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On the order of the high-dimensional Cochrane sum and its mean value ☆

Xu Zhefeng*, Zhang Wenpeng

Department of Mathematics, Northwest University, Xi'an, Shaanxi, PR China

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Abstract

The main purpose of this paper is to study the high-dimensional Cochrane sum and give a sharp estimate of its order by using properties of hyper-Kloosterman sum and the mean value theorems of Dirichlet L-functions.

Keywords: High-dimensional cochrane sum; Hyper-Kloosterman sum; Asymptotic formula; Order; Mean value

1. Introduction

For a positive integer q and an arbitrary integer h, the classical Dedekind sum $S(h,q)$ is defined by

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

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* Corresponding author.

E-mail address: zfxu@nwu.edu.cn (X. Zhefeng).

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where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

In October 2000, during his visit to Xi'an, Professor Todd Cochrane introduced a sum analogous to the Dedekind sum as follows:

$$C(h, q) = {\sum_{a=1}^{q}}' \left(\left(\frac{\overline{a}}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where $\overline{a}$ is defined by the equation $a\overline{a} \equiv 1 \bmod q$ and ${\sum_{a=1}^{q}}'$ denotes the summation over all $1 \leqslant a \leqslant q$, such that $(a, q) = 1$. Then he suggested us to study the arithmetical properties and mean value distribution properties of $C(h, q)$. About these problems, the second author [10] obtained a mean-square value formula for $C(h, q)$. That is,

$$\sum_{h=1}^{q} C^2(h, q) = \frac{5}{144} \phi^2(q) \prod_{p^\alpha \| q} \frac{(p+1)^2/(p^2+1) + 1/p^{3\alpha}}{1 + 1/p + 1/p^2} + O\left(q \exp\left(\frac{4 \ln q}{\ln \ln q} \right) \right),$$

where $\exp(y) = e^y$, $\phi(q)$ is the Euler function and $\prod_{p^\alpha \| q}$ denotes the product over all prime divisors of q with $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$. In the same manner, we define a high-dimensional Cochrane sum as follows:

$$C(h, k, q) = {\sum_{a_1=1}^{q}}' {\sum_{a_2=1}^{q}}' \cdots {\sum_{a_k=1}^{q}}' \left(\left(\frac{a_1}{q} \right) \right) \left(\left(\frac{a_2}{q} \right) \right) \cdots \left(\left(\frac{a_k}{q} \right) \right) \left(\left(\frac{h\overline{a_1 a_2 \cdots a_k}}{q} \right) \right).$$

In [5], Mordell introduced the hyper-Kloosterman sum

$$K(h, k, q) = \sum_{\substack{a_1, \cdots, a_k \bmod q \\ (a_1, q) = \cdots = (a_k, q) = 1}} e\left(\frac{a_1 + \cdots + a_k + h\overline{a_1} \cdots \overline{a_k}}{q} \right),$$

where $e(y) = e^{2\pi i y}$, a high-dimensional generalization of the Kloosterman sum. The applications of hyper-Kloosterman sum were found both in the estimation of Fourier coefficients of Maass forms [2] and in the work on Selberg's eigenvalue conjecture [4]. More interesting connections between hyper-Kloosterman sum and the Heilbronn sum can be found in [6]. In this paper, we shall obtain an estimate of the order of $C(h,k,q)$ by using bounds on hyper-Kloosterman sum, study the asymptotic properties of $C(h,k,q)$, and give a mean-square value formula in the case of $q = p^{\alpha}$. That is, we shall prove the following theorems:

Theorem 1. *Let a positive integer $q \geqslant 2$, an integer h satisfy $(h,q) = 1$. Then for any fixed positive integer k with $(q, k(k+1)) = 1$, we have the estimate*

$$|C(h,k,q)| \ll \frac{2^{(k+1)^2}}{\pi^{k+1}} q^{\frac{k}{2}} d(q) \left(2^{k+2}k\right)^{\omega(q)} \ln^{k+1} q,$$

where $\omega(q)$ denotes the number of different prime divisors of q and $d(q)$ is the divisor function.

