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On geometric independency trees for points in the plane

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Abstract

A plane spanning tree is a tree drawn in the plane so that its edges are closed straight-line segments and no two edges intersect internally, and no three of its vertices are collinear. In this paper, we present several results on a plane spanning tree T such that the graph obtained from T by adding a line segment between any two end-vertices of T is self-intersecting.

Keywords: Discrete geometry; A geometric graph; A spanning tree

1. Introduction

A *geometric graph* is a graph drawn in the plane whose edges are straight-line segments. Let U be a set of n points in the plane. If no three points are collinear, then we say that U is *in general position*. In this paper, we suppose that all sets of points in the plane are in general position. For a set U of points in the plane we denote by $K(U)$ the complete geometric graph whose vertex set is U. For two points a and b in the plane, we denote by ab the closed straight-line segment joining a to b. If an edge e joins a to b, these vertices are called *end-points* of e. In this paper, an edge e with end-points a and b is the same as the straight-line segment ab, and the term

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line e means the line which contains the edge e. A vertex with degree 1 is called an *end-vertex*.

A non-self-intersecting spanning tree of $K(U)$ (i.e., a spanning tree T of $K(U)$ such that no two edges in $E(T)$ intersect except at their common end-point) is said to be a *plane spanning tree* (*on U*). We call a plane spanning tree with two end-vertices a *plane Hamilton path*.

Plane spanning trees have been frequently studied. For example, Ikebe et al. [3] showed that any rooted tree with n vertices can be embedded as a plane spanning tree on U, with the root being mapped onto an arbitrary specified point of U. In [4], the following theorem was presented.

Theorem 1.1 (Károlyi et al. [4]). *Let U be a set of points in the plane in general position and G be a geometric graph whose vertex set is U. If G does not have a plane spanning tree, then the complementary geometric graph of G contains a plane spanning tree.*

Rivera-Campo [5] showed that a geometric graph G contains a plane spanning tree if the subgraph of G induced by any vertex subset with five vertices of $V(G)$ has a plane spanning tree (on the vertex subset). In this paper, we shall introduce a new kind of plane spanning tree, called a *geometric independency tree*.

Let G be a connected graph. If the end-vertices of a spanning tree of G are pairwise non-adjacent in G, then the spanning tree is called an *independency tree*. It is easy to see that a graph which does not have an independency tree is Hamiltonian. Furthermore, Böhme et al. [1] characterized the graphs containing an independency tree.

Theorem 1.2 (Böhme et al. [1]). *A connected graph does not have an independency tree if and only if the graph is isomorphic to a cycle, a complete graph or a balanced complete bipartite graph.*

We consider a geometric version of an independency tree as follows. Let a and b be two vertices of a geometric graph G. The two vertices a and b *see each other* or *a sees b* if there does not exist an edge e in $E(G)$ such that the edge ab and e intersect internally. (Such an edge e is called a *shield between a and b*.) A *geometric independency tree* (*on U*) is a plane spanning tree T (on U) such that no two end-vertices of T see each other.

In the next section, we shall determine the configurations of a set U of n points in which there does not exist a geometric independency tree on U. Moreover, we shall show that any geometric independency tree on U does not have more than $\lfloor n/2 \rfloor$ end-vertices and prove that given a set U of $n \geqslant 5$ points in the plane, there exists a geometric independency tree T on U such that T has at least $\lceil n/6 \rceil$ end-vertices.

All notation and terminology not explained here are given in [2].

2. Main results

Let U be a set of n points in general position in the plane. Let conv(U) denote the convex hull of U, which is the smallest convex set containing U. Denote by ∂U the set of points of U lying on the boundary of conv(U). First we show the following fundamental lemma.

Lemma 2.1. *For any two points a and b in U, there exists a plane Hamilton path with end-vertices a and b.*

Proof. We use induction on n. Let p be the middle point of the edge ab. We divide the plane into two half-planes by a line px, where x is a point in U which is neither a nor b. Let A be the set of points in U in the closed half-plane which contains a. Set $B = U\setminus(A\setminus\{x\})$. By the hypothesis there exist a non-self-intersecting Hamilton path P_1 for the set A with ends a and x and a non-self-intersecting Hamilton path P_2 for the set B with ends x and b. Since $\mathrm{conv}(A)\cap\mathrm{conv}(B)=\{x\}$, then $P_1\cup P_2$ is a non-self-intersecting Hamilton path on U. □

Let $\gamma(T)$ be the number of end-vertices of a geometric independency tree T. A geometric independency tree T with $\gamma(T)=2$ is called a *geometric independency Hamilton path*. By using Lemma 2.1 we prove the following proposition.

