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Collapses, products and LC manifolds

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ABSTRACT

The quantum physicists Durhuus and Jonsson (1995) [9] introduced the class of "locally constructible" (LC) triangulated manifolds and showed that all the LC 2- and 3-manifolds are spheres. We show here that for each $d > 3$ some LC d-manifolds are not spheres. We prove this result by studying how to collapse products of manifolds with one facet removed.

1. Introduction

Collapses are a classical notion in Combinatorial Topology, originally introduced in the Thirties by Whitehead [14], extensively studied in the Sixties by Bing, Cohen, Lickorish and Zeeman among others, yet also at the center of recent works such as [1] and [8].

Given a polytopal (or a regular CW) complex, a collapse is a move that cancels two faces and yields a smaller complex which is topologically a strong deformation retract of the starting one. Any complex that is collapsible (i.e. transformable into a point via a sequence of collapses) is thus also contractible. Conversely, every shellable contractible complex is collapsible.

However, not all contractible complexes are collapsible: A famous two-dimensional counterexample is given by Zeeman's dunce hat [15]. According to the work of Whitehead [14] and Cohen [7], a complex C is contractible if and only if some collapsible complex D collapses also onto C. In fact, one can construct a collapsible triangulated 3-ball with only 8 vertices that collapses onto a copy of the dunce hat [3]. Cohen's result is obtained by taking *products*: Zeeman [15] first noticed that the product of the dunce hat with a segment $\mathbb{I}$ is polyhedrally collapsible and asked whether the same holds for any contractible 2-complex. (The question, known as *Zeeman's conjecture*, is still open [10]. For polyhedral collapsibility, see e.g. [11, pp. 42–48].)

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Cohen [7, Corollaries 3 and 4] showed that the product of any contractible d-complex C with the q-dimensional cube $\mathbb{I}^q$ polyhedrally collapses onto a point, provided $q \geqslant \max(2d, 5)$. At the same time, $C \times \mathbb{I}^q$ collapses onto C (cf. Corollary 2.2).

It was first discovered by Bing [5] that some triangulations of 3-balls are not collapsible. For each $d \geqslant 3$, Lickorish [12] proved that some triangulated d-balls of the form $S - \Delta$ (with S a d-sphere and Δ a facet of S) are not collapsible. Bing's and Lickorish's claim were recently strengthened by the author and Ziegler [4, Theorem 2.19], who showed that for each $d \geqslant 3$ certain triangulated d-balls of the form $S - \Delta$ do not even collapse onto any $(d-2)$-dimensional subcomplex of S. These three results were all obtained via knot theory. In fact, a 3-ball may contain arbitrarily complicated three-edge-knots in its 1-skeleton. Depending on how complicated the knot is, one can draw sharp conclusions on the collapsibility of the 3-ball and of its successive suspensions.

In the nineties, two quantum physicists, Durhuus and Jonsson [9], introduced the term "*LC d-manifold*" to describe a manifold that can be obtained from a tree of d-polytopes by repeatedly identifying two combinatorially equivalent adjacent $(d-1)$-faces in the boundary ($d \geqslant 2$). Plenty of spheres satisfy this bizarre requirement: In fact, all shellable and all constructible d-spheres are LC (cf. [4]). At the same time, simplicial LC d-manifolds are only exponentially many when counted with respect to the number of facets, while arbitrary (simplicial) d-manifolds are much more numerous [2, Chapter 2].

Durhuus and Jonsson noticed that the class of LC d-manifolds coincides with the class of all d-spheres for $d = 2$. But what about higher dimensions?

For $d = 3$, they were able to prove one of the two inclusions, namely, that all LC 3-manifolds are spheres [9, Theorem 2]. The other inclusion does not hold: For each $d \geqslant 3$, some d-spheres are not LC, as established in [4]. The examples of non-LC spheres are given by 3-spheres with a three-edge-knot in their 1-skeleton (provided the knot is sufficiently complicated!) and by their successive suspensions.

