_{ij}) \in L^r_{\mathrm{loc}}(\mathbb{R}^N,dx)$ $(j=1,\dots,n)$ as the functions q_{ij} $(i,j=1,\dots,N)$ are locally Hölder continuous and ϱ is locally uniformly positive.

The following lemma is essential in what follows.

Lemma 3.3. *Let r' be the conjugate exponent of r. Then, for any nonnegative function $\phi \in W^{1,r'}(\mathbb{R}^N, dx)$ with compact support, we have*

$$\int_{\mathbb{R}^N} (\beta \cdot \nabla\phi)\, \mu(dx) \leqslant \alpha \int_{\mathbb{R}^N} \phi\, \mu(dx). \tag{3.1}$$

Proof. We begin the proof observing that an integration by parts shows that

$$\int_{\mathbb{R}^N} f \mathcal{A}_0\phi\, \mu(dx) = \int_{\mathbb{R}^N} \operatorname{div}(\varrho Q\nabla\phi) f\, dx = -\int_{\mathbb{R}^N} \varrho(Q\nabla\phi)\cdot\nabla f\, dx = -\int_{\mathbb{R}^N} (Q\nabla\phi)\cdot\nabla f\, \mu(dx),$$

for every $f \in C^1(\mathbb{R}^N)$ and every $\phi \in C_c^\infty(\mathbb{R}^N)$. In particular, taking $f \equiv 1$ we get

$$\int_{\mathbb{R}^N} \mathcal{A}_0\phi\, \mu(dx) = 0. \tag{3.2}$$

Taking Hypothesis 2.1(iii) into account and integrating both the sides of the equality $\beta \cdot \nabla\phi = \mathcal{A}\phi - \mathcal{A}_0\phi$, we get (3.1) for all nonnegative functions $\phi \in C_c^\infty(\mathbb{R}^N)$.

To prove (3.1) in the general case, it suffices to observe that any nonnegative function $\phi \in W^{1,r'}(\mathbb{R}^N)$ with compact support is the limit in $W^{1,r'}(\mathbb{R}^N)$ of a sequence $\{\phi_n\} \in C_c^\infty(K)$ of nonnegative functions, where K is a suitable neighborhood of the support of ϕ, and use the dominated convergence theorem. □

We can now prove the following result.

Theorem 3.4. *Let $1 < p < +\infty$. Assume that Hypothesis 3.1 holds and there exists a positive function $V \in C^1(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and*

$$\frac{\beta \cdot \nabla V}{V \log V} \geqslant -C \quad \text{and} \quad \frac{(Q\nabla V)\cdot\nabla V}{(V\log V)^2} \in L^\infty\big(\mathbb{R}^N, dx\big), \tag{3.3}$$

for some positive constant C. Then, the closure of the operator $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ on $L^p(\mathbb{R}^N, \mu)$ generates a sub-Markovian strongly continuous semigroup. In particular, $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ is $L^p(\mathbb{R}^N, \mu)$ unique.

Proof. Since $V(x)$ tends to $+\infty$ as $|x| \to +\infty$, up to replacing V with $V + c$ for a suitable positive constant c, we can assume, without loss of generality, that $V(x) > 1$ for any $x \in \mathbb{R}^N$.

Let p' be the conjugate exponent of p. Fix $\lambda > \alpha + \frac{1}{p}$ and let $g \in L^{p'}(\mathbb{R}^N, \mu)$ be such that

$$\int_{\mathbb{R}^N} (\lambda\phi - \mathcal{A}\phi) g\, \mu(dx) = 0,$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. We claim that $g \equiv 0$.

By [7, Corollary 2.10], the function $F := g\varrho$ belongs to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N, dx)$. In particular, since $r > N$, F is continuous and locally bounded in $\mathbb{R}^N$. Hence, g is locally bounded in $\mathbb{R}^N$ as well, ϱ being everywhere positive and continuous in $\mathbb{R}^N$. Moreover $\nabla \frac{1}{\varrho} = -\frac{\nabla\varrho}{\varrho^2} \in L^r_{\mathrm{loc}}(\mathbb{R}^N, dx)$. Therefore, $g = F \cdot \frac{1}{\varrho} \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N, dx)$.

