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On Thompson's Simple Group

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Recently, Thompson [9] constructed a new simple group E of order $2^{15} \circ 3^{10}5^{3}7^{2} \circ 13 \circ 19 \circ 31 = 90,745,943,887,872,000$. In particular, E contains only one conjugacy class of involutions, and if e is an involution in E then $C_E(e)$ is a (nonsplit) extension of an extra-special 2-group of order 2^9 by $\mathscr{A}_9$, the alternating group of degree 9.

The aim of this paper is to prove the following result.

THEOREM. *Let G be a finite group which contains an involution z. Let $H = C_G(z)$ and suppose that $G \neq H \cdot O(G)$ and H satisfies:*

(i) *$J = O_2(H)$ is extra-special of order 2^9;*

(ii) *$H/J \cong \mathscr{A}_9$, the alternating group of degree 9. Then G is a simple group with the same order and the same conjugacy classes as Thompson's simple group E.*

COROLLARY (Thompson). *If G is a finite group satisfying the assumptions of the theorem then $G \cong E$.*

(This follows immediately from Thompson's paper [9].)

The notation in this paper will follow Gorenstein [4]. In addition we will use:

$X * Y$: a central product of the groups X and Y;

$x \sim_X y$: x is conjugate to y in X;

Z_n : the cyclic group of order n;

D_n : the dihedral group of order n;

Q_{2^n} : the (generalized) quaternion group of order 2^n;

SD_{2^n} : the semidihedral group of order 2^n;

$\mathscr{A}_n$, Σ_n : the alternating, symmetric groups of degree n.

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1. Preliminary Results

Proposition 1 [4, pp. 105, 328]. *If x is an involution in the finite group X and $x \notin O_2(X)$, then x inverts an elements of odd order in $X^{\#}$.*

Proposition 2 [10]. *Let $Y = \{1, y_1, y_2, y_3\}$ be a 4-group of automorphisms of the finite group X of odd order. Then*

$$|X| \cdot |C_X(Y)|^2 = |C_X(y_1)| \cdot |C_X(y_2)| \cdot |C_X(y_3)|.$$

Recall that a p-group P is *extra-special* if $P' = Z(P) = \Phi(P)$, and $|P'| = p$.

Proposition 3 [4, Theorem 5.5.2]. *An extra-special 2-group J is the central product of $n \geqslant 1$ non-Abelian groups of order 8. (Thus J has order 2^{2n+1}.)*
Further,

$$J \cong D_8 * D_8 * \cdots * D_8 \qquad (J \text{ of type } +)$$

or

$$J \cong D_8 * \cdots * D_8 * Q_8 \qquad (J \text{ of type } -).$$

In the rest of this section we prove some simple but useful results under the following assumption:

Hypothesis 1. Let G be a finite group, let z be an involution in G, let $H = C_G(z)$ and let $J = O_2(H)$ be extra-special of order 2^{2n+1}, $n \geqslant 1$.
Let $P \neq 1$ be a p-subgroup of H(p an odd prime), and let t be an involution in $H - \langle z \rangle$ with $t \sim_G z$.

Proposition 4. *Under Hypothesis 1, each of $C_J(P)$, $[P, J]$ is extraspecial or equal to $\langle z \rangle = Z(J)$.*

Proof. This follows immediately from $C_J(P) \cdot [P, J] = J$ [4, Theorem 5.3.5] and the three subgroups lemma [4, Theorem 2.2.3].

Proposition 5. *Suppose that Hypothesis 1 holds, that $t \in C_H(P)$, and that P satisfies:*

(*) *If $y^{-1}Py \subseteq H$ with $y \in G$, then there exists $h \in H$ such that $y^{-1}Py = h^{-1}Ph$.*

Then $t \sim_{N(P)} z$, and further if $N_G(P) = N_H(P) \circ C_G(P)$ then $t \sim_{C(P)} z$.

Proof. Obvious. (Note that if P is a Sylow p-subgroup of H then P satisfies (*).)

LEMMA 6. *Let H satisfy Hypothesis 1 with $C_H(J) = \langle z \rangle$. If $x \in H - J$ and $|J : C_J(x)| = 2$ then $x^2 \notin \langle z \rangle$.*

Proof. Suppose $x^2 \in \langle z \rangle$. By Proposition 1 (applied to $H/\langle z \rangle$) x inverts an element r of odd order in H. Hence r has order 3, $[r, J] \cong Q_8$, and $\langle x, [r, J]\rangle \cong Q_{16}$ or SD_{16}. In any case $C(x) \cap [r, J] = \langle z \rangle$, which contradicts $|J : C_J(x)| = 2$.

PROPOSITION 7 (Janko). *Suppose Hypothesis 1 holds and in addition $C_H(J) = \langle z \rangle$ and $t \in J - \langle z \rangle$. Then $O_2(C_H(t))/C_J(t)$ contains a normal elementary Abelian subgroup of order $\geqslant 2^{n-1}$ if J is of type $+$, and of order $\geqslant 2^n$ if J is of type $-$.*

Proof. From Lemma 6 we have $\Omega_1(Z(T)) = \langle t, z \rangle$, where T is a Sylow 2-subgroup of $C_H(t)$. It follows that $N_G(\langle t, z\rangle)/C_H(t) \cong \Sigma_3$. Let T^* be a Sylow 2-subgroup of $N_G(\langle t, z \rangle)$ which contains T, but $T^* \not\subseteq H$. If $x \in T^* - T$ then x normalizes $C_J(t)^x \cap C_J(t)$. Hence $C_J(t)^x \cap C_J(t)$ is elementary Abelian, and so of order $\leqslant 2^{n+1}$ (J of type $+$), $\leqslant 2^n$ (J of type $-$). Thus $C_J(t)/C_J(t)^x \cap C_J(t) \cong C_J(t)^x C_J(t)/C_J(t)$ is elementary of order $\geqslant 2^{n-1}$ (type $+$) or $\geqslant 2^n$ (type $-$). The proposition follows as $C_J(t) \lhd O_2(C_H(t)) \lhd N_G(\langle t, z \rangle)$.

2. SOME PROPERTIES OF $H/J \cong \mathscr{A}_9$

In this section we use the bar convention for $H/J = \bar{H}$ and use $\leftrightarrow$ to denote the correspondence of elements in the isomorphism $\bar{H} \cong \mathscr{A}_9(H, J, z$ as in the statement of the theorem.)

Let u, v be elements of minimal order in H with $\bar{u} \leftrightarrow (14)(25)(36)(78)$ and $\bar{v} \leftrightarrow (15)(24)$. (Recall that $\mathscr{A}_9$ has precisely two classes of involutions.) Thus $C_H(\bar{u}) \cong (D_8 * D_8) \circ \Sigma_3$ and $C_H(\bar{v}) \cong (Z_2 \times Z_2 \times \mathscr{A}_5) \circ Z_2$.

Let $B = \langle b \rangle$ be a Sylow 5-subgroup of H with $\bar{b} \leftrightarrow (12345)$. Thus $C_H(\bar{b}) \cong Z_5 \times \mathscr{A}_4$ and $N_{\bar{H}}(\bar{B})/C_{\bar{H}}(\bar{B}) \cong Z_4$.

Let $H_3 = \langle c_i \mid i = 1, 2, 3, 4 \rangle$ be a Sylow 3-subgroup of H where $\bar{c}_1 \leftrightarrow (789)$, $\bar{c}_2 \leftrightarrow (123)(456)$, $\bar{c}_3 \leftrightarrow (123)(465)(789)$, and $\bar{c}_4 \leftrightarrow (147)(268)(359)$. Therefore we have $C_{\bar{H}}(\bar{c}_1) \cong Z_3 \times \mathscr{A}_6$, $C_{\bar{H}}(\bar{c}_3) = \bar{H}_3$ and $C_{\bar{H}}(\bar{c}_2) = \bar{C}\langle \bar{u} \rangle$, where $C = \langle c_1, c_2, c_3 \rangle$ is the only elementary Abelian subgroup of order 27 in H_3. Further, $N_{\bar{H}}(\bar{H}_3) = \bar{H}_3\langle \bar{u} \rangle$ and $\Phi(\bar{H}_3) = \langle \bar{c}_2, \bar{c}_3 \rangle$. Let $\bar{f}_i$, $i = 1, 2$ represent the two conjugacy classes of elements of order 9 in $\bar{H}$, with $f_i^3 = c_3$, $i = 1, 2$. Note that $C_{\bar{H}}(\bar{f}_i) = \langle \bar{f}_i \rangle$ and f_i is conjugate to each element in $\langle f_i \rangle - \langle c_3 \rangle$ in $N_H(H_3)$ $i = 1, 2$.