Theorem 2. *Let p be a prime and α be a positive integer. Then we have the asymptotic formula*

$$\sum_{h=1}^{p^{\alpha}}{}' |C(h,k,p^{\alpha})|^2 = \frac{\zeta^{2k+1}(2)\left[\left(1-\frac{1}{p}\right)p^{(\alpha+1)(k+2)} + p^{2k+3} - p^{k+2}\right]}{2\pi^{2k+2}p^{\alpha}(p^{k+2}-1)} \times \left(\frac{p^2-1}{p^2}\right)^{2k+1} \prod_{p_1 \neq p}\left(1 - \frac{1-C_{2k}^{k}}{p_1^2}\right) + O_k\left(p^{\alpha k+\varepsilon}\right),$$

where $\zeta(s)$ denotes the Riemann-zeta function, ε is any fixed positive number, $\prod_{p_1 \neq p}$ denotes the products over all primes p_1 with $p_1 \neq p$, $C_m^n = m!/n!(m-n)!$ and O_k denotes the O-symbols depend only on k.

In particular, from this theorem we immediately deduce:

Corollary. *For any prime p, we have the asymptotic formula*

$$\sum_{h=1}^{p-1} |C(h,k,p)|^2 = \frac{\pi^{2k}}{2\cdot 6^{2k+1}} p^{k+1} \prod_{p_1 \neq p}\left(1 - \frac{1-C_{2k}^{k}}{p_1^2}\right) + O_k\left(p^{k+\varepsilon}\right). \tag{1}$$

From (1) we know that $|C(h,k,p)| \gg p^{\frac{k}{2}}$ for some h and thus the bound in Theorem 1 is close to best possible.

2. Some lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. *Suppose χ is an odd character modulo q, generated by the primitive character χ_m modulo m. Then we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{q}{m}\left(\prod_{\substack{p\mid q\\ p\nmid m}}\left(1-\chi_m(p)\right)\right)\left(\sum_{a=1}^{m} a\chi_m(a)\right).$$

Proof. See Lemma 6 of Ref. [9]. □

Lemma 2. *Let h, q be positive integers with $q \geqslant 3$ and $(h, q) = 1$. Then we have*

$$C(h,k,q) = \frac{i^{k+1}}{\pi^{k+1}\,\phi(q)}\sum_{m\mid q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \overline{\chi}(h)\left(\prod_{\substack{p\mid q\\ p\nmid m}}(1-\chi(p))\right)^{k+1} \times \tau^{k+1}(\chi)L^{k+1}(1,\overline{\chi}),$$

where χ denotes a Dirichlet character modulo m, $\sideset{}{^*}\sum_{\chi \bmod m}$ denotes the summation over all primitive characters modulo m,

$$\tau(\chi) = \sum_{b=1}^{m}\chi(b)e\left(\frac{b}{m}\right)$$

denotes the Gauss sum, and $L(s,\chi)$ is the Dirichlet L-function corresponding to χ.

Proof. From the orthogonality relation for characters modulo q we have

$$\begin{aligned}
C(h,k,q) &= \sideset{}{'}\sum_{a_1=1}^{q}\sideset{}{'}\sum_{a_2=1}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\left(\left(\frac{a_1}{q}\right)\right)\left(\left(\frac{a_2}{q}\right)\right)\cdots\left(\left(\frac{a_k}{q}\right)\right)\\
&\quad\times\left(\left(\frac{h\overline{a_1a_2\cdots a_k}}{q}\right)\right)\\
&= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\left\{\sum_{a_1=1}^{q}\chi(a_1)\left(\left(\frac{a_1}{q}\right)\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\cdots\left\{\sum_{a_k=1}^{q}\chi(a_k)\left(\left(\frac{a_k}{q}\right)\right)\right\}\left\{\sum_{a_{k+1}=1}^{q}\chi(a_{k+1})\left(\left(\frac{a_{k+1}h}{q}\right)\right)\right\}\\
&=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\left\{\sum_{a=1}^{q}\chi(a)\left(\frac{a}{q}-\frac{1}{2}\right)\right\}^{k}\\
&\quad\times\left\{\overline{\chi}(h)\sum_{a_{k+1}=1}^{q}\chi(a_{k+1}h)\left(\left(\frac{a_{k+1}h}{q}\right)\right)\right\}\\
&=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(h)\left\{\sum_{a=1}^{q}\chi(a)\left(\frac{a}{q}-\frac{1}{2}\right)\right\}^{k+1}. \qquad (2)
\end{aligned}
$$

