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The relative error in the Pruess method for Sturm–Liouville problems

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Abstract

We consider the Pruess method to solve the Sturm–Liouville eigenvalue problem. Superconvergence of the method for the relative error of an eigenvalue is examined with respect to its index.

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Keywords: Eigenvalue problem; Sturm–Liouville operator; Pruess method; Absolute and relative errors

1. Introduction

The Pruess method has been widely used in numerical computation with Sturm–Liouville problems because of its good convergence behavior. In many practical cases, where there is a need to determine the first n ($n \gg 1$) eigenvalues (e.g., various earth models in [1,32]), it seems to be better than standard techniques, such as finite difference and variational methods. It is known from the theorems of Keller (see [16]) and Fix–Strang–Vainikko (see [28,30,31]) that the eigenvalue approximation deteriorates as the index k increases.

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On the other hand, we know (see [15,17]) that upper bounds for the absolute and relative sensitivity of eigenvalues for the Sturm–Liouville operator in normal form (LNF) are independent of k; thus the problem is perfectly posed for each eigenvalue regardless of its index. Following [11,24] it can be proved that in the case of LNF the Pruess method is consistent with the (absolute, relative) spectral continuity of these kinds of operators. When the Sturm–Liouville equation is in its full form the situation is different. Pruess proved that under mild conditions on the operator coefficients the relative error $\left|(\lambda_k^h - \lambda_k)/\lambda_k\right| = \mathrm{O}(h^{m+1})$, where h denotes the mesh size and m is the degree of polynomials interpolating coefficient functions. The relative error is independent of k.

In this paper we prove that when the approximation of coefficients satisfies a superconvergence condition from [24] (cf. condition (11) in Theorem 6) then the relative accuracy can be doubled with the dependence of k being of order $m+1$. Numerical results are also provided which show the behavior of error estimates.

2. Notation and the case of the Liouville normal form

Suppose that

$$\mathscr{D} = \{u \in H^2(a,b) : \alpha u(a) + \alpha' u'(a) = 0,\ \beta u(b) + \beta' u'(b) = 0\}.$$

The constants $\alpha, \alpha', \beta, \beta'$ are assumed to be real with $\alpha^2 + \alpha'^2 \neq 0$, $\beta^2 + \beta'^2 \neq 0$, and the interval $[a,b]$ is finite.

For $u \in \mathscr{D}$, let

$$\begin{aligned} Lu &= -u'' + qu, \\ \tilde{L}u &= -u'' + \tilde{q}u, \end{aligned}$$

where the real-valued functions $q, \tilde{q}$ are assumed to belong to the piecewise continuous functions $\mathrm{PC}([a,b])$. Now consider the eigenvalue problems for operators L and $\tilde{L}$:

$$Lu = \lambda u, \quad u \in \mathscr{D}, \tag{1}$$

$$\tilde{L}u = \tilde{\lambda} u, \quad u \in \mathscr{D}. \tag{2}$$

It is known that the operators L and $\tilde{L}$ are self-adjoint (see [27]), and eigensystems (1) and (2) with the separated boundary conditions given, are regular and have sequences of simple real distinct eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, $\{\tilde{\lambda}_k\}_{k=1}^{\infty}$ such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \quad \text{and} \quad \tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3 < \cdots,$$

and corresponding sequences of orthonormal eigenfunctions $\{u_k\}_{k=1}^{\infty}$, $\{\tilde{u}_k\}_{k=1}^{\infty}$ such that $\|u_k\|_2 = \|\tilde{u}_k\|_2 = 1$ (see [7,18]). To simplify our discussion it helps to assume that L is positive definite which, in view of the Courant–Fischer–Poincaré min–max theorem, is equivalent to having the lowest eigenvalue $\lambda_1 > 0$. This can be achieved by replacing $q(x)$ by $q(x) + \lambda^*$, where λ^* is such that $\lambda^* + \lambda_1 > 0$; this just shifts all the eigenvalues up by λ^* without essentially changing the problem. Let F denote the bounded symmetric operator $Fu = (\tilde{q} - q)u$ defined for $u \in L^2(a,b)$. With this notation we have the following theorems (see [15,17,25]).