Finally $A = \langle a \rangle$ will denote a Sylow 7-subgroup of H, $\bar{a} \leftrightarrow (1263549)$. Hence $C_{\bar{H}}(\bar{a}) = \bar{A}$ and $N_{\bar{H}}(\bar{A}) = \bar{A}\langle \bar{c}_2, \bar{u} \rangle$, a Frobenius group of order 42.

Remark. There is precisely one (proper) maximal subgroup of $\bar{H}$ which contains $\bar{H}_3$, namely, $N_{\bar{H}}(\bar{C})$. (This property of $\mathscr{A}_9$ will be needed in Section 3.). Note that $N_{\bar{H}}(\bar{C})/\bar{C} \cong \Sigma_4$, and that c_1, c_2, c_3 have 6, 12, 8 conjugates, respectively, in $N_H(C)$.

3. The Fusion of Involutions

For the rest of this paper G will denote a finite group which satisfies the hypothesis of the theorem. In addition the notation introduced in Section 2 will retain its meaning throughout the paper.

By assumption $G \neq H \circ O(G)$, whence Glauberman's theorem [3] yields the following result.

(3.1) There exists an involution $h \in H - \langle z \rangle$ with $h \sim_G z$.

Suppose that H is not 2-constrained, i.e., $C_H(J) \neq \langle z \rangle$. Then $C_H(J)$ covers H/J and $C_H(J) = H'$, with $H'/\langle z \rangle \cong \mathscr{A}_9$. Let $h \in H - \langle z \rangle$ and $h \sim_G z$. As $z \in C_H(h)' \subseteq H'$, it follows that z is conjugate to one of (the involutions) u, uz, v, vz by (3.1). Note that v, vz are involutions only if $H' \cong Z_2 \times \mathscr{A}_9$. If $z \sim_G v$, let S, S^* be Sylow 2-subgroups of $C_H(v)$, $C_G(v)$, respectively, with $| S^* : S | = 2$. Choose $x \in S^* - S$ and set $Y = (JS') \cap (JS')^x$. Since $| S : S' | = 2^{11}$ and $J \cap S' = \langle z \rangle$ we have $| Y | \geqslant 2^8$. Thus Y is not Abelian and so x normalizes $\langle z \rangle = Y'$, which contradicts $z \sim_G v$. The same argument with v replaced by vz shows $z \not\sim_G vz$.

Now let S denote a Syllow 2-subgroup of $C_H(u)$. If $H = J \times Y$, $Y \cong \mathscr{A}_9$, then $Z(S) = \langle u, z \rangle$, $[S', S] = \langle u \rangle$. Burnside's lemma [4, Theorem 7.1.1] yields $z \not\sim_G u$. As $| S : S' | = 2^{11}$, the same argument as above shows $z \not\sim_G uz$ either. Finally, if $H \cong \hat{\mathscr{A}}_9$, the covering group of $\mathscr{A}_9$, then $S \lhd C_H(u)$, $C_H(u) = S \circ P$, where $| P | = 3$, and $C_S(P) = J \times \langle u \rangle$. From $z \sim_G u$ or $z \sim_G uz$ it follows that $N_G(\langle u, z \rangle)/C_H(u) \cong \Sigma_3$. However, this forces $C_S(P) \lhd N_G(\langle u, z \rangle)$, which is impossible. We have shown therefore that $C_H(J) = \langle z \rangle$. Thus if T is a Sylow 2-subgroup of H, $Z(T) = \langle z \rangle$. Sylow's theorem then yields the final part of the next result.

(3.2) The centralizer H is 2-constrained; namely, $C_H(J) = \langle z \rangle$. Also a Sylow 2-subgroup T of H is a Sylow 2-subgroup of G.

We observe that if $x \in J - \langle z \rangle$ then $C_H(x)$ does not cover H/J. For if $C_H(x)/C_J(x) \cong \mathscr{A}_9$ then $C_H(x)$ must act nontrivially on $C_J(x)/\langle x, z \rangle$ (elementary of order 64) by (3.2). Thus $GL(6, 2)$ would have to contain subgroups isomorphic to $\mathscr{A}_9$, which is not the case.

Since $f_1^3 = c_3$ and c_3 acts fixed-point-free on $[c_3, J]/\langle z \rangle$, it follows from Proposition 4 and (3.2) that $[c_3, J] \cong Q_8 * Q_8 * Q_8$ and $| C_J(c_3) | = 8$. If $[H_3, C_J(c_3)] = 1$ then for $x \in C_J(c_3) - \langle z \rangle$ we have $C_H(x) \subseteq C_J(x) \circ N_H(C)$ (by the remark at the end of Section 2). It follows that x has at least 280 conjugates in H, which is not possible. Thus $[H_3, C_H(c_3)] \neq 1$, whence $C_J(c_3) \cong Q_8$.

(3.3) We have $J \cong D_8 * D_8 * D_8 * D_8$ $(\cong Q_8 * Q_8 * Q_8 * Q_8)$, $C_J(c_3) \cong Q_8$, and $C_J(H_3) = \langle z \rangle$.

Note that $J - \langle z \rangle$ contains 270 involutions and 240 elements of order 4. It follows immediately from (3.3) and Proposition 4 that $[a, J] \cong D_8 * D_8 * D_8$ and $C_J(a) \cong D_8$. Thus as $c_2 \in N_H(\langle a \rangle)$, $C_J(c_2) \supseteq C_J(a)$, and $C_J(c_2) \cap [a, J] \supset \langle z \rangle$ [4, Theorem 5.3.14]. Hence $| C_J(c_2)| = 32$. Now $\Phi(H_3) = \langle c_2, c_3 \rangle$, whence $[c_2, C_J(c_3)] = 1$. This implies $[c_3, C_J(c_2)] \cong Q_8$ and $C_J(c_2) \cong Q_8 * Q_8 \cong [c_2, J]$.

(3.4) We have $C_J(c_2) \cong Q_8 * Q_8$ and $C_J(a) \cong D_8$.

Choose $u \in N_H(H_3)$ so that $u^2 \in C_J(H_3) = \langle z \rangle$. Hence $C_C(u) = \langle c_2 \rangle$ and $[u, C] = \langle c_1, c_3 \rangle$ so that $(c_2c_3)^u = c_2c_3^{-1}$. Now $c_2c_3 \sim_H c_2$ and $C_J(c_3) \subseteq C_J(c_2)$, whence $C_J(c_2c_3) \cap [c_2, J] \cong Q_8$. It follows that u interchanges the two quaternion groups in $[c_2, J]$. On the other hand u normalizes $C_J(c_3)$ and $[c_3, C_J(c_2)]$ so that u must act as an outer automorphism on both quaternion subgroups of $C_J(c_2)$ (recall that $[H_3, C_J(c_3)] \neq 1$).

Let $C_J(c_3) = \langle r_1, s_1 \rangle$, $[C_J(c_2), c_3] = \langle r_2, s_2 \rangle$, $C_J(c_2c_3) \cap [c_2, J] = \langle r_3, s_3 \rangle$, and $C_J(c_2c^{-1}) \cap [c_2, J] = \langle r_4, s_4 \rangle$, so that $\langle r_i, s_i \rangle \cong Q_8$, $i = 1, \ldots, 4$.

(3.5) Choosing the r_i, s_i suitably we have

$$r_i^u = r_i^{-1}, \qquad s_i^u = r_i s_i, \qquad i = 1, 2$$

and

$$r_3^u = r_4, \qquad s_3^u = s_4.$$

In particular, $C_J(u) = \langle r_1r_2, r_3r_4, s_3s_4, z \rangle$, which is elementary Abelian of order 16. If u is an involution then all (32) involutions in uJ are conjugate in $\langle u, J \rangle$, and no element in uJ squares to z.

