Discrete Mathematics 312 (2012) 633–636

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Balanced and strongly balanced P_k-designs[☆]

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ARTICLE INFO

Article history:
Available online 30 June 2011

Keywords:
Balanced
Strongly balanced
Simply balanced
G-design

ABSTRACT

Given a graph G, a *G-decomposition* of the complete graph K_v is a set of graphs, all isomorphic to G, whose edge sets partition the edge set of K_v. A G-decomposition of K_v is also called a *G-design* and the graphs of the partition are said to be the *blocks*. A G-design is said to be *balanced* if the number of blocks containing any given vertex of K_v is a constant.

In this paper the concept of *strongly balanced G-design* is introduced and strongly balanced *path-designs* are studied. Furthermore, we determine the spectrum of those path-designs which are balanced, but not strongly balanced.

1. Introduction

Let $K_v = (X, \mathcal{E})$ be the complete graph defined on a vertex set X. Let G be a subgraph of K_v. A G-decomposition of K_v is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of K_v into subsets all of which yield subgraphs that are isomorphic to G. A G-decomposition $\Sigma = (X, \mathcal{B})$ of K_v is also called a G-design of *order* v and the classes of the partition $\mathcal{B}$ are said to be the *blocks* of Σ [4–6].

A G-design is called *balanced* if the *degree* of each vertex $x \in X$ is a constant: in other words, the number of blocks of Σ containing x is a constant [1,3].

In the case $G = P_k$, path with k vertices, a G-decomposition is called a *path-decomposition* or a P_k*-decomposition* or a P_k*-design*.

In this paper we introduce the concept of a *strongly balanced G-design* as a balanced G-design with an additional property. We study some properties and relations between balanced and strongly balanced G-designs and, also, determine the spectrum of those path-designs which are balanced, but not strongly balanced.

2. Main definitions

In what follows, we denote by P_k a path with k vertices. If a P_k has vertices $x_1, x_2, \ldots, x_k$ and edges $\{x_1, x_2\}$, $\{x_2, x_3\}, \ldots, \{x_{k-1}, x_k\}$, it will be denoted by $(x_1, \ldots, x_k)$.

It is well known that a P_k-design of order v, briefly a $P_k(v)$-design, exists if and only if [2]: $v \cdot (v-1) \equiv 0, \bmod 2 \cdot (k-1)$, $v \geq k$.

Let G be a graph with n vertices and let $\Sigma = (X, \mathcal{B})$ be a G-design. If $x \in X$ is a vertex of Σ, the *degree* $d(x)$ of x is the number of blocks of Σ containing x.

A G-design $\Sigma = (X, \mathcal{B})$ is said to be *balanced* if the degree $d(x)$ of a vertex $x \in X$ is a constant.

Observe that if G is a regular graph then a G-design is always balanced, hence the notion of a balanced G-design becomes meaningful only for a non-regular graph G.

☆ Lavoro eseguito nell'ambito del GNSAGA e nell'ambito del MIUR.

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doi:10.1016/j.disc.2011.05.015

Example 1. Let $\Sigma = (X, \mathcal{B})$ be the $P_6(6)$-design defined on Z_6, having the blocks:

$$\mathcal{B}_1 = (0, 1, 5, 2, 4, 3), \qquad \mathcal{B}_2 = (1, 2, 3, 5, 0, 4), \qquad \mathcal{B}_3 = (2, 0, 3, 1, 4, 5).$$

Σ is a balanced P_6-design of order 6. □

Example 2. Let $\Sigma = (X, \mathcal{B})$ be the $P_4(7)$-design defined on $X = Z_7$, having the blocks:

$$B = (i, i + 1, i + 3, i + 6),$$

for every $i \in Z_7$.

Σ is a balanced P_4-design of order 7. □

Example 3. Let $\Sigma = (X, \mathcal{B})$ be the $P_3(4)$-design defined on $X = \{0, 1, 2, 3\}$, having the blocks:

$$B_1 = (0, 1, 2), \qquad B_2 = (0, 2, 3), \qquad B_3 = (0, 3, 1).$$

Σ is not a balanced P_3-design. The vertex 0 has degree three and the vertex 1 has degree two. □

3. Strongly balanced and simply balanced G-designs

Let G be a graph and let $A_1, A_2, \ldots, A_h$ be the orbits of the automorphism group of G on its vertex set. Let $\Sigma = (X, \mathcal{B})$ be a G-design.

We define the degree $d_{A_i(x)}$ of a vertex $x \in X$ as the number of blocks of Σ containing x as an element of A_i.

We say that:

$\Sigma = (X, \mathcal{B})$ is a *strongly balanced G-design if, for every* $i = 1, 2, \ldots, h$, *there exists a constant* C_i *such that*

$$d_{A_i}(x) = C_i,$$

for every $x \in X$.