Theorem 1. *If* $\|\tilde{q} - q\|_\infty \equiv \sup_{x\in[a,b]}|\tilde{q}(x) - q(x)| \leqslant \varepsilon$, *then* $\left|\tilde{\lambda}_k - \lambda_k\right| \leqslant \varepsilon$ *for each* $k = 1, 2, 3, \ldots$

Theorem 2. *Under the above assumption the following inequality holds for each* $k = 1, 2, 3, \ldots,$

$$\left|\frac{\tilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \leqslant \rho\big(L^{-1}F\big), \tag{3}$$

where $\rho(.)$ *denotes the spectral radius.*

Remark 1. The important note here is that the estimation in Theorem 1 is sharp and the bound with the L^2-norm is not valid (see [17]).

Remark 2. From the asymptotics of λ_k it is evident that the relative error is a decreasing function of k. So one could expect it would be possible to strengthen (3), according to the principle of related operators for Riesz operators (see [23]), to $\left|(\tilde{\lambda}_k - \lambda_k)/\lambda_k\right| \leqslant |\lambda_k(L^{-1}F)|, \quad k = 1, 2, \ldots$ However this may not be true in general (see [23] for more details). Similar problems in the finite dimensional space have been recently investigated in [10,13,26].

When using the most prevalent of existing methods for approximating the solution of (1), i.e. conforming finite element [6,8,14,21,22,33] or finite differences [12,29], we reduce the problem (1) to a finite-dimensional algebraic eigenvalue problem; for a numerical treatment of a generalized symmetric matrix eigenproblem we refer the reader to [20]. Unfortunately resulting errors (absolute, relative) are not uniformly bounded with respect to k. To be more precise we can state the following two theorems.

Theorem 3 [16]. *There exists a positive constant C independent of k and h such that for the centered difference scheme* (*CDS*)

$$\left|\lambda_k^h - \lambda_k\right| \leqslant C\lambda_k^2 h^2, \quad k = 1, \ldots, N,$$

where N is the dimension of the algebraic eigenvalue problem.

Theorem 4 [28,30,31]. *If S^h is a finite element space of degree $d-1$, then there is a constant C such that the approximate eigenvalues are bounded for small h by*

$$\lambda_k \leqslant \lambda_k^h \leqslant \lambda_k + C\lambda_k^d h^{2(d-1)}.$$

The constant C does not depend on k and the above estimates are the best possible.

As a direct consequence we see for the CDS and FEM with hat functions that these two methods are of the order of $\mathrm{O}\left(\lambda_k^2 h^2\right)$ and calculating λ_{10} gives us 10^4 worse approximation than λ_1. Yet the upper bounds for operators in LNF are independent of k . This anomaly, called *the phenomenon of finite element eigenvalues* in the book of Fix and Strang [28], is a consequence of the fact that the eigenfunctions become more oscillatory as the eigenvalue λ_k increases. Since each finite dimensional algebraic method relies on a fixed-order piecewise polynomial (or just Taylor's series) approximation of the eigenfunction, it is to be expected that approximation of the Rayleigh quotient, and hence the accuracy of eigenvalues will deteriorate as k increases.

Remark 3. There is a technique of asymptotic correction AAdHP introduced by Paine in 1979 in which λ_k^h is replaced by $\tilde{\lambda}_k^h \doteq \lambda_k^h + \varepsilon_k^h$, where ε_k^h is the (known) exact error for the case $q \equiv 0$, $k = 1, \ldots, N$. One proves that when $q \in C^2$ there exists an α, independent of h, such that $\left|\lambda_k^h - \lambda_k\right| = \mathrm{O}(kh^2)$, $1 \leqslant k \leqslant \alpha N$, $\alpha < 1$. So this kind of correction is not uniformly valid for all eigenvalues but greatly improves the accuracy of eigenvalues computed by finite difference schemes or finite element methods at negligible extra cost. For a fuller treatment we refer the reader to [2–5,19].