From (3.3) and (3.5) we see that $C_J(c_3)$, $\langle u, C_J(c_2) \rangle$ are Sylow 2-subgroups of $C_G(c_3)$, $C_G(c_2)$, respectively. Thus $c_2 \not\sim_G c_3 \not\sim_G c_1$ (recall that $2^3 \mid | C_{\bar{H}}(\bar{c}_1)|$). Suppose that $c_2 \sim_G c_1$. Then $C_J(c_1) = \langle z \rangle$, as $C_J(c_1) \supset \langle z \rangle$ implies $| C_J(c_1)| = 2^7$. Note that all involutions in $C_J(c_2) - \langle z \rangle$ are conjugate in $C_H(c_2)$ and if t is such an involution, $| C(t) \cap C_H(c_2)| = 2^5 \cdot 3$. Also if u is an involution then all involu-volutions in $C_H(c_2) - C_J(c_2)$ are conjugate to u and $| C(u) \cap C_H(c_2)| = 2^3 \cdot 3$. On the other hand, $| C(z) \cap C_G(c_1)| = | C_H(c_1)| = 2^4 \cdot 3^3 \cdot 5$. This is incompatible with the orders of the centralizers listed above. Thus $c_2 \not\sim_G c_1$.

(3.6) The elements c_1, c_2, c_3 lie in distinct conjugate classes of G.

We next argue, by way of contradiction, that $C_J(c_1) = \langle z \rangle$. From Proposition 4, $C_{\bar{H}}(\bar{c}_1) \cong Z_3 \times \mathscr{A}_6$, and $C_J(c_1) \neq \langle z \rangle$, it follows that $C_J(c_1) \cong Q_8 * Q_8 * Q_8$, $[c_1, J] \cong Q_8$. Further, as we may assume $[b, c_1] = 1$, $C(b) \cap C_J(c_1) \cong D_8$ and $C_J(b) \cong D_8 * Q_8$.

Suppose $t \in C_J(c_1)$ and $t \sim_G z$. Then $z \sim t$ in $C_G(c_1)$ by Proposition 5 and (3.6). However, this is impossible by Proposition 7, as $[C_J(c_1), C_H(c_1)] \neq 1$ and $C_H(c_1)/C_J(c_1) \cong Z_3 \times \mathscr{A}_6$. If j is an involution in J then we claim j is conjugate to an involution in $C_J(c_1)$. For if not, as $c_2 \in \langle c_1^{-1}c_2c_3, c_1c_2c_3^{-1} \rangle$, $c_1^{-1}c_2c_3 \sim_H$

$c_1c_2c_3^{-1} \sim_H c_1$, and $|C_J(c_3^{-1}c_2c_3) \cap C_J(c_1c_2c_3^{-1})| = 2^5$, we have $3 \nmid |C_H(j)|$. Thus j has $2 \cdot 3^4 = 162$ conjugates in H and $|C_H(j)/C_J(j)| = 2^6 \cdot 5 \cdot 7$, which is impossible. We conclude that $z \nsim_G j$ for any $j \in J - \langle z \rangle$.

Suppose next that $z \sim_G u$. It follows from Proposition 5 and (3.6) that $u \sim z$ in $C_G(c_2)$. Now $\langle u, r_1r_2, z \rangle$ is a Sylow 2-subgroup of $C_H(c_2) \cap C(u)$ whence there exists a 2-group $Y \subseteq C(u) \cap C_G(c_2)$ with $|Y : \langle u, r_1r_2, z\rangle| = 2$. However, $\{r_1r_2, r_1r_2z\} \lhd Y$ (as these are the only involutions in $\langle u, r_1r_2, z\rangle$ not conjugate to z in G) whence $\langle z \rangle \lhd Y$, a contradiction. This shows $z \nsim uj$ for any involution $uj \in uJ$ (see (3.5)).

Finally we suppose $z \sim_G v$. Let T_1 be a Sylow 2-subgroup of $C_H(v)$, let T_2 be a 2-subgroup of $C_G(v)$ with $|T_2 : T_1| = 2$, and let $x \in T_2 - T_1$. By Proposition 1 we may assume that v inverts b. Clearly $O_{5'}(C_{\bar{H}}(\bar{b}))(\simeq \mathscr{A}_4)$ acts trivially on $[b, J](\simeq D_8 * Q_8)$ whence $O_{5'}(C_{\bar{H}}(\bar{b}))$ acts nontrivially on $C_J(b)$. This implies $[v, C_J(b)] \subseteq \langle z \rangle$ and therefore $C(v) \cap C_J(b) \supset Q^*$, where $Q^* \simeq Q_8$. As $z \nsim_G j$ for any $j \in J - \langle z \rangle$ it follows that $(Q^*)^x \cap C_J(v) = 1$, which contradicts the structure of $T_1/C_J(v)$. This completes the proof that $C_J(c_1) = \langle z \rangle$. It follows immediately (as we may assume $[c_1, b] = 1$) that $C_J(b) = \langle z \rangle$ also.

(3.7) We have $C_J(c_1) = C_J(b) = \langle z \rangle$.

The above results yield that any involution $t \in J - \langle z \rangle$ has at least $2 \cdot 3^3 5 = 270$ conjugates in H. Thus all involutions in $J - z$ are conjugate in H and from the structure of $\mathscr{A}_9$ we have $C_H(t)/C_J(t) \simeq (Z_2 \times Z_2 \times Z_2) \cdot PSL(2, 7)$.

(3.8) All involutions in $J - \langle z \rangle$ are conjugate in H.

For any $s \in J$, s of order four, (3.7) yields that s has $30 \cdot n$ conjugates ($n \geqslant 1$). Thus $3 \mid |C_H(s)|$ and it then follows that we may assume $s \in C_J(c_2)$. Let $u_1 \in N_H(\langle c_2 \rangle) - C_H(c_2)$, $u_1 \sim_H u$, and $\bar{u}_1 \leftrightarrow (14)(26)(35)(78)$. From (3.5) we see that u_1 interchanges the two quarternion subgroups in $C_J(c_2)$, whence all elements of order four in $C_J(c_2)$ are conjugate in $N_H(\langle c_2 \rangle)$. All elements of order four in J are therefore conjugate in H and $|C_H(s)/C_J(s)| = 2^3 \circ 3^3 \circ 7$. The structure of $\mathscr{A}_9$ yields that $C_H(s)/C_J(s) \simeq \mathrm{Aut}(PSL(2, 8))$.

(3.9) All elements of order four in J are conjugate in H. If s is an element of order four in J then $C_H(s)/C_J(s) \simeq \mathrm{Aut}(PSL(2, 8))$.

If v is an involution then we may assume $v \in N_H(B)$ (Proposition 1). If vJ does not contain involutions, choose $v \in N_H(B)$, so that $v^2 \in \langle z \rangle$ in either case. It follows that v has 16 fixed points on $J/\langle z \rangle$, whence $|C_J(v)| \leqslant 32$. For $j \in J$, $(vj)^2 \in \langle z \rangle$ implies $[v, j] \in \langle z \rangle$, whence vJ has at most two classes of elements with square in $\langle z \rangle$. Hence we have $b', c_1' \in C_H(v)$, $b' \sim_H b$, $c_1' \sim_H c_1$, which implies (by (3.7)) that $C_J(v)$ is elementary of order 32 and $v \nsim vz$ in $\langle J, v \rangle$.

(3.10) By choosing $v \in N_H(B)$ we have $v^2 \in \langle z \rangle$ and $C_J(v)$ is elementary of order 32. Further if v is an involution then vJ contains no elements of order four with square z.

Suppose that $z \sim_G t = r_1r_2$. Then $z \sim t$ in $C_G(c_2)$ by (3.6) and Proposition 5. Let $S = \langle C_J(c_2), u\rangle$, a Sylow 2-subgroup of $C_G(c_2)$. As $C_S(t) \cap J \cong Z_2 \times D_8$, it follows (using (3.8)) that $z \sim u$ in $C_G(c_2)$ also. Conversely, suppose $z \sim_G u$. As above $z \sim u$ in $C_G(c_2)$ and as $C_S(u) = \langle u, z, t\rangle$ we have $z \sim t$ in $C_G(c_2)$. (We use (3.5) and the fact that $\{t, tz\}$ can not be a characteristic subset of $C_S(u)$.) In either case our arguments show that $C_G(c_2)$ contains one class of involutions.