The analogy with the aforementioned obstructions to collapsibility is not a coincidence: In fact, the LC d-spheres can be characterized [4, Theorem 2.1] as the d-spheres that collapse onto a $(d-2)$-complex after the removal of a facet. (It does not matter which facet you choose.) This characterization can be easily extended to (closed) manifolds:

A d-manifold M is LC if and only if M minus a facet collapses onto a $(d-2)$-complex.

Exploiting this characterization, in the present paper we prove the following statement:

Main Theorem 1. *The product of LC manifolds is an LC manifold.*

The proof, which is elementary, can be outlined as follows: Suppose a manifold M (resp. M') minus a facet collapses onto a $(\dim M - 2)$-complex C (resp. a $(\dim M' - 2)$-complex C'). We show that the complex obtained by removing a facet from $M \times M'$ collapses onto the complex $(C \times M') \cup (M \times C')$, which is $(\dim M + \dim M' - 2)$-dimensional.

As a corollary, we immediately obtain that some LC 4-manifolds are not spheres, but rather products of two LC 2-spheres. This enables us to solve Durhuus–Jonsson's problem for all dimensions:

Main Theorem 2. *The class of LC 2-manifolds coincides with the class of all 2-spheres.*

The class of LC 3-manifolds is strictly contained in the class of all 3-spheres.

For each $d \geqslant 4$, the class of LC d-manifolds and the class of all d-spheres are overlapping, but none of them is contained in the other.

By the work of Zeeman (see e.g. [6]), for every positive integer d, every shellable or constructible d-manifold is a d-sphere. Thus, the properties of shellability and constructibility are obviously *not* inherited by products. All 2-spheres are LC, constructible and shellable; however, for each $d \geqslant 3$, all shellable d-spheres are constructible, all constructible d-spheres are LC, but some LC d-spheres are not constructible [4]. It is still unknown whether all constructible spheres are shellable.

1.1. Definitions

A *polytopal complex* is a finite, nonempty collection C of polytopes (called the *faces* of C) in some Euclidean space $\mathbb{R}^k$, such that (1) if σ is a polytope in C then all the faces of σ are elements of C and (2) the intersection of any two polytopes of C is a face of both. If d is the largest dimension of a polytope of C, the polytopal complex C is called *d-complex*. An inclusion-maximal face of C is called *facet*. A d-complex is *simplicial* (resp. *cubical*) if all of its facets are simplices (resp. cubes). Given an a-complex A and a b-complex B, the *product* $C = A \times B$ is an $(a+b)$-complex whose nonempty faces are the products $P_\alpha \times P_\beta$, where P_α (resp. P_β) ranges over the nonempty polytopes of A (resp. B). In general, the product of two simplicial complexes is *not* a simplicial complex, while the product of two cubical complexes is a cubical complex.

Let C be a d-complex. An *elementary collapse* is the simultaneous removal from C of a pair of faces (σ, Σ), such that σ is a proper face of Σ and of no other face of C. (This is usually abbreviated as "σ is a free face of Σ"; some complexes have no free faces.) We say the complex C *collapses onto* the complex D, and write $C \searrow D$, if C can be deformed onto D by a finite (nonempty) sequence of elementary collapses. Without loss of generality, we may assume that in this sequence the pairs $((d-1)\text{-face}, d\text{-face})$ are removed first; we may also assume that after that, the pairs $((d-2)\text{-face}, (d-1)\text{-face})$ are removed; and so on. A *collapsible* d-complex is a d-complex that can be collapsed onto a single vertex. If C collapses onto D, then D is a strong deformation retract of C, so C and D have the same homotopy type. In particular, all collapsible complexes are contractible.

The *underlying space* $|C|$ of a d-complex C is the union of all of its faces. A *d-sphere* is a d-complex whose underlying space is homeomorphic to $\{\mathbf{x} \in \mathbb{R}^{d+1}\colon\ |\mathbf{x}| = 1\}$. A *d-ball* is a d-complex with underlying space homeomorphic to $\{\mathbf{x} \in \mathbb{R}^{d}\colon\ |\mathbf{x}| \leqslant 1\}$; a *tree of d-polytopes* is a d-ball whose dual graph is a tree. With abuse of language, by *d-manifold* we will mean any d-complex whose underlying space is homeomorphic to a compact connected topological manifold (without boundary).