3. The Pruess method

This section is concerned with approximating the eigenvalues of the regular Sturm–Liouville problem in the full form

$$-(pu')' + qu = \lambda ru, \quad p > 0, \ \ r > 0, \ \ x \in [a, b], \tag{4}$$

with separable linear homogeneous boundary conditions given in $\mathscr{D}$. About the real-valued coefficient functions we assume that $p \in C^1([a, b])$, $q, \ r \in C([a, b])$. An alternative approach is based on approximating the differential equation itself and on finding approximate eigenfunctions $u_k^h = u_k^h(x)$ in an infinite dimensional subspace of H^1. Specifically we replace p, q and r in (4) by the approximations p_π, q_π and r_π respectively and solve the approximating problem exactly. Let us neglect, for the time being, the difficulties associated with solving the perturbed equation and look at some feature of such an approach.

Potentially, for a given regular approximation, we can obtain an infinite sequence of eigenvalues. It is not clear, however, what effect these perturbations of the coefficients will have on the eigenvalues (for the LNF form we can use Theorems 1 and 2), nor is it obvious that it will be easier to solve than the original problem. Pruess has shown (see [24]) that for piecewise mth order polynomial interpolation of the coefficient on a grid with maximum stepsize h,

$$\left|\lambda_k^h - \lambda_k\right| \leqslant C|\lambda_k| h^{m+1}, \quad h < h_0. \tag{5}$$

An enhanced convergence result, given in the same paper, states that if p_π, q_π, r_π interpolate to p, q and r at the Gaussian points of each mesh interval then (5) improves to

$$\left|\lambda_k^h - \lambda_k\right| \leqslant \mathrm{O}\left(h^{2m+2}\right), \quad \text{as } h \to 0.$$

Unfortunately this estimation does not provide information about the deterioration of the absolute error when k increases. Paine and de Hoog [11] stated without proof that

$$\left|\lambda_k^h - \lambda_k\right| \leqslant \left|\lambda_k^\alpha\right| h^{2m+2}, \quad h < h_0(k), \tag{6}$$

where

$$\alpha = \max\left\{1 + \frac{1}{2}m,\ 3\right\}.$$

For the problems in Liouville normal form it is known that for k, which is not a multiple of the mesh size N the midpoint Pruess method is of order $\left|\lambda_k^h - \lambda_k\right| = \mathrm{O}(h^2)$; on the other hand for $k \simeq lN$ $(l = 1, \ldots, \lceil N/2\pi - 1\rceil)$ then there are the $\mathrm{O}(h)$ peaks (for more details see Theorem 3.1 in [11]). For the case of the problem in full form the situation presents a more delicate problem and for the piecewise constant midpoint (PWCM) approximation bound (6) gives

$$\left|\lambda_k^h - \lambda_k\right| \leqslant C\lambda_k^3 h^2.$$

On the other hand the first simple bound (5) gives

$$\left|\lambda_k^h - \lambda_k\right| \leqslant C\lambda_k h,$$

so the bound (6) must be pessimistic for large k. In this paper we would like to give a sketch of a proof of the de Hoog–Paine estimation (6) of the absolute error for the approximation, which satisfies the superconvergence condition. Yet another approach allows us to strengthen the bound (6) for the lowest degree of approximation. The proof is adapted from [24] so we follow its notations: Π is the set of all partitions of $[a,b]$ having the form $\pi = \{a = x_1 < x_2 < \cdots < x_{N+1} = b\}$, for any $\pi \in \Pi$, let $h = \max_{1 \leqslant n \leqslant N}(x_{n+1} - x_n)$

and $\mathscr{P}_m$ is the space of all polynomials of degree at most m. Since there is some inconvenience with the domain of the differential Sturm–Liouville operator when the function $p = p(x)$ has jump discontinuities we will use a weak variational formulation.