Clearly, since for each vertex $x \in X$ the relation $d(x) = \sum_{i=1}^{h} d_{A_i}(x)$ holds, we have:

Theorem 3.1. *A strongly balanced G-design is a balanced G-design.*

The $P_6(6)$-design of Example 1 is balanced but not strongly balanced, showing that the converse of Theorem 3.1 does not hold.

We say that a G-design is *simply balanced* if it is balanced, but not strongly balanced.

Example 4. Given a kite $K_3 + e$ having vertices a, b, c, d, we denote it by $[a, b, (c, d)]$, where the ordered pair (c, d) contains the vertex c of degree three and the vertex d of degree one.

Let $\Sigma = (X, \mathcal{B})$ be the $(K_3 + e)$-design of order 9, defined on $X = Z_9$, having the blocks:

$$[0, 2, (1, 8)], [0, 3, (6, 1)], [0, 4, (7, 2)],$$
$$[0, 5, (8, 4)], [4, 5, (1, 7)], [6, 8, (2, 5)],$$
$$[7, 8, (3, 1)], [2, 3, (4, 6)], [6, 7, (5, 3)].$$

Σ is a balanced $(K_3 + e)$-design, but it is not strongly balanced: observe that the vertex 0 is never a terminal vertex, while the vertex 1 occurs twice in a terminal position. □

In what follows, we will determine the spectrum for simply balanced P_5-designs and P_6-designs. Our purpose is a preliminary study of the existence of balanced *paths*-designs which are not strongly balanced. The same investigation for $(K_3 + e)$-designs and other designs will be the subject of future work.

4. Simply balanced P_5-designs

It is known that [2].

Theorem 4.1. *A balanced* $P_5(v)$*-design exists if and only if* $v \equiv 1 \bmod 8$, $v \geq 9$.

Proof. The proof can be obtained using the *difference method*. If $v = 8k + 1$, $k \geq 1$, we can consider a $P_5(v)$-design, defined on Z_{8k+1}, having as blocks the translates of k *base blocks*. These can be obtained by distributing in a suitable way the $4k$ differences among different elements of Z_{8k+1} on the $4k$ edges of the k base blocks. □

By the construction indicated in the previous Theorem we obtain $P_5(v)$-designs which are strongly balanced, because the difference method produces these types of designs.

So, we can say that:

Theorem 4.2. *A strongly balanced $P_5(v)$-design exists if and only if* $v \equiv 1 \bmod 8$, $v \geq 9$.

So, we must determine the spectrum of those $P_5(v)$-designs which are balanced, but not strongly balanced.

Theorem 4.3. *There exists a simply balanced $P_5(9)$-design.*

Proof. Let $v = 9$. The system $\Sigma = (Z_9, \mathcal{B})$ having the following blocks:

$$(0, 1, 3, 7, 4),\ (1, 2, 4, 8, 5),\ (2, 3, 0, 5, 6),$$
$$(3, 4, 6, 1, 7),\ (4, 5, 7, 2, 8),\ (0, 6, 8, 3, 5),$$
$$(6, 7, 0, 4, 1),\ (7, 8, 1, 5, 2),\ (8, 0, 2, 6, 3),$$

is a balanced $P_5(9)$-design, it is not strongly balanced. □

Theorem 4.4. *A simply balanced $P_5(v)$-design exists if and only if* $v \equiv 1 \bmod 8$, $v \geq 9$.

Proof (*Construction*). $v \to v + 8$

Let $\Sigma_1 = (X, \mathcal{B}_1)$ be a simply balanced $P_5(9)$-design. Let $\Sigma_2 = (Y, \mathcal{B}_2)$ be a strongly balanced $P_5(v)$-design, where $v = 8k + 1$, $k \geq 1$ and $X \cap Y = \{0\}$. Define a family $\mathcal{F}$ of P_5-paths, containing all the pairs $\{x, y\}$ with $x \in X - \{0\}$ and $y \in Y - \{0\}$, as follows.

Since $v = 8k + 1$, then $|Y - \{0\}| = 8k$ and it is possible to partition the set $Y - \{0\}$ into two classes as follows: $\Gamma_a = \{a_1, a_2, \ldots, a_{4k}\}$, $\Gamma_b = \{b_1, b_2, \ldots, b_{4k}\}$.

Furthermore, since $|X - \{0\}| = 8$, it is possible to partition $X - \{0\}$ into the following two classes: $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$.