A *locally constructible* (LC) d-manifold is a d-manifold obtained from a tree of polytopes by repeatedly identifying a pair of adjacent $(d-1)$-faces of the boundary. ("Adjacent" means here "sharing at least a $(d-2)$-face" and represents a dynamic requirement: after each identification, new pairs of boundary facets might become adjacent and may be glued together.) Equivalently [4, Theorem 2.1], [2, Theorem 5.2.6], an LC d-manifold is a d-manifold that after the removal of a facet collapses onto a $(d-2)$-dimensional subcomplex. For the definition of shellability or constructibility, see e.g. Björner [6, p. 1854].

2. Proof of the main results

In this section, we exploit the characterization of LC manifolds mentioned in the Introduction to prove Main Theorems 1 and 2. In fact:

- Main Theorem 1 will be a straightforward consequence of Corollary 2.4;
- Main Theorem 2 follows directly from Remark 2.7, because we already know that all LC 2- and 3-manifolds are spheres [9, Theorem 2], that all 2-spheres are LC [9] and that some d-spheres are not LC for each $d \geqslant 3$ [4].

Let us start with a classical result on collapses and products:

Proposition 2.1. *(See Cohen [7, p. 254], see also Welker [13, Theorem 2.6].) Let A and B be two polytopal complexes. If A collapses onto a complex C_A then $A \times B$ collapses onto $C_A \times B$.*

Proof. Let $B_1, \ldots, B_M$ be an ordered list of all the faces of B, ordered by weakly decreasing dimension. Let (σ_1^A, Σ_1^A) be the first pair of faces appearing in the collapse of A onto C_A. We perform the M collapses $(\sigma_1^A \times B_1, \Sigma_1^A \times B_1), \ldots, (\sigma_1^A \times B_M, \Sigma_1^A \times B_M)$, in this order. It is easy to check that each of the steps above is a legitimate collapse: When we remove $\sigma_1^A \times B_i$ all the faces of the type $\sigma_1^A \times \beta$ containing $\sigma_1^A \times B_i$ have already been removed, because in the list $B_1, \ldots, B_M$ the face β appears before B_i. On the other hand, σ_1^A is a *free* face of Σ_1^A, thus no face of the type $\alpha \times B_i$ may contain $\sigma_1^A \times B_i$ other than $\Sigma_1^A \times B_i$.

Next, we consider the *second* pair of faces (σ_2^A, Σ_2^A) that appears in the collapse of A onto C_A and we repeat the procedure above, and so on: In the end, the only faces left are those of $C_A \times B$. □

Corollary 2.2. *If A is collapsible, then $A \times B$ collapses onto a copy of B.*

Since the product of the dunce hat with a segment $\mathbb{I}$ is collapsible [15], the collapsibility of both A and B *strictly* implies the collapsibility of $A \times B$.

Now, consider a 1-sphere S consisting of four edges. The 2-complex $S \times S$ is a cubical torus; after the removal of a facet, it collapses onto the union of a meridian and a longitude of the torus. (Topologically, a punctured torus retracts to a bouquet of two circles.) This can be generalized as follows:

Proposition 2.3. *Let A and B be two polytopal complexes. Let Δ_A (resp. Δ_B) be a facet of A (resp. B). If $A - \Delta_A$ collapses onto some complex C_A and if $B - \Delta_B$ collapses onto some complex C_B then $(A \times B) - (\Delta_A \times \Delta_B)$ collapses onto $(A \times C_B) \cup (C_A \times B)$.*

Proof. We start by forming three ordered lists of pairs of faces. Let $(\sigma_1, \Sigma_1), \ldots, (\sigma_U, \Sigma_U)$ be the list of the removed pairs of faces in the collapse of $A - \Delta_A$ onto C_A. (We assume that higher-dimensional faces are collapsed first.) Analogously, let $(\gamma_1, \Gamma_1), \ldots, (\gamma_V, \Gamma_V)$ be the list of all the removed pairs in the collapse of $B - \Delta_B$ onto C_B. Let then $B_1, \ldots, B_W$ be the list of all the faces of B that are not in C_B, ordered by weakly decreasing dimension.