Proposition 2.2. *For any two points a and b in U, there exists a geometric independency Hamilton path with end-vertices a and b if and only if a and b do not see each other on K(U).*

Proof. We can easily see that *only if* part is true by the definition of a geometric independency Hamilton path. Suppose that a and b do not see each other. Then, there exists an edge xy which is a shield between a and b. We divide the plane into two half-planes by the line xy. Let A and B be the sets of points of U in the closed half-planes containing a and b, respectively. By Lemma 2.1, there exist a non-self-intersecting Hamilton path P_1 for the set $A\setminus\{y\}$ with ends a and x and a non-self-intersecting Hamilton path P_2 for the set $B\setminus\{x\}$ with ends y and b. Since $\mathrm{conv}(A\setminus\{y\})\cap\mathrm{conv}(B\setminus\{x\})=\emptyset$, then by connecting P_1 and P_2 with the edge xy we obtain a non-self-intersecting Hamilton path on U with ends a and b. □

We note that every geometric graph with <4 vertices is non-self-intersecting and therefore it cannot contain an geometric independency tree. On the other hand, since the complete graph K_m with $m\geqslant 5$ is not planar, by the above proposition there exists a geometric independency Hamilton path for any set U with $n\geqslant 5$. It is obvious that if $n=4$ and $K(U)$ is non-self-intersecting, i.e., $|\partial U|=3$, then $K(U)$ has no geometric independency tree. Hence, we obtain the following corollary.

Corollary 2.3. *For a set U of n points in general position in the plane with $n\geqslant 4$, there exists a geometric independency tree if and only if U does not satisfy the condition that $n=4$ and $|\partial U|=3$.*

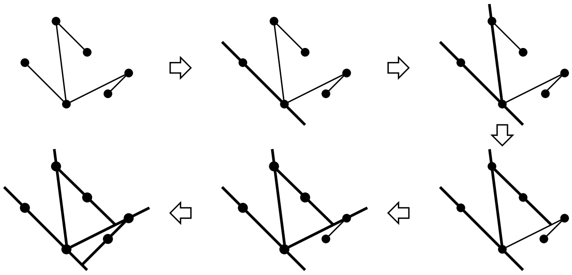

Fig. 1. An example of the procedure in Theorem 2.4.

Next, we consider upper and lower bounds on the number of end-vertices in geometric independency trees. First we mention an upper bound.

Theorem 2.4. *Let U be a set of n points in general position in the plane, where $n \geqslant 5$. Let T be a geometric independency tree on U. Then $\gamma(T) \leqslant \lfloor n/2 \rfloor$.*

Proof. For a given T, we proceed as follows.

(i) Set $E := E(T)$ and $F := E(T)$.
(ii) Choose an edge $e = xy \in E$ such that $y \in \partial U$ and extend it to infinity in the direction of y to form a half-line e'; call x the *free end* of e'.
(iii) Extend e' in the direction of its free end x until it first hits a segment $f \in F$ not passing through x, or (if this does not happen) extend it to infinity; call the resulting segment l.
(iv) Set $E := E \backslash \{e\}$ and $F := (F \backslash \{e\}) \cup \{l\}$.
(v) If $E = \emptyset$, then stop. Otherwise, choose an edge $e = e' = xy \in E$ that has an end-vertex y in common with some edge in $E(T) \backslash E$, and call its other end x the *free end* of $e' = e$. Go to (iii).

The above procedure divides the plane into n regions, because every application of rule (iii) increases the number of regions by exactly one (see Fig. 1). Moreover, the regions are convex, since for every vertex $u \in U$ some edge incident with u has been extended in the direction of u. Each end-vertex of T is in the boundary of at least two regions, and two end-vertices cannot be in the boundary of the same region. Thus $\gamma(T) \leqslant n/2$. □

In addition, this bound is tight. Let U be a set of n points in the plane such that $U = \partial U$. Let the vertices of U be labelled $u_1, u_2, \ldots, u_n$ in clockwise order round the boundary cycle. Let T be the spanning tree on U whose edges are $u_{2i-1}u_{2i}$ and $u_{2i-1}u_{2i+1}$ for every integer i with $1 \leqslant i < \lfloor n/2 \rfloor$, and $u_{n-1}u_n$ if n is even, and $u_{n-2}u_{n-1}$ and $u_{n-1}u_n$ if n is odd (see Fig. 2). This tree T is a geometric independency tree whose end-vertices are u_{2i} with $1 \leqslant i < \lfloor n/2 \rfloor$ and u_n.