Suppose $z \sim_G t \sim_G u$ and let u_1 be an involution in $N_H(\langle c_2\rangle) - C_H(c_2)$, $\bar{u}_1 \leftrightarrow (14)(26)(35)(78)$. As above (in the proof of (3.9)) u_1 interchanges the two quaternion subgroups $\langle r_1, s_1\rangle$, $\langle r_2, s_2\rangle$ of $C_J(c_2)$. Without loss, let $r_1^{u_1} = r_2$, whence $(r_1u_1)^2 = r_1r_2 = t$. Hence there exists $y \in \langle S, u_1\rangle - S$ with $y^2 = z$. It follows from $u_1 \sim_H u$ and (3.5) that $y \in uu_1C_J(c_2)$, $\overline{uu_1} \leftrightarrow (23)(56)$. As $uu_1J \sim_H vJ$ and because of (3.10) we see that vJ does not contain involutions. Thus $z \sim_G t \sim_G u$ implies G contains one class of involutions.

Suppose now that $z \sim_G v$, and take $b', c_1' \in C_H(v)$ as above. Then we must have $C_H(c_1') = \langle c_1'\rangle \times \langle z\rangle \times Y$, where $Y \cong \mathscr{A}_6$. Now as before, $z \sim v$ in $C_G(c_1')$. Hence, as $T_0 = C(v) \cap C_H(c_1') \cong Z_2 \times D_8$, we have $T_0' = \langle vz\rangle$ and $vz \not\sim_G z$. Set $\langle z, v, v'\rangle = O_2(C_H(b'))$ and note that as v is an involution, $\langle z, v, v'\rangle$ is elementary (of order 8). Thus $C_J(v') = C_J(v)$, whence $V = \langle v, v', C_J(v)\rangle$ is elementary of order 2^7. As $v \not\sim_H vz$, $C_H(v)\, J/J = C_{\bar{H}}(\bar{v})$ and $|\, C_H(v)\colon C_H(\langle v, v'\rangle)| = 2$ with $C_H(\langle v, v'\rangle)/V \cong \mathscr{A}_5$. Further $N_H(V) = J \cdot C_H(v)\langle c_1''\rangle$, where $\langle c_1''\rangle\langle z, v, v'\rangle = O_{5'}(C_H(b')) \cong Z_2 \times \mathscr{A}_4$, which implies that both v and vz have 48 conjugates in $N_H(V)$. Also all involutions in $C_J(v) - \langle z\rangle$ are conjugate in $N_H(V)$. Recall that $t \not\sim_G z \not\sim_G u$ under the assumption $z \sim_G v$. This implies V contains precisely 49 involutions conjugate to z in G.

Let T_1, T_2 be Sylow 2-subgroups of $C_H(v)$, $C_H(\langle v, v'\rangle)$, respectively, with $|\, T_1 : T_2 \,| = 2$; let T_1^* be a 2-subgroup of $C_G(v)$ with $|\, T_1^*\colon T_1 \,| = 2$; and let $x \in T_1^* - T_1$. If $V^x \neq V$ then $V^x \subseteq T_2$ and V^x covers $T_2/C_J(v)$. However, this implies that V^x contains 25 involutions conjugate to z in G, a contradiction. Hence $V^x = V$ and $N_G(V) \supset N_H(V)$. Clearly $(C_J(v) - \langle z\rangle) \ntriangleleft N_G(V)$, so vz has 79 conjugates in $N_G(V)$, obviously a contradiction. We have shown therefore that $z \not\sim v$ and (as we could replace v by vz above) that z is not conjugate to any involution in vJ. By (3.1), (3.5), (3.8) either $z \sim_G t$ or $z \sim_G u$. We have therefore completely determined the fusion of involutions in G.

(3.11) The group G contains only one conjugacy class of involutions; more precisely, $z \sim_G t \sim_G u$ and the coset vJ does not contain involutions.

We complete this section with three results which follow from the proof of (3.11).

(3.12) We have $C_G(c_2)/\langle c_2\rangle \cong G_2(3)$.

Proof. From (3.3), (3.5), (3.7), $C_H(c_2)/\langle c_2\rangle$ is isomorphic to the centralizer of an involution in $G_2(3)$. Further, as $z \sim u$ in $C_G(c_2)$, $C_G(c_2)$ contains no subgroup of index two. The result of Janko [7] yields $C_G(c_2)/\langle c_2 \rangle \cong G_2(3)$, as required.

(3.13) We have $C_G(a) = \langle a \rangle \times L$, where $L \cong PSL(2, 7)$.

Proof. Let $\langle a_1 \rangle$ be a Sylow 7-subgroup of $C_G(c_2)$. Then $N_G(\langle c_2 \rangle) \cap C_G(a_1) \cong Z_7 \times \Sigma_3$, which implies $a_1 \sim_G a$ (by (3.11)), and $\langle c_2 \rangle$ is a Sylow 3-subgroup of $C_G(a)$. Since $C_H(a) \cong Z_7 \times D_8$, (3.11), Proposition 5, and the Gorenstein-Walter result [5] yield $C_G(a) = \langle a \rangle \times L$, where $L \cong PSL(2, 7)$.

(3.14) There is an elementary Abelian subgroup F of order 32, $F \subseteq J$ with $N_G(F)/F \cong GL(5, 2)$. In particular, $31 \mid |G|$.

Proof. Let t be an involution in $C_J(a)$ and let S be a Sylow 2-subgroup of $C_H(t)$. Then $Z(S) = \langle t, z \rangle$ which implies that $N_G(\langle t, z \rangle)/C_H(t) \cong \Sigma_3$ (using (3.11)). It follows that $N_G(\langle t, z \rangle)/O_2(C_H(t)) \cong \Sigma_3 \times PSL(2, 7)$ (see the proof of (3.8)), where $O_2(C_H(t))/C_J(t) \cong Z_2 \times Z_2 \times Z_2$. As $\langle t, z \rangle \subset O_2(C_H(t))' \subset C_J(t)$ it follows that $F = O_2(C_H(t))'$ is elementary Abelian of order 32. Let $\langle c_0, c_2 \rangle$ be a Sylow 3-subgroup of $N(A) \cap N_G(\langle t, z \rangle)$ with $\langle c_0 \rangle \subseteq C_G(a)$. Then $F = \langle t, z \rangle \times C_F(c_0)$ and if $\langle t_1 \rangle = C_F(\langle c_0, c_2 \rangle)$, t_1 has 28 conjugates in $N_G(\langle t, z \rangle)$. It follows from (3.11) and the structure of H that $F \subseteq O_2(C_G(t_1))$, whence $N_G(F) \supset N_G(\langle t, z \rangle)$. Thus z has 31 conjugates in $N_G(F)$ and $|N_G(F)| \mid 2^{15} \cdot 3^2 \cdot 7 \cdot 31$. From $C_G(F) = F$ and the fact that $GL(5, 2)$ is simple we get $N_G(F)/F \cong GL(5, 2)$.

4. The 3-Structure of G

Throughout this section S will denote a Sylow 2-subgroup of $N_H(C)$ with $u \in S$. From (3.11) it follows that $S \cap O_{3,3'}(N_H(C))$ is a quaternion group and so $S \cong SD_{16}$ (as u is an involution). In particular $u \sim_S uz$ and $N_H(C)/C \cong GL(2, 3)$.

(4.1) If $K = O_3(C_G(C))$ then $K = C \times D$, where $D = [z, K]$ is elementary Abelian of order 9. In addition $N_G(C) = K \cdot N_H(C)$, $N_G(C)/K \cong N_G(D)/C_G(D) \cong GL(2, 3)$.

Proof. As C char H_3 and $3^7 \mid |G|$ (by (3.12)), $N_G(C) \not\subseteq H$. Clearly $\langle z \rangle$ is a Sylow 2-subgroup of $C_G(C)$ so H covers $N_G(C)/C_G(C)$ and $C_G(C)$ has a normal 2-complement. From the structure of $G_2(3)$ (see [7]), $|C_G(C) \cap C_G(c_2)| \mid 3^5 \cdot 2$. This forces $|C_G(C)| = 3^5 \cdot 2$ and $C_G(C) = K \cdot \langle z \rangle$, where $K = O_3(C_G(C))$. The result follows as K is Abelian (because C is a minimal normal subgroup of $N_G(C)$).

(4.2) We have $R = O_3(C_G(c_1))$ is elementary of order 3^5 and $C_G(c_1)/R \cong SL(2, 9)$.