Let

$$\mathscr{V} = \begin{cases} H^1(a,b), & \text{when} \quad \alpha' \neq 0, \quad \beta' \neq 0, \\ \{u \in H^1(a,b) : u(b) = 0\}, & \text{when} \quad \alpha' \neq 0, \quad \beta' = 0, \\ \{u \in H^1(a,b) : u(a) = 0\}, & \text{when} \quad \alpha' = 0, \quad \beta' \neq 0, \\ H_0^1(a,b), & \text{when} \quad \alpha' = 0, \quad \beta' = 0. \end{cases}$$

The variational problem of (4) is to find a number λ and $u \neq 0$ in $\mathscr{V}$ such that

$$a(u,v) = \lambda(u,v) \quad \forall_{v\in\mathscr{V}}, \tag{7}$$

where

$$a(u,v) = -\alpha^{\#}p(a)u(a)v(a) + \beta^{\#}p(b)u(b)v(b) + \int_a^b \left(pu'v' + quv\right) \mathrm{d}x,$$

$$(u,v) = \int_a^b ruv \,\mathrm{d}x,$$

where $\alpha^{\#} = 0$, if $\alpha' = 0$, else $\alpha^{\#} = \alpha/\alpha'$; similarly $\beta^{\#} = 0$, when $\beta' = 0$ and $\beta^{\#} = \beta/\beta'$ in the other case.

Definition 1. For $p, q, r \in C^{m+1}([a,b])$, by the mth degree approximate problem (Pruess method) we mean that of finding λ_k^h, $u^h \neq 0$ in $\mathscr{V}$ such that

$$a_\pi\left(u^h, v\right) = \lambda^h\left(u^h, v\right)_\pi \quad \forall_{v\in\mathscr{V}}, \tag{8}$$

where the bilinear form $a_\pi(.,.)$ is defined as follows

$$a_\pi(u,v) = -\alpha^{\#}p(a)u(a)v(a) + \beta^{\#}p(b)u(b)v(b) + \int_a^b \left(p_\pi u'v' + q_\pi uv\right) \mathrm{d}x,$$

and the scalar product

$$(u,v)_\pi = \int_a^b r_\pi uv \,\mathrm{d}x.$$

The functions p_π, q_π, r_π are defined by the following rule:

- let Q be a continuous linear projection from $C([-1,1])$ into $\mathscr{P}_m$. For each fixed partition $\pi \in \Pi$ consisting of the intervals (x_n, x_{n+1}), $n = 1, 2, \ldots, N$, we define $p_\pi(x)$, $q_\pi(x)$, $r_\pi(x)$, on the union of these intervals, by formulas

$$p_\pi(x) = Q(p(x(t))), \quad q_\pi(x) = Q(q(x(t))), \quad r_\pi(x) = Q(r(x(t))),$$

where for $x \in (x_n, x_{n+1}):\quad x = x(t) = \frac{1}{2}(x_{n+1} - x_n)(t+1) + x_n$, for suitable $t \in (-1, 1)$.

This definition is more suitable in our settings, since the space $\mathscr{V}$ remains the same for variational eigenvalue formulation for forms $a(.,.)$ and $a_\pi(.,.)$, thus we can use min–max theorems simultaneously. Moreover, as $\mathscr{V} \subset L^2(a,b)$ is continuously, densely and compactly embedded and forms $a(.,.)$, $a_\pi(.,.)$ are symmetric, bounded, H^1-coercive, the problems (5) and (6) have countably many eigenvalues λ, $\lambda^h \in \mathbb{R}$ which may only have an accumulation point at ∞. From the spline analysis we see that for sufficiently small $h < h_0$ this will be a regular approximation of the Sturm–Liouville equation, i.e. $p_\pi > 0$, $r_\pi > 0$ on $[a,b]$. Hence the approximate eigenvalues λ_k^h are simple, real and distinct. Thus considering q_π as a small perturbation of the potential q, we immediately have from Lemma 1 in [24] and Theorem 1

$$\left|\lambda_k^h - \lambda_k\right| \leqslant Ch^{m+1} \quad \forall_k, \tag{9}$$

provided that Eq. (4) is in the Liouville normal form and $q \in C^{m+1}([a,b])$. Due to uniform estimation of the absolute error we can say that this kind of approximation is consistent with the spectral continuity of the Sturm–Liouville operator. When the problem (4) is in the full form we state the following (see [24]).