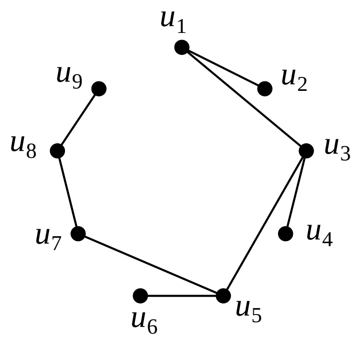

Fig. 2. A geometric independency tree with $\lfloor n/2 \rfloor$ end-vertices, where $n=9$.

Next, we consider a lower bound on the number of end-vertices. In Corollary 2.3, we mentioned that for a set U such that $n=4$ and $|\partial U|=3$, $K(U)$ does not have any geometric independency tree.

Theorem 2.5. *Let U be a set of n points in general position in the plane, where $n \geqslant 5$. Then there exists a geometric independency tree T on U such that $\gamma(T) \geqslant \lceil n/6 \rceil$.*

Proof. Let $U_0 = U$. For $i=1,2,\ldots$ proceed as follows. If $U_{i-1}=\emptyset$, set $j=i-1$ and stop. Otherwise, define B_i to be the bounding cycle of $\mathrm{conv}(U_{i-1})$ if $|U_{i-1}| \geqslant 3$, or to be $K(U_{i-1})$ if $|U_{i-1}| \leqslant 2$, and define $U_i = U_{i-1} \setminus V(B_i)$. Then $B_1,\ldots,B_{j-1}$ are cycles, B_j is a cycle or K_1 or K_2, and $U = \bigcup_{1 \leqslant i \leqslant j} V(B_i)$.

If $j=1$, then remember the example which states that the upper bound $\lfloor n/2 \rfloor$ of end-vertices is tight (see Fig. 2).

Suppose $j \geqslant 2$. Let $K=\{1,2,\ldots,j\}$. Let M comprise all even or all odd numbers in K, chosen so that

$$\sum_{i \in M} |V(B_i)| \geqslant \sum_{i \in M^c} |V(B_i)|,$$

where $M^c = K \setminus M$. If $\sum_{i \in M} |V(B_i)| > \sum_{i \in M^c} |V(B_i)|$ and $M=\{1,3,\ldots\}$, or $\sum_{i \in M} |V(B_i)| = \sum_{i \in M^c} |V(B_i)|$ and $|M|=|M^c|$, then we say that U is *Type 1* and set $k=j$ and $C_i = B_i$ for all integer i with $1 \leqslant i \leqslant k$. For convenience, define C_0 to be empty. Otherwise, we say that U is *Type 2* and set $k=j-1$ and $C_i = B_{i+1}$ for all integer i with $0 \leqslant i \leqslant k$. Let M_d be the set of odd integers i with $1 \leqslant i \leqslant k$ and M_e be the set of even integers i with $2 \leqslant i \leqslant k$. For all i with $0 \leqslant i \leqslant k$, let $n_i = |V(C_i)|$. Then, it is easy to see that $\sum_{i \in M_d} n_i \geqslant \sum_{i \in M_e \cup \{0\}} n_i$ and we note that

$$\text{if } |M_d| \neq |M_e| \text{ then } \sum_{i \in M_d} n_i > \sum_{i \in M_e \cup \{0\}} n_i. \quad (1)$$

We shall construct an independency tree with $\sum_{i \in M_d} \lfloor n_i/2 \rfloor$ end-vertices. This is enough because $\sum_{i \in M_d} \lfloor n_i/2 \rfloor \geqslant n/6$, as we will show.

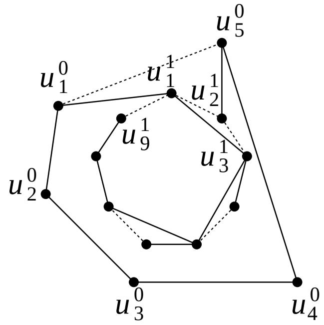

Fig. 3. The case that $k=1$ and U is Type 2.