Let us now fix $\phi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ and observe that

$$\lambda \int_{\mathbb{R}^N} \phi g\, \mu(dx) = \int_{\mathbb{R}^N} (\mathcal{A}\phi) g\, \mu(dx)$$
$$= -\int_{\mathbb{R}^N} (Q\nabla\phi)\cdot \nabla g\, \mu(dx) + \int_{\mathbb{R}^N} (\beta \cdot \nabla\phi) g\, \mu(dx). \tag{3.4}$$

By density, the equality (3.4) extends to every $\phi \in W^{1,r'}(\mathbb{R}^N, dx) \cap L^\infty(\mathbb{R}^N, dx)$ with compact support. Since $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N, dx) \subset W^{1,r'}_{\mathrm{loc}}(\mathbb{R}^N, dx)$, formula (3.4) holds true for every $\phi \in W^{1,2}(\mathbb{R}^N, dx) \cap L^\infty(\mathbb{R}^N, dx)$ with compact support.

Next, let $\zeta : \mathbb{R} \to [0, 1]$ be a decreasing C^1-function such that $\zeta(s) = 1$ if $s \leqslant 1$ and $\zeta(s) = 0$ if $s \geqslant 2$, and define $\zeta_n := \zeta(\frac{\log V}{n})$ for every $n \in \mathbb{N}$. Since $V(x)$ tends to $+\infty$ as $|x| \to +\infty$, each function ζ_n belongs to $C^1_{\mathrm{c}}(\mathbb{R}^N)$. Moreover, $\zeta_n \leqslant 1$ and ζ_n converges to 1 pointwise in $\mathbb{R}^N$ as $n \to +\infty$.

We now consider the cases $p \leqslant 2$ and $p > 2$ separately.

Case $1 < p \leqslant 2$. Let us set $\phi_n = \zeta_n^2 \operatorname{sign}(g)|g|^{p'-1}$ for any $n \in \mathbb{N}$. Let us observe that both the functions ϕ_n and $\phi_n g$ belong to $W^{1,2}(\mathbb{R}^N, dx) \cap L^\infty(\mathbb{R}^N, dx)$ by the local boundedness of g, and are compactly supported in $\mathbb{R}^N$. From (3.4) we obtain

$$\lambda \int_{\mathbb{R}^N} \zeta_n^2 \operatorname{sign}(g)|g|^{p'-1} g\, \mu(dx) + (p'-1)\int_{\mathbb{R}^N} \zeta_n^2 |g|^{p'-2} (Q\nabla g)\cdot\nabla g\, \mu(dx)$$
$$= -\frac{2}{n}\int_{\mathbb{R}^N} \zeta_n \zeta'\big(n^{-1}\log V\big) \operatorname{sign}(g)|g|^{p'-1} \frac{(Q\nabla V)\cdot\nabla g}{V}\, \mu(dx) + \int_{\mathbb{R}^N} (\beta\cdot\nabla\phi_n) g\, \mu(dx)$$
$$= -\frac{2}{n}\int_{\mathbb{R}^N} \zeta_n \zeta'\big(n^{-1}\log V\big) \operatorname{sign}(g)|g|^{p'-1} \frac{(Q\nabla V)\cdot\nabla g}{V}\, \mu(dx)$$
$$+ \frac{p'-1}{p'}\int_{\mathbb{R}^N} \beta\cdot\nabla(\phi_n g)\, \mu(dx) + \frac{1}{p'}\int_{\mathbb{R}^N} \big(\beta\cdot\nabla\zeta_n^2\big)|g|^{p'}\, \mu(dx)$$
$$\leqslant -\frac{2}{n}\int_{\mathbb{R}^N} \zeta_n \zeta'\big(n^{-1}\log V\big) \operatorname{sign}(g)|g|^{p'-1} \frac{(Q\nabla V)\cdot\nabla g}{V}\, \mu(dx)$$
$$+ \frac{\alpha}{p}\int_{\mathbb{R}^N} \zeta_n^2 |g|^{p'}\, \mu(dx) + \frac{2}{p'n}\int_{\mathbb{R}^N} \zeta_n\zeta'\big(n^{-1}\log V\big)|g|^{p'} \frac{\beta\cdot\nabla V}{V}\, \mu(dx).$$