Proof. From (3.11) we see that $C_H(c_1)/\langle c_1\rangle \cong SL(2, 9)$, so a Sylow 2-subgroup of $C_G(c_1)$ is isomorphic to Q_{16}. A result of Brauer and Suzuki [1] yields that $C_G(c_1) = O(C_G(c_1)) \cdot C_H(c_1)$. By (4.1) we have $O(C_G(c_1)) \supset \langle c_1\rangle$. Now $S \subseteq N_H(\langle c_1\rangle)$ and $u \sim_S uz$, so Proposition 2 and the action of $C_H(c_1)$ on $O(C_G(c_1))$ yield $O(C_G(c_1)) = O_3(C_G(c_1)) = R$ with $| R | = 3^5$. Let $C_1 = R \cdot C$, a Sylow 3-subgroup of $C_G(c_1)$. If R is non-Abelian then $\langle c_1\rangle = Z(R) = Z(C_1)$, whence C_1 is a Sylow 3-subgroup of G. This contradicts (3.12) and (3.6). Thus R is Abelian and so elementary Abelian ($R = \langle c_1\rangle \times [z, R]$).

(4.3) Each $d \in D^\#$ is conjugate to c_3 in G. Both $O(N_G(\langle d\rangle)) = O_3(N_G(\langle d\rangle))$ and $M = C_G(D) = O_3(N_G(D))$ have order 3^9, and $N_G(\langle d\rangle)/O(N_G(\langle d\rangle)) \cong GL(2, 3)$. Also, a Sylow 3-subgroup $G_3 = M\langle c_4\rangle$ of $C_G(d)$ is a Sylow 3-subgroup of G, with $Z(G_3) = \langle d\rangle$ and $N_G(G_3) = G_3\langle u, z\rangle$.

Proof. As above, $C_1 = C \cdot R$ is not a Sylow 3-subgroup of G. Now $N(C_1) \cap N_G(\langle c_1\rangle) = C_1 \cdot S$, $Z(C_1) = \langle c_1\rangle \times D$, and c_1, d, $c_1 d$ have 2, 8, 16 conjugates in C_1S (where $d \in D^\#$). Thus $| N_G(C_1): C_1S | = 9$ and $N_G(C_1) = O_3(N_G(C_1)) \cdot S$. Let $M = O_3(N_G(C_1))$ so that $D = Z(M)$. Also, let $D = \langle d\rangle \times \langle d_1\rangle$, where $\langle d\rangle = C_D(u)$ and $\langle d_1\rangle = C_D(uz)$. Clearly $d \sim c_3$, as $2 \cdot 3^9 \mid | C_G(d)|$ (see (4.2) and (3.12)).

Since $\langle r_1, s_1\rangle \cong Q_8$ is a Sylow 2-subgroup of $C_G(c_3)$ the Brauer–Suzuki result [1] yields $C_G(c_3) = O(C_G(c_3)) \cdot C_H(c_3)$. Thus $N_G(\langle c_3\rangle)/O_3(N_G(\langle c_3\rangle)) \cong GL(2, 3)$. Proposition 2 applied to $\langle u, z\rangle$ acting on $C_G(d)$ and the structure of H yield that $O(C_G(d)) = O_3(C_G(d))$ has order 3^9. It follows that $C_G(D) = M$. Without loss choose $c_4 \in C_G(d)$ so that $G_3 = M\langle c_4\rangle$ is a Sylow 3-subgroup of $N_G(D) \cap C_G(d)$ and $\langle d\rangle = Z(G_3)$(see (4.1)). It follows immediately that G_3 is a Sylow 3-subgroup of G.

In order to determine the conjugacy classes of 3-elements of G, we study $N_G(D)$ and $N_G(\langle d\rangle)$ in some detail. Note that $N_G(D) = MS\langle c_4\rangle$, where $\langle C, c_4\rangle = H_3$ and $S\langle c_4\rangle \cong GL(2, 3)$. Let $C_i = C_M(c_i)$, $i = 2, 3$ and recall $C_1 = C \cdot R (\subseteq M)$ is a Sylow 3-subgroup of $C_G(c_1)$ (so $C_1 = C_M(c_1)$ also). We begin by studying $N_G(D)$ and list some properties of this subgroup:

(i) $K \lhd N_G(D)$ and if $x \in C_1 - K$ then $x^m \in R - K$ for some $m \in M$. Further, $R^\#$ only contains elements conjugate to c_1, c_3 in G.

(Note that for $c \in C - \langle c_1\rangle$, $C_R(c) = \langle c_1\rangle \times D = Z(C_1)$; this follows from the action of $C_H(c_1)$ on R. In $N_G(\langle c_1\rangle)$, c_1, d, $c_1 d$ have 2, 80, 160 conjugates, respectively, whence $N_G(R) = N_G(\langle c_1\rangle)$ by (4.3). As $c_1 d \sim_G c_1$, and $d \sim_G c_3$, R contains 162 conjugates of c_1 and 80 conjugates of c_3. Obviously $R \not\sim_G K$, and from above, $R \cap R^m = \langle c_1\rangle \times D$ (for $m \in M - C_1$) whence C_1 is the disjoint union of K and (nine) M-conjugates of $R - (K \cap R)$. Thus $K \lhd M$, whence $K \lhd N_G(D)$ as $N_G(C)$ covers $N_G(D)/C_G(D)$; see (4.1).)

(ii) M/K is elementary Abelian (of order 81), $| C_i | = 3^7$, and $Z(C_i) = \langle c_i, D\rangle$, $i = 1, 2, 3$.

(As z acts fixed-point-free on M/K, M/K is Abelian; as $M/K = C_M(u) \cdot K/K \times C_M(uz)\, K/K$, M/K is elementary Abelian (note that $C_M(u) \sim_G H_3$ and $C_K(u) = \langle c_2, d\rangle = C_M(u)'$). From (4.3) we have $|M: C_i| \geqslant 9$ for each i. Since $Z_2(M) \cap K \lhd N_G(D)$, $K \subseteq Z_2(M)$, whence $|M : C_i| = 9$ as required. In particular c_1, c_2, c_3 have 54, 108, 72 conjugates (respectively) in $N_G(D)$, and C_2 is a Sylow 3-subgroup of $C_G(c_2)$. Finally, $Z(C_1) = \langle c_1, D\rangle$ by (4.3), $Z(C_2) = \langle c_2, D\rangle$ follows from the structure of $G_2(3)$, and $Z(C_3) = \langle c_3, D\rangle$ as c_4 normalizes $C \cap Z(C_3)$.)

(iii) $|C_3 \cap C_2 : K| = 3$ and $C_{M/K}(c_4) = C_3/K$.

(First note that for $m \in M$, $C_K(m) \supset D$. If $m \in M$, $[m, c_4] \in K$ then c_4 normalizes $C_C(m)$ whence $c_3 \in C_C(m)$. Thus $C_3/K = C_{M/K}(c_4)$ as both have order 9. As we chose $c_4 \in C_G(d)$, $[\langle c_4\rangle, D] = \langle d\rangle$, whence $[\langle c_4\rangle, K] = \langle d, c_2, c_3\rangle$. By the result just proved $[\langle c_4\rangle, K] \lhd C_3$, so $C_2 \cap C_3 \supset K$. The result follows immediately from (ii).)

We now introduce some more notation. Let $C_3 = \langle c_{31}, c_{32}, K\rangle$, $C_1 = \langle c_{11}, c_{12}, K\rangle$ (recall $C_3 \cap C_1 = K$) and (as above) $[\langle c_4\rangle, D] = \langle d\rangle$ so that $[\langle c_2\rangle, C_3] = \langle d\rangle$ also. As $\langle u, z\rangle$ normalizes each C_i choose $C_M(u) = \langle d, c_2, c_{31}, c_{11}\rangle$, $C_M(uz) = \langle d_1, c_2, c_{32}, c_{12}\rangle$, and we may suppose u inverts c_{32}, c_{12} and uz inverts c_{31}, c_{11}. Since $[C_3, \langle c_2\rangle] = \langle d\rangle = C_D(u)$, $[c_{32}, c_2] = [c_{32}, c_2]^u = [c_{32}^{-1}, c_2] = [c_{32}, c_2]^{-1}$, whence $[c_{32}, c_2] = 1$; i.e., $C_3 \cap C_2 = \langle c_{32}, K\rangle$. As $u \sim_{N(D)} uz$, we may choose $c_{32} \sim_{N(D)} c_{11}c_{31}^{-1}$. It is now straightforward to determine the conjugacy classes of M/K in $N_G(D)/K$.