Let the vertices of C_i be labelled $u_1^i, u_2^i, \ldots, u_{n_i}^i$ in clockwise order round C_i if $i \in M_{\rm d}$ and in counterclockwise order if $i \in M_{\rm e} \cup \{0\}$. Set $u_0^i = u_{n_i}^i$, $u_{-1}^i = u_{n_i-1}^i$, $u_{n_i+1}^i = u_1^i$ and $u_{n_i+2}^i = u_2^i$. We split into three cases.

Case 1: $k=1$ and U is Type 2.

We will construct a tree with vertex set $V(C_1)$ as in the case when $j=1$, and then replace one edge of it by a path through all the vertices of $V(C_0)$ so that the resulting tree is non-self-intersecting (see Fig. 3). Here are the details. Fix the indices of the points u_i^0 with $1 \leqslant i \leqslant n_0$ in counterclockwise order round C_0. It is easy to see that for some t with $1 \leqslant t \leqslant n_1$ there exist consecutive points u_t^1 and u_{t+1}^1 in C_1 such that $C_1 \cup \{u_1^0 u_t^1, u_{n_0}^0 u_{t+1}^1\}$ is non-self-intersecting. Set $t=1$ and let $C_1 = (u_1^1, u_2^1, \ldots, u_{n_1}^1)$ in clockwise order. Clearly, the following tree T is a geometric independency tree on U.

$$T = \begin{cases} C_0 \cup \{u_{2i-1}^1 u_{2i}^1, u_{2i-1}^1 u_{2i+1}^1 \colon 1 \leqslant i < n_1/2\} & \\ \quad \cup \{u_{n_1-1}^1 u_{n_1}^1, u_1^0 u_1^1, u_{n_0}^0 u_2^1\} \setminus \{u_{n_0}^0 u_1^0, u_1^1 u_2^1\} & \text{if } n_1 \text{ is even,} \\ C_0 \cup \{u_{2i-1}^1 u_{2i}^1, u_{2i-1}^1 u_{2i+1}^1 \colon 1 \leqslant i < (n_1-1)/2\} & \\ \quad \cup \{u_{n_1-2}^1 u_{n_1-1}^1 u_{n_1}^1, u_1^0 u_1^1, u_{n_0}^0 u_2^1\} \setminus \{u_{n_0}^0 u_1^0, u_1^1 u_2^1\} & \text{if } n_1 \text{ is odd.} \end{cases}$$

(Here $u_{n_1-2}^1 u_{n_1-1}^1 u_{n_1}^1$ denotes the path with two edges $u_{n_1-2}^1 u_{n_1-1}^1$ and $u_{n_1-1}^1 u_{n_1}^1$.) Since $n_1 > n_0 \geqslant 3$, we have $\gamma(T) \geqslant \lfloor n_1/2 \rfloor \geqslant \max\{\lfloor n/4 \rfloor, 2\}$. If $n \geqslant 8$, then $\lceil n/6 \rceil \leqslant \lfloor n/4 \rfloor$. If $n \leqslant 7$, then $\lceil n/6 \rceil \leqslant 2$. Hence, we have $\gamma(T) \geqslant \lceil n/6 \rceil$.

Case 2: $k=2$ and U is Type 1.

For each i with $1 \leqslant i \leqslant \lfloor n_1/2 \rfloor$, we will choose a point $w_{2i-1} \in C_2$ and define

$$Q = \begin{cases} C_2 \cup \{w_{2i-1} u_{2i-1}^1 u_{2i}^1 \colon 1 \leqslant i \leqslant n_1/2\} & \text{if } n_1 \text{ is even,} \\ C_2 \cup \{w_{2i-1} u_{2i-1}^1 u_{2i}^1 \colon 1 \leqslant i \leqslant (n_1-1)/2\} \cup \{u_{n_1-1}^1 u_{n_1}^1\} & \text{if } n_1 \text{ is odd.} \end{cases}$$

If $n_2 = 1$, then $n_1 \geqslant 4$ since $n_1 + n_2 = n \geqslant 5$. In this case, for all i with $1 \leqslant i \leqslant \lfloor n_1/2 \rfloor$, define $w_{2i-1} = u_1^2$; then the degree of u_1^2 is at least 2, and Q is a geometric independency tree on U with $\gamma(Q) \geqslant \lfloor n_1/2 \rfloor = \lfloor (n-1)/2 \rfloor > \lceil n/6 \rceil$.