If $\chi(-1)=1$, then it is easy to prove that

$$\sum_{a=1}^{q}\chi(a)\left(\frac{a}{q}-\frac{1}{2}\right)=0. \qquad (3)$$

If $\chi(-1)=-1$, then we have

$$\sum_{a=1}^{q}\chi(a)\left(\frac{a}{q}-\frac{1}{2}\right)=\frac{1}{q}\sum_{a=1}^{q}a\chi(a). \qquad (4)$$

On the other hand, from Theorems 12.11 and 12.20 of Apostol [1] we know that if χ_m is a primitive character modulo m with $\chi_m(-1)=-1$, then

$$\frac{1}{m}\sum_{a=1}^{m}a\chi_m(a)=\frac{i}{\pi}\tau(\chi_m)L(1,\overline{\chi_m}). \qquad (5)$$

Combining (2)–(5), Lemma 1 and the properties of character modulo q we may immediately obtain

$$
\begin{aligned}
C(h,k,q)&=\sideset{}{'}\sum_{a_1=1}^{q}\sideset{}{'}\sum_{a_2=1}^{q}\cdots\sideset{}{'}\sum_{a_k=1}^{q}\left(\left(\frac{a_1}{q}\right)\right)\left(\left(\frac{a_2}{q}\right)\right)\\
&\quad\cdots\left(\left(\frac{a_k}{q}\right)\right)\left(\left(\frac{h\overline{a_1a_2\cdots a_k}}{q}\right)\right)
\end{aligned}
$$

$$= \frac{1}{\phi(q)} \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left\{ \frac{1}{m} \left(\prod_{\substack{p|q \\ p\nmid m}} (1-\chi(p)) \right) \left(\sum_{a=1}^{m} a\chi(a) \right) \right\}^{k+1}$$

$$= \frac{i^{k+1}}{\pi^{k+1}\phi(q)} \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\prod_{\substack{p|q \\ p\nmid m}} (1-\chi(p)) \right)^{k+1} \tau^{k+1}(\chi) L^{k+1}(1, \overline{\chi}).$$

This proves Lemma 2. □

Note: If $q = p^\alpha$ is a prime power, then $m = p^\beta$, $1 \leqslant \beta \leqslant \alpha$. So we have the identity:

$$C(h, k, p^\alpha) = \frac{i^{k+1}}{\pi^{k+1}(p^\alpha - p^{\alpha-1})} \sum_{\beta=1}^{\alpha} \sideset{}{^*}\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=-1}} \overline{\chi}(h) \tau^{k+1}(\chi) L^{k+1}(1, \overline{\chi}). \tag{6}$$

Lemma 3. *Let $q = p^\alpha$ with p a prime such that $p \nmid k+1$ and $p \nmid k$. Then we have the bound*:

$$|K(h, k, q)| \ll k q^{\frac{k}{2}}.$$

Proof. See formula (19) of Ye [7]. □

Lemma 4. *For any positive integer m with $(m, k(k+1)) = 1$, and positive integer n with $(n, m) = 1$, we have*

$$\left| \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \chi(n) \tau^{k+1}(\chi) \right| \ll (2k)^{\omega(m)} \phi(m) m^{\frac{k}{2}}.$$