The desired collapsing sequence for $(A \times B) - (\Delta_A \times \Delta_B)$ consists of $U + 1$ distinct phases:

PHASE 0: We remove from $(A \times B) - (\Delta_A \times \Delta_B)$ the V pairs of faces $(\Delta_A \times \gamma_1, \Delta_A \times \Gamma_1)$, $(\Delta_A \times \gamma_2, \Delta_A \times \Gamma_2)$, $\ldots$, $(\Delta_A \times \gamma_V, \Delta_A \times \Gamma_V)$, in this order. Analogously to the proof of Proposition 2.1, one sees that all these removals are elementary collapses. They wipe away the "Δ_A-layer" of $A \times B$, but not entirely: The faces $\alpha \times \beta$ with β in C_B are still present. What we have written is in fact a collapse of $(A \times B) - (\Delta_A \times \Delta_B)$ onto the complex $((A - \Delta_A) \times B) \cup (\Delta_A \times C_B)$.

PHASE 1: We take the first pair (σ_1, Σ_1) in the first list and we perform the W elementary collapses $(\sigma_1 \times B_1, \Sigma_1 \times B_1)$, $\ldots$, $(\sigma_1 \times B_W, \Sigma_1 \times B_W)$. This way we remove (with the exception of $\Sigma_1 \times C_B$) the Σ_1-layer of $A \times B$, where Σ_1 is the first facet of A to be collapsed away in $A - \Delta_A \searrow C_A$.

⋮

PHASE j: We consider (σ_j, Σ_j) and proceed as in Phase 1, performing W collapses to remove (with the exception of $\Sigma_j \times C_B$) the Σ_j-layer of $A \times B$.

⋮

PHASE U: We consider (σ_U, Σ_U) and proceed as in Phase 1, performing W collapses to remove (with the exception of $\Sigma_U \times C_B$) the Σ_U-layer of $A \times B$.

Eventually, the only faces of $A \times B$ left are the polytopes of $A \times C_B \cup C_A \times B$. □

Corollary 2.4. *Given s polytopal complexes $A_1, \ldots, A_s$, suppose that each A_i after the removal of a facet collapses onto some lower-dimensional complex C_i. Then the complex $A_1 \times \cdots \times A_s$ after the removal of a facet collapses onto*

$$(C_1 \times A_2 \times \cdots \times A_s) \cup (A_1 \times C_2 \times A_3 \times \cdots \times A_s) \cup \cdots \cup (A_1 \times \cdots \times A_{s-1} \times C_s).$$

In particular, if $\dim C_i = \dim A_i - 2$ for each i, then $A_1 \times \cdots \times A_s$ minus a facet collapses onto a complex of dimension $\dim A_1 + \cdots + \dim A_s - 2$.

Proof. It follows from Proposition 2.3, by induction on s. □

Remark 2.5. Proposition 2.1, Proposition 2.3 and Corollary 2.4 can be easily extended to the generality of finite regular CW complexes (see e.g. Björner [6, p. 1860] for the definition).

Example 2.6. Let C be the boundary of the three-dimensional cube $\mathbb{I}^3$; removing a square from C one obtains a collapsible 2-complex. The product $C \times C$ is a cubical 4-manifold homeomorphic to $S^2 \times S^2$ (and not homeomorphic to S^4). The 4-complex obtained by removing a facet from $C \times C$ collapses onto a 2-complex, by Proposition 2.3. Therefore, $C \times C$ is LC. Note that the second homotopy group of $C \times C$ is nonzero. However, as observed by Durhuus and Jonsson [9] [2, Lemma 1.6.3], every LC d-manifold is simply connected.

Remark 2.7. The previous example can be generalized by taking the product of the boundary of the 3-cube $\mathbb{I}^3$ with the boundary of the $(d-1)$-cube $\mathbb{I}^{d-1}$ ($d \geqslant 4$). As a result, one obtains a cubical d-manifold that is homeomorphic to $S^2 \times S^{d-2}$ (and not homeomorphic to S^d). This d-manifold is LC, because the boundary of a $(d-1)$-cube is shellable and LC. In contrast, no manifold homeomorphic to $S^1 \times S^{d-1}$ is LC, because LC manifolds are simply connected.

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