(iv) The following are representatives for the conjugacy classes of M/K in $N_G(D)/K$: $c_{32}K$ (8 conjugates); $c_{31}K$ (8 conjugates); $c_{31}c_{32}K$ (16 conjugates); $c_{11}K$ (24 conjugates); $c_{11}c_{31}K$ (24 conjugates); K.

We now consider the structure of $N_G(\langle d\rangle)$ and (unfortunately) introduce still more notation. Let $N = N_G(\langle d\rangle)$, $O = O_3(N)$, $G_3 = M\langle c_4\rangle$ so that G_3 is a Sylow 3-subgroup of N and let $U \supseteq \langle u, z\rangle$ be a Sylow 2-subgroup of N. Note that $U \cong SD_{16}$ and $Z(U) = \langle u\rangle$.

From the structure of $C_H(c_3)$ and $c_3 \sim_G d$ it follows that $\Omega_1(C_O(u)) = \langle c_2, d\rangle$ and $C_O(u)$ is non-Abelian of order 27. Clearly U normalizes $\langle c_2, d\rangle$ and as $C_N(c_2)$ covers G_3/O, $|C_O(c_2)| = 3^6$. Now $C_M(uz) \subseteq O$, whence $C_O(c_2) = \langle c_{32}, K\rangle = C_3 \cap C_2$.

Since $\langle c_{32}, K\rangle \lhd G_3$ we have $\langle c_{32}, K\rangle \lhd N$. It follows that $D_{23} = \langle c_2, c_3, D\rangle = Z(\langle c_3, K\rangle) \lhd N$, hence $\langle c_3, D\rangle \lhd N$ as well. Finally, $c_3 \sim_N d_1$ yields $|C_O(c_3)| = 3^8$, $C_O(c_3) = \langle C_3, c_4\rangle$, and $C_3 = C_O(\langle c_3, D\rangle) \lhd N$ also.

Clearly $N_G(K) = N_G(D)$ so all (8) nontrivial cosets of D_{23} in $\langle c_{32}, K\rangle$ are conjugate in N; i.e., the coset $c_{32}K$ only contains elements of order three conjugate to c_1, c_2, c_3 in G.

It also follows easily that $[C_3, O] \subseteq D_{23}$ and $(O/D_{23})' = \langle c_{31}\rangle D_{23}$. Hence $c_{31} D_{23}$, $c_1 D_{23}$, $c_{31}c_1 D_{23}$ have 2, 8, 16 conjugates, respectively, in N. (Note that

$c_{31}c_1 D_{23} \sim_N c_{31}c_{32} D_{23} \sim_N c_{31}c_{32}c_1 D_{23}$.) We now show that all elements in $c_{31} D_{23}$ are conjugate in G_3. From the structure of $C_M(u) = \langle c_{31}, c_{11}, c_2, d\rangle$ we have $[C_M(u), \langle c_{31}\rangle] = \langle c_2, d\rangle$. Further, as $[c_{31}, c_1] \in D$, $[c_{31}, c_1]^u = [c_{31}, c_1]^{-1}$ implies $[c_{31}, c_1] = d_1$ say. Finally, as $C_3' \lhd N$, $C_3' \supset D$, which implies $C_3' = \langle c_3, D\rangle$ or $[c_{31}, c_{32}] \in \langle c_3, D\rangle - D$. A similar argument shows that all elements in $c_{31}c_1 D_{23}$ are conjugate in G_3.

From the above remarks it follows that each element in the cosets $c_{31}K$, $c_{31}c_{32}K$ is conjugate in G either to c_{31} or $c_{31}c_1$. If we choose f_1 so that $[f_1, C_J(c_3)] = 1$, then clearly $c_{31} \sim_G f_1$. Further $|C_N(c_{31})| = 3^6 \cdot 2^3$, $(c_{31} c_1)^3 = d^{\pm 1}$, $|C_N(c_{31}c_1)| = 3^6$, and $c_{31} \not\sim_G c_{31}c_1$.

By assumption $c_{31}c_{11}^{-1} \sim_{N(D)} c_{32}$ and $\langle c_{32}, c_2, d\rangle \sim_G C$ so $c_{31}c_{11} \sim_G f_2$ (note that $c_{31}c_{11} \in G_3 - O$). We show now that $[\langle c_{31}c_{11}\rangle, M] = K$. As above, $[C_M(u), \langle c_{31}c_{11}\rangle] = \langle c_2, d\rangle$ and $[c_{11}c_{31}, c_1] \in D - \langle d\rangle$. Since R has 9 conjugates in M, c_{12} has 81 conjugates in M. Now $[\langle c_{12}\rangle, K] = D$, $[c_{11}, c_{12}] = 1$, $[c_{12}, c_{32}] \in \langle c_2, d_1\rangle$, and $[\langle c_{12}\rangle, M] \subseteq D_{23}$. Thus $[c_{31}, c_{12}] \in D_{23} - \langle c_2, D\rangle$, whence $[c_{31}c_{11}, c_{12}] \in D_{23} - \langle c_2, D\rangle$. Finally $[c_{31}c_{11}, c_{32}] \in c_1^{\pm 1} D_{23}$ which proves $[\langle c_{31}c_{11}\rangle, M] = K$. Thus all elements of $c_{31}c_{11}K$ are conjugate to $c_{31}c_{11}$ in M, and $|C_N(c_{31}c_{11})| = 2 \cdot 3^4$.

From (i) above, $c_{11}K$ only contains elements of order three conjugate to one of c_1, c_2, c_3 in G. Hence we have determined that if $m \in M^{\#}$ then m is conjugate in G to one of $c_1, c_2, c_3, f_1, f_2, c_{31}c_1$.

It remains to consider $G_3 - M$. In $N_G(G_3) = G_3\langle u, z\rangle$, $c_4(M \cap O)$, $c_{11}(M \cap O)$, $c_{11}c_4(M \cap O)$ have 2, 2, 4 conjugates, respectively. As all nontrivial cosets of C_3 in M are conjugate to $c_{12}C_3$ it is only necessary to consider $c_{11}c_4(M \cap O)$. From the structure of $G_3\langle u, z\rangle / D_{23}$ it follows that $c_4c_{11}(M \cap O)$ consists of three conjugacy classes of cosets modulo D_{23} with representatives $c_4c_{11} D_{23}$, $c_4c_{11}c_{32} D_{23}$, $c_4c_{11}c_{32}^{-1} D_{23} \sim_N (c_4c_{11}c_{32})^{-1} D_{23}$. Further, as $[\langle c_4\rangle, M] \nsubseteq C_{M/K}(c_4)$, $(c_4c_{11}c_{32}^{\epsilon}) D_{23}$ has order 9, $\epsilon = 0, 1, -1$. It is easily seen that $C_N(c_4c_{11}c_{32}^{\epsilon}) \subseteq \langle c_{31}, K\rangle$ and that $C_K(c_4c_{11}c_{32}^{\epsilon}) = \langle d\rangle$. An easy computation yields that for each $\epsilon \in \{0, \pm 1\}$, $[c_4c_{11}c_{32}^{\epsilon}, c_{31}] \in \langle c_{21}, D\rangle - D$ whence $(c_4c_{11}c_{31}^{\epsilon})^3 \in c_{31}^{\alpha}c_1^{\beta} D_{23}$, $\alpha, \beta \in \{\pm 1\}$; i.e., $(c_4c_{11}c_{32}^{\epsilon})^3 \sim_G c_{31}c_1$. Thus $C_G(c_4c_{11}c_{32}^{\epsilon}) = \langle c_4c_{11}c_{32}^{\epsilon}\rangle$ of order 27, $\epsilon \in \{0, \pm 1\}$.

(4.4) Let $f_i \in C_G(c_3)$, $i = 3, 4, 5$ with $f_3 \sim_G c_{31}c_1$, $f_4 \sim_G c_4c_{11}$, $f_5 \sim_G c_4c_{11}c_{31}$, and $f_3{}^3 = c_3$, $f_4{}^3 = f_5{}^3 = f_3$. Then the group G contains precisely three classes of elements of order 3 with representatives c_1, c_2, c_3; three classes of elements of order 9 with representatives f_1, f_2, f_3; and three classes of elements of order 27 with representatives f_4, f_5, f_5^{-1}. Further, $|C_G(f_i)| = 2^3 \cdot 3^6$, $2 \cdot 3^4$, 3^6, 3^3, 3^3 for $i = 1, 2, 3, 4, 5$, respectively.