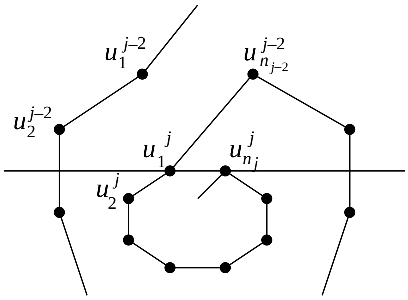

Fig. 4. A path P.

Next, suppose that $n_2 \geqslant 2$. If $n_1 = 3$, then $n = 5$ or 6. We know that for both cases there exists a geometric independency tree with at least two end-vertices by Corollary 2.3. Hence, we may assume that $n_1 \geqslant 4$. Let $W = \{w_{2i-1}: 1 \leqslant i \leqslant \lfloor n_1/2 \rfloor\}$. We choose w_{2i-1} for all i with $1 \leqslant i \leqslant \lfloor n_1/2 \rfloor$ so that:

(i) Q is non-self-intersecting.
(ii) Subject to (i), W has a pair of points w_a and w_{a+2} such that w_a and w_{a+2} are adjacent in C_2.

We claim that we can choose such points. Suppose that no two points in W are adjacent in C_2. Since $n_1 \geqslant n_2$, either there exist two distinct indices a and b such that $w_a = w_b$, or there exist two indices a and b such that w_a and w_b are distance 2 apart around C_2 and $n_1 - n_2 \leqslant 1$. However, for both cases we can take another point in C_2 and obtain the set of points satisfying (i) and (ii) because each point in C_1 sees at least two consecutive points in C_2. Let $T = Q$ if $n_2 = 2$ and $T = Q \setminus \{w_a w_{a+2}\}$ if $n_2 \geqslant 3$. Then T is a geometric independency tree on U. In a similar way as in Case 1, we have $\gamma(T) \geqslant \lceil n/6 \rceil$.

Case 3: Cases 1 and 2 do not apply. Then, either $k = 2$ and U is Type 2, or else $k \geqslant 3$.

We first fix the indices of the points u_i^2 with $1 \leqslant i \leqslant n_2$ in counterclockwise order round C_2. For all i and j with $1 \leqslant i \leqslant n_j$ and $j \in M_{\mathrm{e}} \setminus \{2\}$, we label the points u_i^j in C_j in counterclockwise order so that all points in $\bigcup_{j \leqslant l \leqslant k} V(C_l)$ are in the closed half-plane which is bounded by the line $u_{n_j}^j u_1^j$ and does not contain $u_{n_{j-2}}^{j-2}$ (see Fig. 4). Then, we can easily see that the path

$$P = \bigcup_{j \in M_{\mathrm{e}}} C_j \cup \{u_{n_{j-2}}^{j-2} u_1^j : j \in M_{\mathrm{e}} \setminus \{2\}\} \setminus \{u_{n_j}^j u_1^j : j \in M_{\mathrm{e}}\},$$

is non-self-intersecting; it has u_1^2 as one end-point.

We attach the vertices in C_1 to P. Let A be the half-plane which is bounded by the line $u_{n_2}^2 u_1^2$ and contains the point u_2^2, and let B be the other half-plane. Let l be the index such that u_l^1 lies in B and u_{l+1}^1 lies in A (see Fig. 5). Then, it is easily seen that $P \cup \{u_1^2 u_l^1 u_{l+1}^1\}$ is non-self-intersecting. We may assume that $l = 1$. It is easy to see that we can choose points $w_{2i-1} \in C_2$ for all i with $2 \leqslant i \leqslant \lfloor n_1/2 \rfloor$ so that P_{I}^2 is

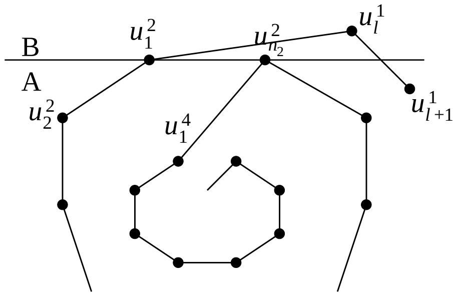

Fig. 5. Choosing u_l^1 in the cycle C_1.