Proof. Let m have the factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. For $r = 1$, note that $\tau(\chi) = 0$, when χ is a non-primitive character modulo p^α, $\alpha \geqslant 2$. (See [8, Lemma 2]). So we get

$$\begin{aligned} \sideset{}{^*}\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi) &= \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi) \\ &= \sideset{}{'}\sum_{a_1=1}^{p^\alpha} \sideset{}{'}\sum_{a_2=1}^{p^\alpha} \cdots \sideset{}{'}\sum_{a_{k+1}=1}^{p^\alpha} e\left(\frac{a_1 + a_2 + \cdots + a_{k+1}}{p^\alpha} \right) \\ &\quad \times \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \chi(n a_1 a_2 \cdots a_{k+1}) \end{aligned}$$

$$= \frac{\phi(p^\alpha)}{2} \sideset{}{'}\sum_{\substack{a_1=1 \\ na_1a_2\cdots a_{k+1}\equiv 1(p^\alpha)}}^{p^\alpha} \sideset{}{'}\sum_{a_2=1}^{p^\alpha} \cdots \sideset{}{'}\sum_{a_{k+1}=1}^{p^\alpha} e\left(\frac{a_1+a_2+\cdots+a_{k+1}}{p^\alpha}\right)$$

$$-\frac{\phi(p^\alpha)}{2} \sideset{}{'}\sum_{\substack{a_1=1 \\ -na_1a_2\cdots a_{k+1}\equiv 1(p^\alpha)}}^{p^\alpha} \sideset{}{'}\sum_{a_2=1}^{p^\alpha} \cdots \sideset{}{'}\sum_{a_{k+1}=1}^{p^\alpha} e\left(\frac{a_1+a_2+\cdots+a_{k+1}}{p^\alpha}\right)$$

$$= \frac{\phi(p^\alpha)}{2}\left(K(\overline{n}, k, p^\alpha) - K(-\overline{n}, k, p^\alpha)\right). \tag{7}$$

From Lemma 3 and (7) we have

$$\left| \sideset{}{^*}\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi) \right| \ll k\phi(p^\alpha)p^{\frac{k\alpha}{2}}. \tag{8}$$

Similarly, we can also get

$$\left| \sideset{}{^*}\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=1}} \chi(n)\tau^{k+1}(\chi) \right| \ll k\phi(p^\alpha)p^{\frac{k\alpha}{2}}. \tag{9}$$

For the case $r = 2$, i.e., $m = p_1^{\alpha_1} p_2^{\alpha_2}$, from Theorem 13.3.1 of Chengdong and Chenbiao [3], we have

$$\sideset{}{^*}\sum_{\substack{\chi \bmod p_1^{\alpha_1} p_2^{\alpha_2} \\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi)$$

$$= \left(\sideset{}{^*}\sum_{\substack{\chi_1 \bmod p_1^{\alpha_1} \\ \chi_1(-1)=-1}} \chi_1\left(np_2^{(k+1)\alpha_2}\right)\tau^{k+1}(\chi_1) \right)$$

$$\times \left(\sideset{}{^*}\sum_{\substack{\chi_2 \bmod p_2^{\alpha_2} \\ \chi_2(-1)=1}} \chi_2\left(np_1^{(k+1)\alpha_1}\right)\tau^{k+1}(\chi_2) \right)$$

$$+\left(\sideset{}{^*}\sum_{\substack{\chi_1 \bmod p_1^{\alpha_1}\\ \chi_1(-1)=1}} \chi_1\left(np_2^{(k+1)\alpha_2}\right)\tau^{k+1}(\chi_1)\right)$$

$$\times\left(\sideset{}{^*}\sum_{\substack{\chi_2 \bmod p_2^{\alpha_2}\\ \chi_2(-1)=-1}} \chi_2\left(np_1^{(k+1)\alpha_1}\right)\tau^{k+1}(\chi_2)\right).$$

From (8) and (9), we then obtain

$$\left|\sideset{}{^*}\sum_{\substack{\chi \bmod p_1^{\alpha_1}p_2^{\alpha_2}\\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi)\right| \ll k^2\phi(p_1^{\alpha_1}p_2^{\alpha_2})p_1^{\frac{k\alpha_1}{2}}p_2^{\frac{k\alpha_2}{2}}.$$