Remark. Recall that $C_2 = \langle c_{32}, c_{31}c_{11}^{-1}, K\rangle$ is a Sylow 3-subgroup of $C_G(c_2)$ and $[C_2, K] \subseteq D$. A simple computation yields $[c_{32}, c_{31}c_{11}^{-1}] \in \langle c_1, c_3, D\rangle$; i.e., $c_2 \notin C_2'$. Hence Gaschutz' theorem [6, S.I.17.4] yields $C_G(c_2) \simeq Z_3 \times G_2(3)$, a result not needed in this paper.

5. The Conjugacy Classes of π-Elements of G

Let $\pi = \{2, 3, 5, 7, 13\}$ and let π' denote the complementary set of primes.

Set $C_H(b) = B \times \langle v_1, v_2\rangle\langle c_1\rangle$, $N_H(B) = \langle w\rangle C_H(b)$ with $\langle v_1, v_2\rangle \cong Q_8$ and $w^2 \in C(c_1) \cap N_H(B)$. Thus w^2 is of order four and without loss we choose $w^2 = v$ and $v_1{}^w = v_1^{-1}$, $v_2{}^w = v_1 v_2$ (note that $N_H(B)/\langle B, v\rangle \cong \Sigma_4$). Hence as $v_1 \sim_H v_1^{-1}$ and $v_1 J \sim_H vJ$ we see that vJ contains precisely one class of elements of order four with square z(in H).

(5.1) The group G contains two classes of elements of order four with representatives r_1, $v(|C_G(v)| = 2^9 \cdot 3 \cdot 5)$ and two classes of elements of order eight with representatives us_1, w where $(us_1)^2 = r_1$, $w^2 = v$, and $|C_G(ur_1)| = 2^7 \cdot 3$, $|C_G(w)| = 2^5 \cdot 3$.

Proof. The statement about the elements of order four follows from Section 3 and the remarks above. As $C_H(r_1)/C_J(r_1) \cong \mathrm{Aut}(PSL(2, 8))$, if $h^2 = r_1$ and $h \in H$ then h is conjugate to an element in $us_1 C_J(r_1)$. It is straightforward to verify (see (3.5)) that $us_1 C_J(r_1)$ contains only one class of elements of order 8 with square r_1 in $\langle us_1, C_J(r_1)\rangle$ and that $|C_H(us_1)| = 2^7 \cdot 3$.

It remains to consider $wC_J(v)$ (using the notation above). As $w \in N_H(B) - C_H(B)$, $C_J(w)$ is elementary of order 8 and $wC_J(v)$ contains two classes of elements with square v in $\langle w, C_J(v)\rangle$. These classes have representatives w, wz. However, as $w^{v_1} = wz$ and $v_1 \in C_H(v)$ $(v_1 \in C_H(b))$ we have that there is one class of elements of order 8 with square v and also $|C_H(w)| = 2^5 \cdot 3(|C_{\bar{H}}(\bar{w})| = 2^3 \cdot 3)$.

(5.2) We have $N_G(B) = O_5(C_G(b)) \cdot N_H(B)$ and $O_5(C_G(b))$ is either non-Abelian of order 5^3 and of exponent 5 or equal to B. In any case $O_5(C_G(b))$ is a Sylow 5-subgroup of G and G contains one class of elements of order 5.

Proof. If $C_G(b) \subseteq H$ there is nothing to prove, so suppose $C_G(b) \not\subseteq H$. The Brauer–Suzuki theorem [1] yields $C_G(b) = O(C_G(b)) \cdot C_H(b)$ and as $C_G(b) \cap C_G(c_1) \subseteq H$, $O(C_G(b))$ is a $\{2, 3\}'$-group. Hence Proposition 2 applied to the 4-group $\langle vv_1, z\rangle$ acting on $O(C_G(b))$ yields that $O(C_G(b))$ is of order 5^3 (note that $vv_1 \sim_{N(B)} vv_1 z$).

Set $C(vv_1) \cap O(C_G(b)) = \langle b_1\rangle(\sim_G\langle b\rangle)$. If $O(C_G(b))$ is non-Abelian then b_1 has 120 conjugates in $N_G(B)$ and we are done. It remains to show $O(C_G(b))$ is not Abelian. If $O(C_G(b))$ is Abelian, b, b_1, bb_1 have 4, 24, 96 conjugates, respectively, in $N_G(B)$. It follows that $N_G(B) = N(O(C_G(b)))$, which implies $O(C_G(b))$ is a Sylow 5-subgroup of G. This contradicts Burnside's lemma [4, Theorem 7.1.1], however, as $b \sim_G b_1$. Thus $O(C_G(b))$ is non-Abelian as required.

(5.3) A Sylow 7-subgroup of G has order 7^2 and G has either one class of elements of order 7 with representative a or two classes of elements of order 7 with representatives a, a_2, where $|C_G(a_2)| = 7^2$.

Proof. Recall from (3.13) that $C_G(a) = \langle a\rangle \times L$, where $L \cong PSL(2, 7)$. Let

$\langle a_1 \rangle$ be a Sylow 7-subgroup of L with $u \in N(\langle a_1 \rangle) \cap N_G(A)$. Then $3 \mid | C_G(a_1)|$ so $a \sim_G a_1$ and u inverts a_1. Let $A_1 = \langle a, a_1 \rangle$, a Sylow 7-subgroup of $N_G(A)$. We see that $N(A_1) \cap N_G(A)/A_1 \cong Z_2 \times Z_3 \times Z_3$ and a, a_1, aa_1, aa_1^{-1} have 6, 6, 18, 18 conjugates, respectively, in $N_G(A) \cap N_G(A_1)$. As $a \sim_G a_1$, $7 \nmid | N_G(A_1): A_1 |$, whence A_1 is a Sylow 7-subgroup of G. By Burnside's lemma [4, Theorem 7.1.1] $a \sim a_1$ in $N_G(A_1)$, which leads to two cases:

(α) $a \sim_G aa_1$ and all elements of $A_1^{\#}$ are conjugate in $N_G(A_1)$; i.e., $| N_G(A_1): A_1 | = 2^4 \cdot 3^2$ and G has one class of elements of order 7.

(β) $a \not\sim_G aa_1$, so a, aa_1 have 12, 36 conjugates, respectively, in $N_G(A_1)$; i.e., $| N_G(A_1)| = 2^2 \cdot 3^2 \cdot 7^2$ and G has two classes of elements of order 7 with representatives $a, a_2 = aa_1$.

It remains to show that in case (β), $C_G(a_2) = A_1$. By Burnside's transfer theorem [4, Theorem 7.4.3] $C_G(a_2)/\langle a_2 \rangle$ has a normal 7-complement $X/\langle a_2 \rangle$. However, $\langle u, a \rangle (\cong D_{14})$ acts on $X/\langle a_2 \rangle$ and both u, a must act fixed-point-free. Thus $X = \langle a_2 \rangle$ as required.

(5.4) If $\langle l \rangle$ is a Sylow 13-subgroup of $C_G(c_2)$ then $\langle l \rangle$ is a Sylow 13-subgroup of G. Further, $N_G(\langle l \rangle) \subseteq N_G(\langle c_2 \rangle)$ so that $N_G(\langle l \rangle)/\langle c_2 \rangle$ is a Frobenius group of order $13 \cdot 12$ (and clearly $N_G(\langle l \rangle)$ covers $N_G(\langle c_2 \rangle)/C_G(c_2)$).

Proof. It follows from the structure of $G_2(3)$ and the Frattini argument that $N\langle l \rangle \cap N_G(\langle c_2 \rangle)/\langle c_2 \rangle$ is a Frobenius group of order $13 \cdot 12$. The structure of H now yields that a Sylow 3-subgroup Y of $N(\langle l \rangle) \cap N_G(\langle c_2 \rangle))$ is elementary (of order 9). Burnside's transfer theorem [4, Theorem 7.4.3] yields that $C_G(l)$ has a normal 3-complement X. As X is Y-invariant, X is a π-group and so X is a 13-group, by our previous results.