In the second equality and in the last inequality, we have used, respectively, the formula

$$g\nabla\big(\zeta_n^2 \operatorname{sign}(g)|g|^{p'-1}\big) = \frac{p'-1}{p'}\nabla\big(\zeta_n^2|g|^{p'}\big) + \frac{1}{p'}|g|^{p'}\nabla\zeta_n^2$$

and Lemma 3.3, being $\phi_n g \geqslant 0$. Hence,

$$\begin{aligned}
&\left(\lambda - \frac{\alpha}{p}\right)\int_{\mathbb{R}^N} \zeta_n^2 \operatorname{sign}(g)|g|^{p'-1} g\,\mu(dx) + (p'-1)\int_{\mathbb{R}^N} \zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g\,\mu(dx)\\
&\quad\leqslant -\frac{2}{n}\int_{\mathbb{R}^N} \zeta_n\zeta'\big(n^{-1}\log V\big)\operatorname{sign}(g)|g|^{p'-1}\frac{(Q\nabla V)\cdot\nabla g}{V}\,\mu(dx)\\
&\qquad+\frac{2}{p'n}\int_{\mathbb{R}^N} \zeta_n\zeta'\big(n^{-1}\log V\big)|g|^{p'}\frac{\beta\cdot\nabla V}{V}\,\mu(dx).
\end{aligned}\tag{3.5}$$

Note that both the two terms in the first side of (3.5) are positive and the last side of (3.5) is finite. It follows that the function $\zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g$ belongs to $L^1(\mathbb{R}^N,\mu)$. Taking this fact into account, we can now estimate

$$\begin{aligned}
&\frac{2}{n}\int_{\mathbb{R}^N}\left|\zeta_n\zeta'\big(n^{-1}\log V\big)\operatorname{sign}(g)|g|^{p'-1}\frac{(Q\nabla V)\cdot\nabla g}{V}\right|\mu(dx)\\
&\quad\leqslant \frac{2}{n}\int_{\mathbb{R}^N}\zeta_n\big|\zeta'\big(n^{-1}\log V\big)\big||g|^{p'-1}\frac{((Q\nabla V)\cdot\nabla V)^{\frac12}}{V}\big((Q\nabla g)\cdot\nabla g\big)^{\frac12}\,\mu(dx)\\
&\quad= 2\int_{\mathbb{R}^N}\left(\zeta_n|g|^{\frac{p'-2}{2}}\big((Q\nabla g)\cdot\nabla g\big)^{\frac12}\right)\left(\frac{1}{n}\big|\zeta'\big(n^{-1}\log V\big)\big||g|^{\frac{p'}{2}}\frac{((Q\nabla V)\cdot\nabla V)^{\frac12}}{V}\right)\mu(dx)\\
&\quad\leqslant \int_{\mathbb{R}^N}\zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g\,\mu(dx)\\
&\qquad+\frac{1}{n^2}\int_{\mathbb{R}^N}\big|\zeta'\big(n^{-1}\log V\big)\big|^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx).
\end{aligned}\tag{3.6}$$

Hence, replacing (3.6) into (3.5), we get

$$\begin{aligned}
&\left(\lambda - \frac{\alpha}{p}\right)\int_{\mathbb{R}^N} \zeta_n^2 \operatorname{sign}(g)|g|^{p'-1} g\,\mu(dx) + (p'-2)\int_{\mathbb{R}^N} \zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g\,\mu(dx)\\
&\quad\leqslant \frac{1}{n^2}\int_{\mathbb{R}^N}\big|\zeta'\big(n^{-1}\log V\big)\big|^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx)\\
&\qquad+\frac{2}{p'n}\int_{\mathbb{R}^N} \zeta_n\zeta'\big(n^{-1}\log V\big)|g|^{p'}\frac{\beta\cdot\nabla V}{V}\,\mu(dx).
\end{aligned}\tag{3.7}$$