For general $m = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, let χ_i denote the Dirichlet character modulo $p_i^{\alpha_i}$. Then by the same method, we can write

$$\sideset{}{^*}\sum_{\substack{\chi \bmod p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}\\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi)$$

$$= \sum_{t_1=\pm1}\sum_{\substack{t_2=\pm1\\ t_1t_2\cdots t_r=-1}}\cdots\sum_{t_r=\pm1}\prod_{i=1}^{r}\left(\sideset{}{^*}\sum_{\substack{\chi_i \bmod p_i^{\alpha_i}\\ \chi_i(-1)=t_i}} \chi_i\left(n\left(\frac{m}{p_i^{\alpha_i}}\right)^{k+1}\right)\tau^{k+1}(\chi_i)\right).$$

Noting that

$$\sum_{t_1=\pm1}\sum_{\substack{t_2=\pm1\\ t_1t_2\cdots t_r=-1}}\cdots\sum_{t_r=\pm1} 1 = 2^{r-1} = 2^{\omega(m)-1},$$

combining (8) and (9), we can easily obtain the estimate

$$\left|\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \chi(n)\tau^{k+1}(\chi)\right| \ll (2k)^{\omega(m)}\phi(m)m^{\frac{k}{2}}.$$

This completes the proof of Lemma 4. □

Lemma 5. *Let q, h be integers with $q \geqslant 2$ and $(h, q) = 1$. For any fixed positive integer k with $(q, k(k+1)) = 1$, we have the estimate*

$$\left| \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\prod_{\substack{p|q \\ p\nmid m}} (1-\chi(p)) \right)^{k+1} \tau^{k+1}(\chi) L^{k+1}(1, \overline{\chi}) \right|$$

$$\ll 2^{(k+1)^2} \phi(q) q^{\frac{k}{2}} d(q) \left(2^{k+2}k\right)^{\omega(q)} \ln^{k+1} q.$$

Proof. For any non-principal character $\chi \bmod q$ and any parameter $N \geqslant q$, applying Abel's identity we have

$$L^{k+1}(1, \overline{\chi}) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n) d_{k+1}(n)}{n} = \sum_{1 \leqslant n \leqslant N} \frac{\overline{\chi}(n) d_{k+1}(n)}{n} + \int_N^{\infty} \frac{A(y, \overline{\chi})}{y^2} \, dy,$$

where $d_{k+1}(n)$ denotes the $(k+1)$th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \cdots n_{k+1} = n$ in positive integers $n_1, n_2, \cdots, n_{k+1}$) and $A(y, \overline{\chi}) = \sum_{N<n\leqslant y} \overline{\chi}(n) d_{k+1}(n)$. So we have

$$\left| \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\prod_{\substack{p|q \\ p\nmid m}} (1-\chi(p)) \right)^{k+1} \tau^{k+1}(\chi) L^{k+1}(1, \overline{\chi}) \right|$$

$$\leqslant \left| \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{d|q/m} \mu(d)\chi(d) \right)^{k+1} \tau^{k+1}(\chi) \sum_{1 \leqslant n \leqslant N} \frac{\overline{\chi}(n) d_{k+1}(n)}{n} \right|$$

$$+ \left| \sum_{m|q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h) \left(\sum_{d|q/m} \mu(d)\chi(d) \right)^{k+1} \tau^{k+1}(\chi) \int_N^{\infty} \frac{A(y, \overline{\chi})}{y^2} \, dy \right|. \quad (10)$$

Note that

$$\sum_{\substack{\chi \bmod m \\ \chi \neq \chi_0}} | A(y, \chi) |^2 \ll y^{2-\frac{4}{2^{k+1}}+\varepsilon} \phi^2(m)$$