non-self-intersecting, where

$$P_{\mathrm{I}}^2 = \begin{cases} P \cup \{u_1^2 u_1^1 u_2^1\} \cup \{w_{2i-1} u_{2i-1}^1 u_{2i}^1 \colon 2 \leqslant i \leqslant n_1/2\} & \text{if } n_1 \text{ is even}, \\ P \cup \{u_1^2 u_1^1 u_2^1\} \cup \{w_{2i-1} u_{2i-1}^1 u_{2i}^1 \colon 2 \leqslant i \leqslant (n_1 - 1)/2\} & \\ \quad \cup \{u_{n_1-1}^1 u_{n_1}^1\} & \text{if } n_1 \text{ is odd}. \end{cases}$$

Note that $P_{\mathrm{I}}^2 \cup \{u_s^1 v\}$ is self-intersecting for all s with $2 \leqslant s \leqslant n_1$ and all $v \in \bigcup_{3 \leqslant i \leqslant k} V(C_i)$. Thus, P_{I}^2 is a geometric independency tree on $V(P_{\mathrm{I}}^2)$. If U is Type 2, in a similar way as in Case 1, we replace an edge of P_{I}^2 by a path passing through all points in C_0. Choose a pair of consecutive points u_l^0 and u_{l+1}^0 in C_0 with $1 \leqslant l \leqslant n_0$. Then there exist two consecutive points u_m^1 and u_{m+1}^1 with $1 \leqslant m \leqslant n_1$ such that $P_{\mathrm{I}}^2 \cup \{u_l^0 u_{m+1}^1, u_{l+1}^0 u_m^1\}$ is non-self-intersecting. If $u_m^1 u_{m+1}^1 \in E(P_{\mathrm{I}}^2)$, then set $t = m$ and $u_l^0 = u_{n_0}^0$ and we label the points u_i^0 with $1 \leqslant i \leqslant n_0$ in counterclockwise order. Otherwise, we have $u_{m-1}^1 u_m^1 \in E(P_{\mathrm{I}}^2)$. If $P_{\mathrm{I}}^2 \cup \{u_l^0 u_m^1, u_{l+1}^0 u_{m-1}^1\}$ is non-self-intersecting, then set $t = m - 1$ and $u_l^0 = u_{n_0}^0$ and fix the indices of the points in C_0 in counterclockwise order. Otherwise, let a be the smallest positive integer such that $P_{\mathrm{I}}^2 \cup \{u_{l+a}^0 u_{m-1}^1\}$ is non-self-intersecting. Then, set $t = m - 1$ and $u_{l+a}^0 = u_1^0$ and fix the indices of the points in C_0 in counterclockwise order. We note that $P_{\mathrm{I}}^2 \cup \{u_{l+a-1}^0 u_m^1, u_{l+a}^0 u_{m-1}^1\}$ is non-self-intersecting. We define

$$P_{\mathrm{II}}^2 = P_{\mathrm{I}}^2 \cup C_0 \cup \{u_{n_0}^0 u_{t+1}^1, u_1^0 u_t^1\} \setminus \{u_{n_0}^0 u_1^0, u_t^1 u_{t+1}^1\}.$$

We can easily see that P_{II}^2 is a geometric independency tree on $V(P_{\mathrm{II}}^2)$. We define $P^2 = P_{\mathrm{I}}^2$ if U is Type 1 and $P^2 = P_{\mathrm{II}}^2$ if U is Type 2.

Next, we attach to P^2 all points in C_j for all integers $j \in M_{\mathrm{d}} \setminus \{1, k\}$. For $m \in M_{\mathrm{e}} \setminus \{2\}$, we recursively define P^m as follows, so that $P^2 \subseteq P^{m-2} \subset P^m$ and P^m is non-self-intersecting.

There is exactly one edge $e \in E(C_{m-1})$ which intersects the edge $u_{n_{m-2}}^{m-2} u_1^m$ of P^{m-2}. We may assume $e = u_{n_{m-1}}^{m-1} u_1^{m-1}$. Suppose that P^{m-2} is non-self-intersecting. Clearly, there exists a point x_1^m in C_m such that $P^{m-2} \cup \{x_1^m u_1^{m-1}\}$ is non-self-intersecting. We claim that if $x_1^m = u_1^m$ then $P^{m-2} \cup \{u_{n_m}^m u_1^{m-1}\}$ is also non-self-intersecting. (We prove this claim later.) In this case, redefine $x_1^m = u_{n_m}^m$. It is easy to see that we can choose

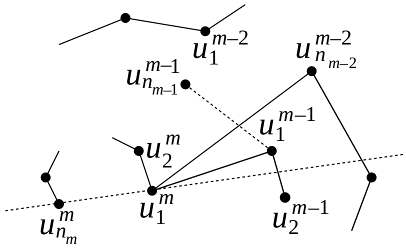

Fig. 6. The case that $u_{n_m}^m$ and u_1^{m-1} do not see each other. (We show that this case cannot arise.)