Since $\zeta' \leqslant 0$, $\zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g \geqslant 0$ and $p' \geqslant 2$, from (3.7) and the conditions (3.3), we get

$$\begin{aligned}\left(\lambda-\frac{\alpha}{p}\right)\int_{\mathbb{R}^N}\zeta_n^2|g|^{p'}\,\mu(dx) &\leqslant \frac{1}{n^2}\int_{\mathbb{R}^N}\bigl|\zeta'\bigl(n^{-1}\log V\bigr)\bigr|^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx)\\ &\quad+\frac{2}{p'n}\int_{\mathbb{R}^N}\zeta_n\zeta'\bigl(n^{-1}\log V\bigr)|g|^{p'}\frac{\beta\cdot\nabla V}{V}\,\mu(dx)\\ &\leqslant \frac{1}{n^2}\int_{\mathbb{R}^N}\bigl|\zeta'\bigl(n^{-1}\log V\bigr)\bigr|^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx)\\ &\quad+\frac{2C}{p'n}\int_{\mathbb{R}^N}\zeta_n\bigl|\zeta'\bigl(n^{-1}\log V\bigr)\bigr||g|^{p'}\log V\,\mu(dx). \end{aligned}\tag{3.8}$$

Observing that $\zeta'(n^{-1}\log V(x)) = 0$ if $V(x) \notin [e^n, e^{2n}]$ and taking conditions (3.3) into account, we can estimate

$$\frac{2C}{p'n}\zeta_n\bigl|\zeta'\bigl(n^{-1}\log V\bigr)\bigr||g|^{p'}\log V \leqslant \frac{4C}{p'}\|\zeta'\|_\infty|g|^{p'}$$

and

$$\frac{1}{n^2}\bigl|\zeta'\bigl(n^{-1}\log V\bigr)\bigr|^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2} \leqslant 4\|\zeta'\|_\infty^2\left\|\frac{(Q\nabla V)\cdot\nabla V}{(V\log V)^2}\right\|_\infty|g|^{p'}.$$

Since $|g|^{p'} \in L^1(\mathbb{R}^N,\mu)$, we can apply Fatou's lemma in (3.8) which yields

$$\left(\lambda-\frac{\alpha}{p}\right)\int_{\mathbb{R}^N}|g|^{p'}\,\mu(dx) \leqslant 0,$$

thereby implying that $g \equiv 0$.

Case $p > 2$. For all $m \in \mathbb{N}$, let $\phi_m(x) = x(x^2+m^{-1})^{\frac{p'-2}{2}}$ for any $x \in \mathbb{R}$. Clearly, $\phi_m \in C^\infty(\mathbb{R})$. For all $n, m \in \mathbb{N}$, set $u_{n,m} = \zeta_n^2\phi_m(g)$, where ζ_n is as above. Then, $u_{n,m}$ has compact support and is bounded by the local boundedness of g. Moreover, as

$$\nabla u_{n,m} = \frac{2}{n}\zeta_n\zeta'\bigl(n^{-1}\log V\bigr)\phi_m(g)\frac{\nabla V}{V} + \zeta_n^2\phi_m'(g)\nabla g, \quad n, m \in \mathbb{N}, \tag{3.9}$$

and $g \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N, dx)$, the function $u_{n,m}$ belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N, dx)$ for any $n, m \in \mathbb{N}$. Hence, we can apply (3.4), replacing ϕ with $u_{n,m}$, which yields

$$\lambda \int_{\mathbb{R}^N} u_{n,m} g\,\mu(dx) + (p'-1) \int_{\mathbb{R}^N} \zeta_n^2 g^2 \left(g^2 + \frac{1}{m}\right)^{\frac{p'-4}{2}} (Q\nabla g)\cdot\nabla g\,\mu(dx)$$

$$+ \frac{1}{m} \int_{\mathbb{R}^N} \zeta_n^2 \left(g^2 + \frac{1}{m}\right)^{\frac{p'-4}{2}} (Q\nabla g)\cdot\nabla g\,\mu(dx)$$