Theorem 5. *If p, q, $r \in C^{m+1}$, then for mth degree approximate problems we have that* $\exists_{C>0} \forall_{\pi:\ h<h_0}$

$$\left|\frac{\lambda_k^h - \lambda_k}{\lambda_k}\right| \leqslant Ch^{m+1} \quad \forall_k. \tag{10}$$

It is known that for the same degree of approximation the rates of convergence for the approximate eigenvalues can be doubled by a propitious choice of projections (for example Q can be the map which takes $f \in C([-1,1])$ into the mth degree polynomial, which interpolates f on the set of zeroes of the $(m+1)$st degree Legendre polynomial). Now we are in position to state the first theorem about superconvergence.

Theorem 6. *If p, q, $r \in C^{2m+2}$ and the projection Q satisfies*

$$\exists_{M>0} \forall_{f \in C^{2m+2}([-1,1])} \left| \int_{-1}^{1} t^i (1 - Q) f \,\mathrm{d}t \right| \leqslant M \left\| f^{(2m+2-i)} \right\|_\infty, \quad i = 0, 1, \ldots, m, \tag{11}$$

then for the Pruess method we have

$$\left|\frac{\lambda_k^h - \lambda_k}{\lambda_k}\right| \leqslant C k^{\max\{m+1,\ 5\}} h^{2m+2}, \quad h < h_0(k), \tag{12}$$

for each $k = 1, 2, \ldots$

The proof will be divided into two lemmas. For fixed $\pi \in \Pi$, let

$$A = \begin{pmatrix} 0 & \frac{1}{p} \\ q - \lambda_k r & 0 \end{pmatrix}, \quad A_\pi = \begin{pmatrix} 0 & \frac{1}{p_\pi} \\ q_\pi - \lambda_k^h r_\pi & 0 \end{pmatrix},$$

and by V we denote unique solution of $V' = AV$, $V(a) = I$; notice that $V = V(x)$ is nonsingular on $[a, b]$. Then we can recall the following lemma (see [24]):

Lemma 1. *If* $p, q, r \in C^{m+1}([a, b])$, *then for sufficiently small* $h = h(k)$, *for the respective* u_k^h *from mth degree approximate problem we have the representation*

$$\begin{aligned}&\begin{pmatrix} u_k \\ p u_k' \end{pmatrix}(x) - \begin{pmatrix} u_k^h \\ p_\pi \left(u_k^h\right)' \end{pmatrix}(x) \\ &\quad = V(x) \int_a^x V^{-1}(t)(A_\pi - A)(t) \begin{pmatrix} u_k^h \\ p_\pi \left(u_k^h\right)' \end{pmatrix}(t)\, \mathrm{d}t.\end{aligned} \tag{13}$$

Moreover

$$\left\|u_k - u_k^h\right\|_\infty = \mathrm{O}\left(\lambda_k h^{m+1}\right) \quad \forall_k.$$

Lemma 2 deals with the observation that the forms $D(v) = (v, v)$, $N(v) = a(v, v)$ will be no longer approximated uniformly over $\mathscr{V}$ but only locally to the eigenfunction with a decreasing accuracy as $k \to \infty$.

Lemma 2. *If* $p, q, r \in C^{2m+2}([a, b])$ *and the projection* Q *satisfies condition* (11), *then there exists a constant* $K > 0$ *independent of* k *such that for* $\pi \in \Pi$, *when* $v = u_k$ *or* $v = u_k^h$

$$|D[v] - D_\pi[v]| \leqslant K \lambda_k^{m+1/2} h^{2m+2}, \tag{14}$$

$$|N[v] - N_\pi[v]| \leqslant K \lambda_k^{m+3/2} h^{2m+2}. \tag{15}$$

Proof. The basic idea of the proof is to expand $w = v^2$ in a Taylor series about $z_n = \frac{1}{2}(x_n + x_{n+1})$. We assume that $p \neq p_\pi$, $q \neq q_\pi$, $r \neq r_\pi$, other wise the inequalities are trivial. So we have