Let $N_G(\langle l \rangle) = X \cdot Y \cdot V$, where $Y \cdot V$ is a Hall $\{2, 3\}$-subgroup of $N_G(\langle l \rangle)$ (of order $2^2 \cdot 3^2$). Without loss we may assume $Y \cdot V \subseteq C_H(c_2)$ and further, $Y \subseteq C$. Then $V = \langle v' \rangle$, $Y = \langle c_2, c_1' \rangle$, where $[v', c_1'] = 1$, $v' \sim_H v$, and $c_1' \sim_H c_1$. Hence $(c_2 c_1')^{v'} = c_2^{-1} c_1'$ and so $c_2 c_1' \sim_H c_3$. It follows immediately from (4.2) and (4.3) that $X = \langle l \rangle$. This completes the proof of (5.4).

We conclude this section by listing the classes of π-elements of G. First, set $| G | = g$ and let g_σ denote the σ-part of g for any set of primes σ. We have showed that there are two possibilities for g_π:

Case I. $g_\pi = 2^{15} \circ 3^{10} \circ 5^3 \circ 7^2 \circ 13.$

Case II. $g_\pi = 2^{15} \circ 3^{10} \circ 5 \circ 7^2 \circ 13.$

6. The Order and π'-Classes of G

From the class equation for G and the table of classes of π-elements of G we obtain a congruence for $g_{\pi'}$ in each of the four cases of Section 5:

$$g_{\pi'} \equiv 589 = 19 \circ 31 \ (g_\pi) \qquad \text{in Case I}(\alpha),$$

$$g_{\pi'} \equiv 6, 288, 482, 304, 589 \ (g_\pi) \qquad \text{in Case I}(\beta),$$

$$g_{\pi'} \equiv 1, 232, 542, 546, 309 \ (g_\pi) \qquad \text{in Case II}(\alpha),$$

$$g_{\pi'} \equiv 4, 376, 783, 698, 309 \ (g_\pi) \qquad \text{in Case II}(\beta).$$

Let

$$\sigma = \{ p \in \pi' \mid G \text{ contains a strongly real element of order } p\}.$$

TABLE I

Conjugacy Classes of π-Elements of G

x	$\lvert x \rvert$	$\lvert C_G(x) \rvert$	x	$\lvert x \rvert$	$\lvert C_G(x) \rvert$
z	2	$2^{15} \cdot 3^4 \cdot 5 \cdot 7$	zb	10	$2^3 \cdot 3 \cdot 5$
r_1	4	$2^{11} \cdot 3^3 \cdot 7$	vb	20	$2^2 \cdot 5$
v	4	$2^9 \cdot 3 \cdot 5$	zbc_1	30	$2 \cdot 3 \cdot 5$
us_1	8	$2^7 \cdot 3$	$(zbc_1)^{-1}$	30	$2 \cdot 3 \cdot 5$
w	8	$2^5 \cdot 3$	za	14	$2^3 \cdot 7$
zc_1	6	$2^4 \cdot 3^3 \cdot 5$	r_1a	28	$2^2 \cdot 7$
zc_2	6	$2^6 \cdot 3^3$	c_1	3	$2^4 \cdot 3^7 \cdot 5$
zc_3	6	$2^3 \cdot 3^4$	c_2	3	$2^6 \cdot 3^7 \cdot 7 \cdot 13$
r_1c_2	12	$2^5 \cdot 3^2$	c_3	3	$2^3 \cdot 3^{10}$
$(r_1c_2)^{-1}$	12	$2^5 \cdot 3^2$	f_1	9	$2^3 \cdot 3^6$
r_1c_3	12	$2^2 \cdot 3^3$	f_2	9	$2 \cdot 3^4$
vc_1	12	$2^3 \cdot 3$	f_3	9	3^6
zf_1	18	$2^3 \cdot 3^2$	f_4	27	3^3
zf_2	18	$2 \cdot 3^2$	f_5	27	3^3
us_1c_2	24	$2^3 \cdot 3$	f_5^{-1}	27	3^3
$(us_1c_2)^{-1}$	24	$2^3 \cdot 3$	c_1b	15	$2 \cdot 3 \cdot 5$
wc_1	24	$2^3 \cdot 3$	$(c_1b)^{-1}$	15	$2 \cdot 3 \cdot 5$
$(wc_1)^{-1}$	24	$2^3 \cdot 3$	c_2a_1	21	$3 \cdot 7$
r_1f_1	36	$2^2 \cdot 3^2$	c_2l	39	$3 \cdot 13$
s_1f_1	36	$2^2 \cdot 3^2$	$(c_2l)^{-1}$	39	$3 \cdot 13$
$(s_1f_1)^{-1}$	36	$2^2 \cdot 3^2$	l	13	$3 \cdot 13$
a	7	$2^3 \cdot 3 \cdot 7^2$			
b	5	$2^3 \cdot 3 \cdot 5^3$	(Case I)		
		$2^3 \cdot 3 \cdot 5$	(Case II)		
a_1	7	7^2	Case (β) only		

(6.1) There exist disjoint subsets $\sigma_1, \sigma_2, ..., \sigma_n$ of σ and subgroups P_i of G such that

(i) $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n$.

(ii) P_i is an Abelian Hall σ_i-subgroup of G and a T.I. set.

(iii) $P_i = C_G(x)$ for all $x \in P_i^{\#}$.

(iv) $N_G(P_i)$ is a Frobenius group with complement of even order.

Proof. See Lyons [8].

It follows from (6.1) that G contains precisely $\lambda_i = |P_i| - 1/|N_G(P_i): P_i|$ classes of σ_i-elements. By a result of Brauer and Fowler [2, (29.7)] $g/|H|^2 \leqslant 30 + \sum_{i=1}^n \lambda_i$, as G has at most 30 classes of real π-elements (see Section 5). Thus $g_{\pi'} \leqslant (|H|^2/g_\pi)(30 + \sum_{i=1}^n \lambda_i)$, from which it follows that

$$g_{\pi'} < 57\left(30 + \sum_{i=1}^{n} \lambda_i\right) \quad \text{in} \quad \text{Case I,}$$

$$g_{\pi'} < 1401\left(30 + \sum_{i=1}^{n} \lambda_i\right) \quad \text{in} \quad \text{Case II.} \tag{*}$$

Now $\lambda_i \leqslant |P_i|/2$, $(|P_i|, |P_k|) = 1$ for $i \neq k$, $|P_1| \cdots |P_n| \mid g_{\pi'}$, and each $|P_i|$ is a π'-number. These facts and Eq. (*) yield $n \leqslant 2$ in Case I and $n \leqslant 4$ in Case II.

Suppose $g_{\pi'} > g_\pi$, so that by (*) σ is nonempty. Clearly we may assume λ_1 is the largest of the λ_i. Then our assumption and (*) yield $\lambda_1 > |H| \geqslant |C_G(x)|$ for any nonidentity π-element x (in both cases).

From (6.1) it follows that there exists a set of λ_1 exceptional characters χ_i which coincide on conjugacy classes not meeting $P_1^{\#}$ (see [2]). In particular the χ_i are rational valued on π-element of G. Hence the orthogonality relations and the fact that $\lambda_1 \geqslant |C_G(x)|$ for any nontrivial π-element x of G yield $\chi_i(x) = 0$. Thus $g_\pi \mid \chi_i(1)$ and so $g \geqslant g_\pi^2 \cdot \lambda_1$, or $\lambda_1 g_\pi < g_{\pi'}$. Combining this with (*) yields $\lambda_1(g_\pi - 2.57) < 57.30$ in Case I, and $\lambda_1(g_\pi - 4.1401) < 1401.30$ in Case II. Both inequalities are clearly impossible, so we conclude $g_{\pi'} < g_\pi$, whence Case I(α) holds (for example by (3.14)).

(6.2) The order of G is $2^{15} \circ 3^{10} \circ 5^3 \circ 7^2 \circ 13 \circ 19 \circ 31$.

Finally the π'-classes of G are determined immediately by Sylow's theorem.

(6.3) The Sylow 19-normalizer is a Frobenius group of order 18.19 and the Sylow 31-normalizer is a Frobenius group of order 15.31. The group G contains one class of elements of order 19 and two (nonreal) classes of elements of order 31.

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