(see [11, Lemma 4]). Applying Cauchy's inequality we have

$$\begin{aligned}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} | A(y,\overline{\chi}) | &\ll \sum_{\chi\neq\chi_0} | A(y,\overline{\chi}) |\\ &\ll \left(\phi(m)\sum_{\chi\neq\chi_0} | A(y,\overline{\chi}) |^2\right)^{\frac{1}{2}}\\ &\ll y^{1-\frac{1}{2^k}+\varepsilon}\phi^{3/2}(m). \qquad (11)\end{aligned}$$

Thus from (11) we get

$$\begin{aligned}&\left|\sum_{m|q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \overline{\chi}(h)\left(\sum_{d|q/m}\mu(d)\chi(d)\right)^{k+1}\tau^{k+1}(\chi)\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\right|\\ &\ll q^{\frac{k+1}{2}}\sum_{m|q} d^{k+1}\left(\frac{q}{m}\right)\int_N^\infty \frac{1}{y^2}\left(\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} |A(y,\overline{\chi})|\right)dy\\ &\ll q^{\frac{k+1}{2}}\sum_{m|q} d^{k+1}\left(\frac{q}{m}\right)\phi^{\frac{3}{2}}(m)\int_N^\infty \frac{y^{1-\frac{1}{2^k}+\varepsilon}}{y^2}\,dy\\ &\ll q^{\frac{k+1}{2}} d^{k+2}(q)\phi^{\frac{3}{2}}(q)N^{-\frac{1}{2^k}+\varepsilon}, \qquad (12)\end{aligned}$$

assuming $\varepsilon < \frac{1}{2^k}$.

Next we estimate the first term in (10).

$$\begin{aligned}&\sum_{m|q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \overline{\chi}(h)\left(\sum_{d|q/m}\mu(d)\chi(d)\right)^{k+1}\tau^{k+1}(\chi)\sum_{1\leqslant n\leqslant N}\frac{\overline{\chi}(n)d_{k+1}(n)}{n}\\ &=\sum_{m|q}\sum_{\substack{d_1|\frac{q}{m}\\ (d_1,m)=1}}\mu(d_1)\sum_{\substack{d_2|\frac{q}{m}\\ (d_2,m)=1}}\mu(d_2)\cdots\sum_{\substack{d_{k+1}|\frac{q}{m}\\ (d_{k+1},m)=1}}\mu(d_{k+1})\\ &\quad\times\sum_{\substack{1\leqslant n\leqslant N\\ (n,m)=1}}\frac{d_{k+1}(n)}{n}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}}\chi(\overline{h}\overline{n}d_1d_2\cdots d_{k+1})\tau^{k+1}(\chi)\end{aligned}$$

and so from Lemma 4 we get

$$\left|\sum_{m|q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}} \overline{\chi}(h)\left(\sum_{d|q/m}\mu(d)\chi(d)\right)^{k+1}\tau^{k+1}(\chi)\sum_{1\leqslant n\leqslant N}\frac{\overline{\chi}(n)d_{k+1}(n)}{n}\right|$$

$$\ll \sum_{m|q}\sum_{d_1|q/m}|\mu(d_1)|\sum_{d_2|q/m}|\mu(d_2)|\cdots\sum_{d_{k+1}|q/m}|\mu(d_{k+1})|\sum_{\substack{1\leqslant n\leqslant N\\ (n,m)=1}}\frac{d_{k+1}(n)}{n}$$

$$\times\left|\sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}}\chi(\overline{h}\overline{n}d_1d_2\cdots d_{k+1})\tau^{k+1}(\chi)\right|$$

$$\ll \ln^{k+1}N\sum_{m|q}2^{(k+1)\omega(\frac{q}{m})}(2k)^{\omega(m)}\phi(m)m^{\frac{k}{2}}$$

$$\ll \phi(q)q^{\frac{k}{2}}d(q)\left(2^{k+2}k\right)^{\omega(q)}\ln^{k+1}N. \tag{13}$$