$$\begin{aligned}\int_{x_n}^{x_{n+1}} &(r - r_\pi) w \,\mathrm{d}x \\ &= \sum_{j=0}^{m} \left(\frac{w^{(j)}(z_n)}{j!} \right) \int_{x_n}^{x_{n+1}} (x - z_n)^j (r - r_\pi)(x) \,\mathrm{d}x \\ &\quad + \frac{1}{(m+1)!} \int_{x_n}^{x_{n+1}} (x - z_n)^{m+1} (r - r_\pi)(x) w^{(m+1)} \left(\xi_x^{(n)} \right) \mathrm{d}x,\end{aligned}$$

where $\xi_x^{(n)} \in (x_n, x_{n+1})$. By the change of variable it follows that

$$\left| \int_{x_n}^{x_{n+1}} (x - z_n)^j (r - r_\pi)(x) \,\mathrm{d}x \right| \leqslant M \left(\frac{h}{2} \right)^{2m+3} \left\| r^{(2m+2-j)} \right\|_\infty .$$

In addition

$$\left| \int_{x_n}^{x_{n+1}} (x - z_n)^{m+1} (r - r_\pi)(x) w^{(m+1)} \left(\xi_x^{(n)} \right) \mathrm{d}x \right| \leqslant C \left\| w^{(m+1)} \right\|_\infty h^{2m+3},$$

hence

$$|D[v] - D_\pi[v]| \leqslant C_2 \sum_{k=1}^{N} \sum_{j=0}^{m+1} \frac{\left\| w^{(j)} \right\|_\infty}{j!} h^{2m+3},$$

where the constant C_2 is independent of h and λ_k. Moreover it follows from the proof of error characterization (13) and from Eq. (4) as well as the differential equation corresponding to (8) that for $v = u_k$ or u_k^h the norms $\left\| w^{(j)} \right\|_\infty$ are bounded independent of h. By the Leibnitz formula and finite asymptotic expansions of eigenvalues (see [9]),

$$|D[v] - D_\pi[v]| \leqslant C_3 \, h^{2m+2} \sum_{j=0}^{m+1} \frac{\lambda_k^{j/2}}{j!} \leqslant K \, \lambda_k^{m+1/2} \, h^{2m+2}.$$

A similar bound for $|N[v] - N_\pi[v]|$ can be established from an analogous argument. $\square$

It is now a simple matter to prove Theorem 6 by the general reasoning given in the proof of Lemma 3 in [24], observing that for $j \neq k$ we have $(u_k, u_j^h)_\pi^2 = \mathrm{O}(\lambda_k^2 h^{2m+2})$ for sufficiently small h.

The bound in the next theorem is a bit sharper for the smallest degree of Pruess method, e.g., for the PWCM approximation.

Theorem 7. *Under the assumption of the previous theorem the following inequality holds, for each* $k = 1, 2, \ldots$

$$\left|\frac{\lambda_k^h - \lambda_k}{\lambda_k}\right| \leqslant Ch^{2m+2} \max\left(1, k^{m+1}\right), \quad h < h_0. \tag{16}$$

Proof. Our proof starts with the error characterization

$$\lambda_k - \lambda_k^h = \left\{ \int_a^b (p - p_\pi) u_k' u_k^{h\prime} \, \mathrm{d}x + \int_a^b (q_\pi - q) u_k u_k^h \right. \\ \left. + \lambda_k \int_a^b (r_\pi - r) u_k u_k^h \, \mathrm{d}x \right\} \Big/ \int_a^b r_\pi u_k u_k^h \, \mathrm{d}x,$$

which can be easily established from integration by parts and is valid for general boundary conditions given in $\mathscr{D}$. A small change of proof of Lemma 2, by taking $w = u_k u_k^h$, completes the proof. □

Remark 4. From a practical viewpoint we are mostly interested in the case when $m = 0$ (cf. Section 4) as well as the correct dependence of (absolute, relative) errors of k. Therefore, in this case the method of proof of Theorem 7 seems to be better adapted to our theory. On the other hand, the estimation (12) in Theorem 6 is a revised version of the Paine–de Hoog inequality, which is consistent with (16) for $m \geqslant 4$. The rate of superconvergence was first given by Pruess in [24].