$$= -\frac{2}{n} \int_{\mathbb{R}^N} \zeta_n \zeta'\left(n^{-1}\log V\right) g \left(g^2 + \frac{1}{m}\right)^{\frac{p'-2}{2}} \frac{(Q\nabla V)\cdot\nabla g}{V}\,\mu(dx)$$

$$+ \int_{\mathbb{R}^N} (\beta\cdot\nabla u_{n,m}) g\,\mu(dx).$$

Since $\zeta_n^2(g^2 + \frac{1}{m})^{\frac{p'-4}{2}} (Q\nabla g)\cdot\nabla g \geqslant 0$ for any $m, n \in \mathbb{N}$, we obtain

$$\lambda \int_{\mathbb{R}^N} u_{n,m} g\,\mu(dx) + (p'-1) \int_{\mathbb{R}^N} \zeta_n^2 g^2 \left(g^2 + \frac{1}{m}\right)^{\frac{p'-4}{2}} (Q\nabla g)\cdot\nabla g\,\mu(dx)$$

$$\leqslant -\frac{2}{n} \int_{\mathbb{R}^N} \zeta_n \zeta'\left(n^{-1}\log V\right) g \left(g^2 + \frac{1}{m}\right)^{\frac{p'-2}{2}} \frac{(Q\nabla V)\cdot\nabla g}{V}\,\mu(dx)$$

$$+ \int_{\mathbb{R}^N} (\beta\cdot\nabla g)\zeta_n^2 \left(g^2 + \frac{1}{m}\right)^{\frac{p'-4}{2}} \left((p'-1)g^2 + \frac{1}{m}\right) g\,\mu(dx)$$

$$+ \int_{\mathbb{R}^N} \left(\beta\cdot\nabla\zeta_n^2\right) g^2 \left(g^2 + \frac{1}{m}\right)^{\frac{p'-2}{2}} \mu(dx). \tag{3.10}$$

Observe that for every $m, n \in \mathbb{N}$, we have

$$\begin{aligned} &\zeta_n \left|\zeta'\left(n^{-1}\log V\right)\right| |g| \left(g^2 + m^{-1}\right)^{\frac{p'-2}{2}} \left|(Q\nabla V)\cdot\nabla g\right| V^{-1} \\ &\quad \leqslant \zeta_n |g|^{p'-1} \|\zeta'\|_\infty \left\| |Q\nabla V| \right\|_{L^\infty(\mathrm{supp}(\zeta_n))} |\nabla g|, \end{aligned} \tag{3.11}$$

and, since $p' < 2$,

$$\begin{aligned} \left|(\beta\cdot\nabla g)g\right| \zeta_n^2 \left(g^2 + m^{-1}\right)^{\frac{p'-4}{2}} \left((p'-1)g^2 + m^{-1}\right) &\leqslant \left|(\beta\cdot\nabla g)g\right| \zeta_n^2 \left(g^2 + m^{-1}\right)^{\frac{p'-2}{2}} \\ &\leqslant \left|(\beta\cdot\nabla g)\right| \left(g^2 + m^{-1}\right)^{\frac{p'-1}{2}}. \end{aligned} \tag{3.12}$$

Moreover,

$$\left|\left(\beta\cdot\nabla\zeta_n^2\right)\right|g^2\left(g^2+m^{-1}\right)^{\frac{p'-2}{2}}\leqslant\left|\left(\beta\cdot\nabla\zeta_n^2\right)\right|\left(g^2+1\right)^{\frac{p'}{2}}. \tag{3.13}$$

Note that the right-hand sides of (3.11)–(3.13) belong to $L^1(\mathbb{R}^N,\mu)$. Therefore, letting m tend to $+\infty$ in both the sides of (3.10), by the Fatou's lemma and dominated convergence we get