Taking $N = q^{2^{k+1}}$, $\varepsilon < \frac{1}{2^{k+2}}$, noting that $d^{k+2}(q) \ll q^{(k+2)\varepsilon}$, combining (10), (12) and (13), we have

$$\left|\sum_{m|q}\ \sideset{}{^*}\sum_{\substack{\chi \bmod m\\ \chi(-1)=-1}}\overline{\chi}(h)\left(\prod_{\substack{p|q\\ p\nmid m}}(1-\chi(p))\right)^{k+1}\tau^{k+1}(\chi)L^{k+1}(1,\overline{\chi})\right|$$

$$\ll 2^{(k+1)^2}\phi(q)q^{\frac{k}{2}}d(q)\left(2^{k+2}k\right)^{\omega(q)}\ln^{k+1}q.$$

This proves Lemma 5. □

Lemma 6. *Let q be an integer with $q\geqslant 3$. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^{2k+2} = \frac{\phi(q)}{2}\zeta^{2k+1}(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k+1}$$

$$\times\prod_{p\nmid q}\left(1-\frac{1-C_{2k}^k}{p^2}\right)+O_k(q^{\varepsilon}).$$

Proof. Let $A(y,\chi)=\sum_{N<n\leqslant y}\chi(n)d_{k+1}(n)$, where $N>q^{2^k}$ is a positive integer. From the properties of the Dirichlet characters we have

$$\begin{aligned}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^{2k+2} &= \frac{1}{2}\sum_{\chi \bmod q}(1-\chi(-1))\,|L(1,\chi)|^{2k+2}\\ &= \frac{1}{2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}|L(1,\chi)|^{2k+2}\\ &\quad -\frac{1}{2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(-1)\,|L(1,\chi)|^{2k+2}\\ &\equiv R_1-R_2. \qquad (14)\end{aligned}$$

Applying the Lemma 6 of Zhang et al. [11] we have

$$R_1=\frac{\phi(q)}{2}\zeta^{2k+1}(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k+1}\prod_{p\nmid q}\left(1-\frac{1-C_{2k}^k}{p^2}\right)+O(q^{\varepsilon}). \qquad (15)$$

From Abel's identity and a similar method of proving Lemma 6 of Zhang et al. [11] we have

$$\begin{aligned}R_2 &= \frac{1}{2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(-1)\left|\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\right|^{2k+2}\\ &= \frac{1}{2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(-1)\left|\sum_{n=1}^{N}\frac{\chi(n)d_{k+1}(n)}{n}+\int_N^{+\infty}\frac{A(y,\chi)}{y^2}\,dy\right|^2\\ &= \frac{1}{2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(-1)\left|\sum_{n=1}^{N}\frac{\chi(n)d_{k+1}(n)}{n}\right|^2\\ &\quad +O\left(\left(\sum_{n=1}^{N}\frac{d_{k+1}(n)}{n}\right)\int_N^{+\infty}\frac{1}{y^2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}|A(y,\chi)|\,dy\right)\end{aligned}$$

$$+O\left(\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \left|\int_N^{+\infty} \frac{A(y,\chi)}{y^2}dy\right|^2\right)$$

$$\ll \phi(q) \sideset{}{'}\sum_{n=1}^{N} \sideset{}{'}\sum_{\substack{m=1 \\ m\equiv -n \bmod q}}^{N} \frac{d_{k+1}(n)d_{k+1}(m)}{nm} + \phi^{\frac{3}{2}}(q)N^{-\frac{1}{2^k}}\ln^{k+2} N \ll_k q^{\varepsilon}, \qquad (16)$$

where we have taken $N = q^{3\cdot 2^{k-1}}$ and used the estimate $d_{k+1}(n) \ll_k n^{\varepsilon}$. Then the Lemma 6 follows from (14)–(16).