One question still unanswered is what kind of assumption on the coefficient functions should be imposed to keep the relative error of eigenvalues bounded independently of k, when the approximation of coefficients satisfies a superconvergence condition (11), i.e. when the following estimation holds

$$\left|\frac{\lambda_k^h - \lambda_k}{\lambda_k}\right| \leqslant Ch^{2m+2},$$

for each $k = 1, 2, \ldots$. Our numerical results indicate that for a certain class of problems (e.g., when $pr = K$ (const)) and piecewise constant approximation the relative error is in fact $\mathrm{O}(h^2)$ although there appears to be no clear structure in the error.

4. Computational results

To provide numerical confirmation of the preceding results we present an example. For $m > 0$ we can see that the above bounds present a considerable improvement over the bounds obtained when finite difference and variational schemes are employed. However, in order to solve (8) it is necessary to obtain the fundamental solutions on each subinterval. If $m > 0$ however there does not in general exist a closed form for these fundamental solutions which is

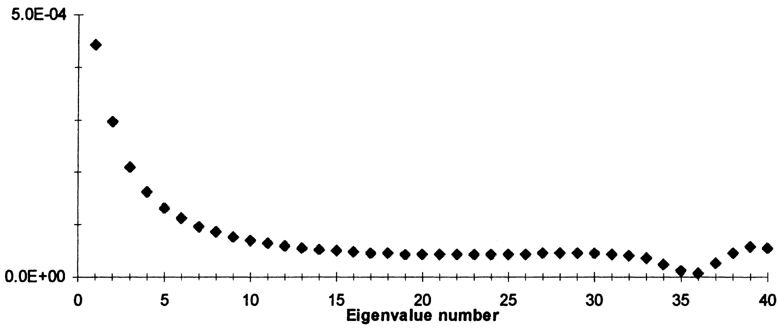

Fig. 1.

computationally convenient. Of course, a series solution could be developed, but this means that we are approximating the fundamental solutions by means of a piecewise polynomial and this is exactly the thing we are trying to avoid. It is when $m = 0$ that computationally tractable schemes can be constructed by means of $\sin(.)$ and $\sinh(.)$; for $m = 1$ we can also try to use Airy functions $Ai(.)$, $Bi(.)$. In our numerical example we have chosen $m = 0$, the most widely used case. The projection will be given by $(Qf)(t) = f(0)$.

Example 1. Consider equation $-(x^2 \cdot u')' = \lambda u$ subject to Dirichlet's BC $u(1) = u(e^{\pi}) = 0$. For this Sturm–Liouville operator the eigenvalues are known $\lambda_k = k^2 + 0.25$, whereas λ_k^h we computed by the Pruess method with projection given above and mesh index $N = 220$. We are interested in the behavior of $\left|(\lambda_k^h - \lambda_k)/\lambda_k\sqrt{\lambda_k}\right|$, which is given for first 40 eigenvalues. We obtained rate of $\mathrm{O}(h^2)$ convergence without any growth when eigenvalue λ_k increases. Note that the dependence of the considered error on k remains almost constant for $k \geqslant 10$. See Fig. 1.

5. Conclusion

To sum up, we have indicated that the upper bounds of (absolute, relative) error of eigenvalues for operators in LNF are independent of k. Yet for virtually all discretizations the large eigenvalues of the approximation diverge (relatively) from the true Sturm–Liouville eigenvalues. It is also known that in the finite dimensional space setting (e.g., Standard Finite Difference or Finite Element eigenvalue approximations) for small eigenvalue high relative accuracy is far more demanding. It was then natural to discuss the relative accuracy in the case of the Pruess method. Theorem 7 provides a sharpened estimation of the relative error of eigenvalues for the most important cases of approximation when $m = 0, 1$.

Despite the serious disadvantage of using finite dimensional approximations some aspects seem to be of independent interest. One of them, raised by Professor Beresford Parlett is, when is it fruitful to make a connection between tridiagonal eigenproblems and Sturm–Liouville eigenproblems? This could shed some new light on a converse of Theorem 3.

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