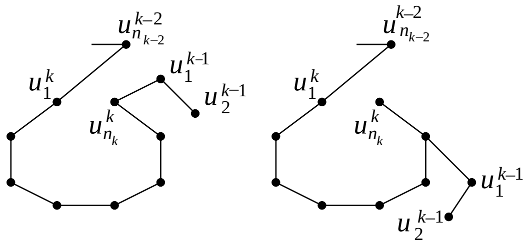

Fig. 7. The case that $k\in M_e\setminus\{2\}$.

points $x_{2i-1}^m\in C_m$ for all i with $2\leqslant i\leqslant\lfloor n_{m-1}/2\rfloor$ so that P^m is non-self-intersecting, where

$$P^m=\begin{cases} p^{m-2}\cup\{x_{2i-1}^m u_{2i-1}^{m-1}u_{2i}^{m-1}: 1\leqslant i\leqslant n_{m-1}/2\} & \text{if } n_{m-1} \text{ is even,}\\ p^{m-2}\cup\{x_{2i-1}^m u_{2i-1}^{m-1}u_{2i}^{m-1}: 1\leqslant i\leqslant (n_{m-1}-1)/2\} & \\ \quad\cup\{u_{n_{m-1}-1}^{m-1}u_{n_{m-1}}^{m-1}\} & \text{if } n_{m-1} \text{ is odd.}\end{cases}$$

Claim 1. *Suppose that P^{m-2} is non-self-intersecting for some $m\in M_e\setminus\{2\}$. If $P^{m-2}\cup\{u_1^m u_1^{m-1}\}$ is non-self-intersecting, then $P^{m-2}\cup\{u_{n_m}^m u_1^{m-1}\}$ is also non-self-intersecting.*

Proof. Suppose that $P^{m-2}\cup\{u_1^m u_1^{m-1}\}$ is non-self-intersecting and $Q=P^{m-2}\cup\{u_{n_m}^m u_1^{m-1}\}$ is self-intersecting. Then $u_{n_m}^m$ and u_1^{m-1} do not see each other on Q and the edge $u_{n_{m-2}}^{m-2}u_1^m$ is a shield between $u_{n_m}^m$ and u_1^{m-1}. This contradicts the fact that all points in $\bigcup_{m\leqslant l\leqslant k}V(C_l)$ are in the closed half-plane which is bounded by the line $u_{n_m}^m u_1^m$ and does not contain $u_{n_{m-2}}^{m-2}$ (compare with Figs. 4 and 6). □

Suppose that $k\in M_e$. Then, remark that the point $u_{n_k}^k$ may be an end-vertex of P^k. In this case, it is also easy to see that $u_{n_k}^k$ does not see any other end-vertex (see the right side of Fig. 7). Hence, P^k is a geometric independency tree on $V(P^k)$. Let $T=P^k$. If $k\in M_d$, then take a vertex u in C_k such that $P^{k-1}\cup\{u_{n_{k-1}}^{k-1}u\}\cup C_k$ is non-

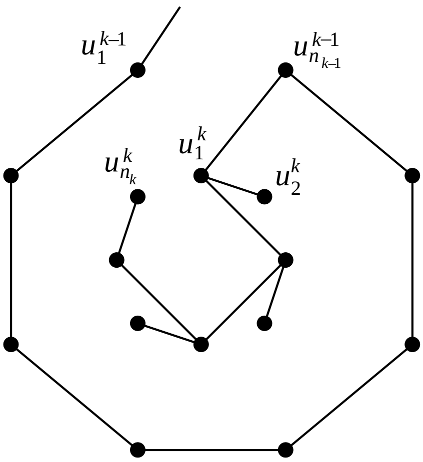

Fig. 8. The case that $k \in M_\mathrm{d}$.