3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. From Lemmas 2 and 5, we have

$$\begin{aligned}|C(h,k,q)| &= \frac{1}{\pi^{k+1}\phi(q)} \\ &\quad\times\left|\sum_{m\mid q}\sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \overline{\chi}(h)\left(\prod_{\substack{p\mid q \\ p\nmid m}}(1-\chi(p))\right)^{k+1} \tau^{k+1}(\chi)L^{k+1}(1,\overline{\chi})\right| \\ &\ll \frac{2^{(k+1)^2}}{\pi^{k+1}} q^{\frac{k}{2}} d(q)\left(2^{k+2}k\right)^{\omega(q)}\ln^{k+1} q.\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From formula (6) and the orthogonality relation for characters modulo p^{α}, we get

$$\begin{aligned}&\sideset{}{'}\sum_{h=1}^{p^{\alpha}} \left|C(h,k,p^{\alpha})\right|^2 \\ &= \sideset{}{'}\sum_{h=1}^{p^{\alpha}} \left|\frac{i^{k+1}}{\pi^{k+1}(p^{\alpha}-p^{\alpha-1})}\sum_{\beta=1}^{\alpha}\sideset{}{^*}\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=-1}} \overline{\chi}(h)\tau^{k+1}(\chi)L^{k+1}(1,\overline{\chi})\right|^2 \\ &= \frac{1}{\pi^{2k+2}\left(p^{\alpha}-p^{\alpha-1}\right)^2}\sum_{\beta_1=1}^{\alpha}\sum_{\beta_2=1}^{\alpha}\sideset{}{^*}\sum_{\substack{\chi_1 \bmod p^{\beta_1} \\ \chi_1(-1)=-1}}\sideset{}{^*}\sum_{\substack{\chi_2 \bmod p^{\beta_2} \\ \chi_2(-1)=-1}}\end{aligned}$$

$$\times \sum_{h=1}^{p^\alpha}{}' \overline{\chi_1}\chi_2(h)\tau^{k+1}(\chi_1)\overline{\tau(\chi_2)}^{k+1} L^{k+1}(1,\overline{\chi_1})L^{k+1}(1,\chi_2)$$

$$= \frac{1}{\pi^{2k+2}\left(p^\alpha - p^{\alpha-1}\right)} \sum_{\beta=1}^{\alpha} p^{(k+1)\beta} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=-1}}^{*} |L(1,\overline{\chi})|^{2k+2}$$

$$= \frac{\pi^{-(2k+2)}}{p^\alpha - p^{\alpha-1}} \sum_{\beta=1}^{\alpha} p^{(k+1)\beta}$$

$$\times \left(\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=-1}} |L(1,\overline{\chi})|^{2k+2} - \sum_{\substack{\chi \bmod p^{\beta-1} \\ \chi(-1)=-1}} |L(1,\overline{\chi})|^{2k+2} \right).$$

Now taking $q = p^\alpha$, from Lemma 6, we immediately obtain

$$\sum_{h=1}^{p^\alpha}{}' |C(h,k,p^\alpha)|^2$$

$$= \frac{\pi^{-(2k+2)}\zeta^{2k+1}(2)}{p^\alpha - p^{\alpha-1}} \left[p^{k+1}\frac{p-1}{2} + \sum_{\beta=2}^{\alpha} p^{(k+1)\beta} \left(\frac{\phi(p^\beta) - \phi(p^{\beta-1})}{2} \right) \right]$$

$$\times \left(\frac{p^2-1}{p^2} \right)^{2k+1} \prod_{p_1 \neq p} \left(1 - \frac{1 - C_{2k}^k}{p_1^2} \right) + O\left(p^{\alpha k+\varepsilon}\right)$$

$$= \frac{\zeta^{2k+1}(2)\left[\left(1 - \frac{1}{p}\right) p^{(\alpha+1)(k+2)} + p^{2k+3} - p^{k+2} \right]}{2\pi^{2k+2} p^\alpha (p^{k+2} - 1)} \left(\frac{p^2-1}{p^2} \right)^{2k+1}$$

$$\times \prod_{p_1 \neq p} \left(1 - \frac{1 - C_{2k}^k}{p_1^2} \right) + O\left(p^{\alpha k+\varepsilon}\right).$$

This completes the proof of Theorems 2.

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