$$\begin{aligned}
&\lambda\int_{\mathbb{R}^N}\zeta_n^2|g|^{p'}\,\mu(dx)+(p'-1)\int_{\mathbb{R}^N}\zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g\,\mu(dx)\\
&\quad\leqslant\frac{2}{n}\int_{\mathbb{R}^N}\zeta_n\left|\zeta'\left(n^{-1}\log V\right)\right||g|^{p'-1}V^{-1}(Q\nabla V)\cdot\nabla g\,\mu(dx)\\
&\qquad+\frac{2}{n}\int_{\mathbb{R}^N}\zeta_n\zeta'\left(n^{-1}\log V\right)|g|^{p'}V^{-1}(\beta\cdot\nabla V)\,\mu(dx)\\
&\qquad+(p'-1)\int_{\mathbb{R}^N}\zeta_n^2\operatorname{sign}(g)|g|^{p'-1}(\beta\cdot\nabla g)\,\mu(dx)\\
&\quad=\frac{2}{n}\int_{\mathbb{R}^N}\zeta_n\left|\zeta'\left(n^{-1}\log V\right)\right||g|^{p'-1}V^{-1}(Q\nabla V)\cdot\nabla g\,\mu(dx)\\
&\qquad+(p'-1)\int_{\mathbb{R}^N}(\beta\cdot\nabla g)\zeta_n^2|g|^{p'-2}g\,\mu(dx)+\int_{\mathbb{R}^N}\left(\beta\cdot\nabla\zeta_n^2\right)|g|^{p'}\,\mu(dx).
\end{aligned}$$

On the other hand, using Young inequality, we get

$$\begin{aligned}
&\frac{2}{n}\left|\zeta_n\zeta'\left(n^{-1}\log V\right)\right||g|^{p'-1}V^{-1}(Q\nabla V)\cdot\nabla g\\
&\quad\leqslant\frac{2}{n}\zeta_n\left|\zeta'\left(n^{-1}\log V\right)\right||g|^{p'-1}V^{-1}\left((Q\nabla V)\cdot\nabla V\right)^{\frac{1}{2}}\left((Q\nabla g)\cdot\nabla g\right)^{\frac{1}{2}}\\
&\quad\leqslant 2\left(\zeta_n|g|^{\frac{p'-2}{2}}\left((Q\nabla g)\cdot\nabla g\right)^{\frac{1}{2}}\right)\left(\frac{1}{n}\left|\zeta'\left(n^{-1}\log V\right)\right||g|^{\frac{p'}{2}}V^{-1}\left((Q\nabla V)\cdot\nabla V\right)^{\frac{1}{2}}\right)\\
&\quad\leqslant\varepsilon\zeta_n^2|g|^{p'-2}(Q\nabla g)\cdot\nabla g+\frac{1}{\varepsilon n^2}\left(\zeta'\left(n^{-1}\log V\right)\right)^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2},
\end{aligned}$$

for any $\varepsilon>0$. Therefore, taking $\varepsilon=p'-1$ we get

$$\begin{aligned}
\lambda\int_{\mathbb{R}^N}\zeta_n^2|g|^{p'}\,\mu(dx)&\leqslant\frac{1}{(p'-1)n^2}\int_{\mathbb{R}^N}\left(\zeta'\left(n^{-1}\log V\right)\right)^2|g|^{p'}\frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx)\\
&\quad+(p'-1)\int_{\mathbb{R}^N}(\beta\cdot\nabla g)\zeta_n^2|g|^{p'-1}\operatorname{sign}(g)\,\mu(dx)\\
&\quad+\int_{\mathbb{R}^N}\left(\beta\cdot\nabla\zeta_n^2\right)|g|^{p'}\,\mu(dx)
\end{aligned}$$

$$= \frac{1}{(p'-1)n^2} \int_{\mathbb{R}^N} \big(\zeta'\big(n^{-1}\log V\big)\big)^2 |g|^{p'} \frac{(Q\nabla V)\cdot\nabla V}{V^2}\,\mu(dx)$$
$$+ \left(1-\frac{1}{p'}\right) \int_{\mathbb{R}^N} \big(\beta\cdot\nabla\big(|g|^{p'}\big)\big)\zeta_n^2\,\mu(dx)$$
$$+ \int_{\mathbb{R}^N} \big(\beta\cdot\nabla\zeta_n^2\big)|g|^{p'}\,\mu(dx). \tag{3.14}$$