self-intersecting, and let $u_1^k = u$. We define

$$T = \begin{cases} P^{k-1} \cup \{u_{2i-1}^k u_{2i}^k, u_{2i-1}^k u_{2i+1}^k : 1 \leqslant i < n_k/2\} & \\ \quad \cup \{u_{n_k-1}^k u_{n_k}^k, u_{n_{k-1}}^{k-1} u_1^k\} & \text{if } n_k \text{ is even,} \\ P^{k-1} \cup \{u_{2i-1}^k u_{2i}^k, u_{2i-1}^k u_{2i+1}^k : 1 \leqslant i < (n_k-1)/2\} & \\ \quad \cup \{u_{n_k-2}^k u_{n_k-1}^k u_{n_k}^k, u_{n_{k-1}}^{k-1} u_1^k\} & \text{if } n_k \text{ is odd.} \end{cases}$$

This is also a geometric independency tree on U (see Fig. 8).

We calculate $\gamma(T)$ for the geometric independency tree T constructed above. Note that $|M_\mathrm{d}| = k/2$ or $(k+1)/2$. We can easily see that $\gamma(T) \geqslant \sum_{i \in M_\mathrm{d}} \lfloor n_i/2 \rfloor$. Suppose that $k > n/3$. Remark that if $k \in M_\mathrm{d}$ then for $V(C_k)$ there exists at least one end-vertex of T even if $n_k = 1$. Hence,

$$\gamma(T) \geqslant \sum_{i \in M_\mathrm{d}} \left\lfloor \frac{n_i}{2} \right\rfloor \geqslant \frac{k}{2} > \frac{n}{6}.$$

Next, we suppose that $k \leqslant n/3$. If $|M_\mathrm{d}| = k/2$, then we have

$$\gamma(T) \geqslant \sum_{i \in M_\mathrm{d}} \left\lfloor \frac{n_i}{2} \right\rfloor \geqslant \frac{1}{2} \sum_{i \in M_\mathrm{d}} (n_i - 1) \geqslant \frac{1}{2} \left(\frac{n}{2} - \frac{k}{2} \right) \geqslant \frac{n}{6}. \tag{2}$$

Assume that $|M_\mathrm{d}| = (k+1)/2$. Since $|M_\mathrm{d}| \neq |M_\mathrm{e}|$, we have $\sum_{i \in M_\mathrm{d}} n_i > \sum_{i \in M_\mathrm{e} \cup \{0\}} n_i$ by (1), that is, $\sum_{i \in M_\mathrm{d}} n_i \geqslant (n+1)/2$ by the definition of Types 1 and 2. Thus, (2) again holds. The proof is now complete (Fig. 9). □

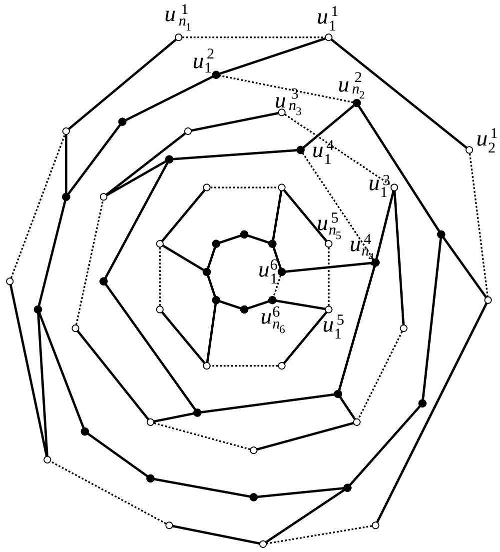

Fig. 9. An example of geometric independency tree on U (U is Type 1).

3. Conclusion

We propose the following problem.

Problem 3.1. *For each integer $k \geqslant 2$, find a characterization of those sets U of points for which there is no geometric independency tree with k end-vertices.*

In Corollary 2.3, the case $k=2$ was done. The authors found a characterization for the case $k=3$ but do not know for $k \geqslant 4$.

In conclusion, we present the following conjecture.

Conjecture 3.2. Let U be a set of points and X be the set of geometric independency trees on U, and we define $T_{\max}$ and $T_{\min}$ as follows:

$$T_{\max} = \max_{T \in X} \gamma(T), \quad T_{\min} = \min_{T \in X} \gamma(T).$$

Then, $K(U)$ has a geometric independency tree with k end-vertices for every integer k with $T_{\min} \leqslant k \leqslant T_{\max}$.

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