Arguing as in the case $p \leqslant 2$, we can show that

$$\int_{\mathbb{R}^N} \big(\beta\cdot\nabla\big(|g|^{p'}\big)\big)\zeta_n^2\,\mu(dx) \leqslant \alpha \int_{\mathbb{R}^N} |g|^{p'}\zeta_n^2\,\mu(dx) - \int_{\mathbb{R}^N} \big(\beta\cdot\nabla\zeta_n^2\big)|g|^{p'}\,\mu(dx), \tag{3.15}$$

for any $n \in \mathbb{N}$. Hence, replacing (3.15) into (3.14), observing that $\zeta'(n^{-1}\log V)$ pointwise vanishes as $n \to +\infty$ and applying Fatou's lemma, we obtain that $g \equiv 0$. Thus $(\lambda - \mathcal{A})(C_c^\infty(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N,\mu)$. The assertion now follows as in the proof of Theorem 2.2. □

Proposition 3.5. *Assume that Hypothesis* 3.1 *holds and there exists a strictly positive function* $V \in C^1(\mathbb{R}^N)$ *such that* $\lim_{|x|\to+\infty} V(x) = +\infty$ *and*

$$\frac{\beta\cdot\nabla V}{V\log V} \in L^1\big(\mathbb{R}^N,\mu\big) \quad \textit{and} \quad \frac{(Q\nabla V)\cdot\nabla V}{(V\log V)^2} \in L^1\big(\mathbb{R}^N,\mu\big).$$

Then, the closure of the operator $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ *on* $L^1(\mathbb{R}^N,\mu)$ *generates a sub-Markovian strongly continuous semigroup. In particular,* $(\mathcal{A}, C_c^\infty(\mathbb{R}^N))$ *is* $L^1(\mathbb{R}^N,\mu)$ *unique.*

Proof. Repeating verbatim the proof of Theorem 3.4 with $p = 2$, we can show that

$$\left(\lambda - \frac{\alpha}{2}\right) \int_{\mathbb{R}^N} \zeta_n^2 |g|^2\,\mu(dx) \leqslant \int_{\mathbb{R}^N} \frac{(\log V)^2}{n^2} \big|\zeta'\big(n^{-1}\log V\big)\big|^2 |g|^2 \frac{(Q\nabla V)\cdot\nabla V}{(V\log V)^2}\,\mu(dx)$$
$$+ \int_{\mathbb{R}^N} \frac{\log V}{n} \zeta_n \zeta'\big(n^{-1}\log V\big)|g|^2 \frac{\beta\cdot\nabla V}{V\log V}\,\mu(dx). \tag{3.16}$$

The integrability assumptions on $(V\log V)^{-1}(\beta\cdot V)$ and $(V\log V)^{-2}(Q\nabla V)\cdot\nabla V$ allow us to apply the Fatou's lemma and still conclude from (3.8) that $\int_{\mathbb{R}^N} g^2\,\mu(dx) = 0$, so that $g \equiv 0$. □

3.1. The case of symmetrizing invariant measures

In this subsection, we consider the case when the measure μ is symmetrizing for the operator $\mathcal{A}$, i.e., the case when $\beta \equiv 0$ in $\mathbb{R}^N$. In this case, μ is an infinitesimally invariant measure for $\mathcal{A}$ (see (3.2)) and we can prove the following result.

Proposition 3.6. *Fix $p > 1$. Assume that Hypothesis* 2.1(i)–(iii) *holds, $b_i \in L^p_{\mathrm{loc}}(\mathbb{R}^N, dx)$ ($i = 1, \dots, N$) and that μ is a symmetrizing invariant measure for the operator $\mathcal{A}$. Let $\Lambda(x)$ denote the maximum eigenvalue of the matrix $Q(x)$ for any $x \in \mathbb{R}^N$, and set $\lambda(s) = \max_{|x|=s} \Lambda(x)$ for any $s \geqslant 0$. Finally, assume that $\lambda^{-1/2}$ is not integrable in a neighborhood of $+\infty$. Then, the closure of the operator $(\mathcal{A}, C^\infty_c(\mathbb{R}^N))$ on $L^p(\mathbb{R}^N, \mu)$ generates a Markov strongly continuous semigroup.*

Proof. As it has been observed in Remark 3.2, the density of μ with respect to the Lebesgue measure belongs to $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^N, dx)$. Thus, Hypothesis 3.1 is satisfied. Let $V : \mathbb{R}^N \to \mathbb{R}$ be any positive C^1-function such that $V(x) = e^{\int_0^{|x|} \lambda^{-1/2}(s)\, ds}$ for any $x \in \mathbb{R}^N$ with $|x| \geqslant 1$. Then,

$$\frac{(Q(x)\nabla V(x)) \cdot \nabla V(x)}{(V(x) \log V(x))^2} = \frac{(Q(x)x) \cdot x}{\lambda(|x|)|x|^2 (\int_0^{|x|} \lambda^{-1/2}(s)\, ds)^2} \leqslant \frac{1}{(\int_0^{|x|} \lambda^{-1/2}(s)\, ds)^2}, \quad |x| \geqslant 1.$$

Hence, conditions (3.3) are satisfied and the assertion follows from Theorem 3.4. □

In the following corollary we specialize our result to the case when $\mathcal{A}$ is a generalized Schrödinger operator, i.e., in the case when

$$\mathcal{A}\phi = \Delta\phi + \frac{\nabla\varrho}{\varrho} \cdot \nabla\phi, \tag{3.17}$$

on smooth functions ϕ.

Corollary 3.7. *Let $p > 1$ and let $\varrho \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N, dx)$ be locally uniformly positive. If there exists $r > N$ such that $\nabla\varrho/\varrho \in L^r_{\mathrm{loc}}(\mathbb{R}^N, dx)$, then the closure of the operator $(\mathcal{A}, C^\infty_c(\mathbb{R}^N))$ on $L^p(\mathbb{R}^N, \mu)$ generates a Markov strongly continuous semigroup, where $\mu(dx) = \varrho\, dx$. If μ is finite, then the result holds also for $p = 1$.*

Proof. It suffices to apply Proposition 3.6 for $p > 1$, and Proposition 3.5 for $p = 1$, with $V(x) = |x|$ for large $|x|$. □

Remark 3.8. Some remarks are in order.

(i) The one-dimensional case has been completely characterized by Eberle in [14].
(ii) Corollary 3.7 allows us to cover also some situations to which [14, Theorem 2.6] does not apply. Indeed, in the case when ϱ is not integrable with respect to the Lebesgue measure, the invariant measure $\mu(dx) = \varrho\, dx$ may not satisfy the condition

$$\limsup_{r \to +\infty} \frac{1}{r^k} \int_{B_r} \varrho(x)\, dx < +\infty, \tag{3.18}$$

whatever $k > 0$ may be, which was one of the main requirement of [14, Theorem 2.6]. For instance, the condition (3.18) is not satisfied when ϱ is any smooth function such that $\varrho(x) = e^{|x|^2}$ for large $|x|$.

Similarly, in the case when the measure μ is finite, our result generalizes the quoted theorem by Eberle for large p's. Indeed, another important requirement in that theorem is that the function $\beta = \frac{\nabla\varrho}{\varrho}$ belongs to $L^r_{\mathrm{loc}}(\mathbb{R}^N, \varrho\, dx)$ for some $r > (1 + N/2)p$. Since, by our assumptions $L^r_{\mathrm{loc}}(\mathbb{R}^N, \varrho\, dx) = L^r_{\mathrm{loc}}(\mathbb{R}^N, dx)$, our result extends [14, Theorem 2.6] when $(1 + N/2)p > N$.

(iii) In the case when $p = 2$ we recover the same result as in [6, Theorem 7].

(iv) A result similar to Corollary 3.7 has been proved by Liskevič [20] and Liskevič and Semenov [21] for $p > 3/2$. Our result generalizes the results by Liskevič and Semenov in the sense that, differently from them, we do not assume any global integrability conditions and our result holds true for any $p > 1$.

(v) Finally, we point out that a detailed discussion of L^1-uniqueness has been given by Stannat in [32].

Acknowledgment

The authors thank G. Metafune for having suggested some improvements in our paper.

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