Let $\Gamma_{\alpha,\beta}$ be the family containing the following blocks:

$$(\alpha_4, a_i, \alpha_1, a_{k+i}, \alpha_3),$$
$$(\alpha_4, a_{k+i}, \alpha_2, a_i, \alpha_3),$$
$$(\alpha_1, a_{2k+i}, \alpha_3, a_{3k+i}, \alpha_2),$$
$$(\alpha_1, a_{3k+i}, \alpha_4, a_{2k+i}, \alpha_2),$$
$$(\beta_4, b_i, \beta_1, b_{k+i}, \beta_3),$$
$$(\beta_4, b_{k+i}, \beta_2, b_i, \beta_3),$$
$$(\beta_1, b_{2k+i}, \beta_3, b_{3k+i}, \beta_2),$$
$$(\beta_1, b_{3k+i}, \beta_4, b_{2k+i}, \beta_2),$$

for every $i = 1, 2, \ldots, k$.

Let $\Gamma_{a,b}$ be the family containing the following blocks:

$$(b_1, \alpha_1, b_{2k+1}, \alpha_2, b_2),$$
$$(b_2, \alpha_1, b_{2k+2}, \alpha_2, b_3),$$
$$\ldots$$
$$(b_{2k-1}, \alpha_1, b_{4k-1}, \alpha_2, b_{2k}),$$
$$(b_{2k}, \alpha_1, b_{4k}, \alpha_2, b_1),$$
$$(b_{2k+1}, \alpha_3, b_1, \alpha_4, b_{2k+2}),$$
$$(b_{2k+2}, \alpha_3, b_2, \alpha_4, b_{2k+3}),$$
$$\ldots$$
$$(b_{4k-1}, \alpha_3, b_{2k-1}, \alpha_4, b_{4k}),$$
$$(b_{4k}, \alpha_3, b_{2k}, \alpha_4, b_{2k+1}),$$
$$(a_1, \beta_1, a_{2k+1}, \beta_2, a_2),$$
$$(a_2, \beta_1, a_{2k+2}, \beta_2, a_3),$$
$$\ldots$$
$$(a_{2k-1}, \beta_1, a_{4k-1}, \beta_2, a_{2k}),$$
$$(a_{2k}, \beta_1, a_{4k}, \beta_2, a_1),$$

$(a_{2k+1}, \beta_3, a_1, \beta_4, a_{2k+2})$,

$(a_{2k+2}, \beta_3, a_2, \beta_4, a_{2k+3})$,

$\dots$

$(a_{4k-1}, \beta_3, a_{2k-1}, \beta_4, a_{4k})$,

$(a_{4k}, \beta_3, a_{2k}, \beta_4, a_{2k+1})$.

If $\mathcal{F} = \Gamma_{\alpha,\beta} \cup \Gamma_{a,b}$, it is possible to verify that the blocks of $\mathcal{F}$ cover all the edges having a vertex in $X - \{0\}$ and the other vertex in $Y - \{0\}$. Furthermore, each vertex $x \in X - \{0\}$ appears in $\mathcal{F}$ k times in the central position, $2k$ times in any of the two medial positions and $2k$ times in any of the two extreme positions of a P_5-block.

Set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}$ and note that the blocks of $\mathcal{B}$ passing through an element of $X - \{0\}$ are only those in $\mathcal{B}_1 \cup \mathcal{F}$. Since Σ_1 is simply balanced one can easily check that $\Sigma = (X \cup Y, \mathcal{B})$ is a simply balanced $P_5(v + 8)$-design. □

5. Simply balanced P_6-designs

It is known that [2].

Theorem 5.1. *A balanced $P_6(v)$-design exists if and only if $v \equiv 1 \bmod 5$, $v \geq 6$.*

Theorem 5.2. *There exist strongly balanced and simply balanced $P_6(6)$-designs.*

Proof. Let $v = 6$. We can verify that:

(i) the system $\Sigma_1 = (Z_6, \mathcal{B}_1)$ with the blocks:

$$(0, 1, 2, 5, 4, 3),\ (1, 5, 3, 0, 2, 4),\ (2, 3, 1, 4, 0, 5),$$

is a strongly balanced $P_6(9)$-design;

(ii) the system $\Sigma_2 = (Z_6, \mathcal{B}_2)$ with the blocks:

$$(0, 1, 2, 4, 5, 3),\ (1, 5, 2, 0, 3, 4),\ (2, 3, 1, 4, 0, 5),$$

is a balanced $P_6(6)$-design, which is not strongly balanced. □

Theorem 5.3. (i) *A simply balanced $P_6(v)$-design exists if and only if $v \equiv 1 \bmod 5$, $v \geq 6$.* (ii) *A strongly balanced $P_6(v)$-design exists if and only if $v \equiv 1 \bmod 5$, $v \geq 6$.*

Proof. From Theorem 5.2, for $v = 6$ there exist simply balanced $P_6(6)$-designs and strongly balanced $P_6(6)$-designs.

Construction $v \to v + 5$

Observe that if $\Sigma = (X, \mathcal{B})$ is a strongly balanced $P_6(v)$-design with $v = 5k + 1$, necessarily all vertices have degree k in any position of P_6. This is not true in any simply balanced $P_6(v)$-design.

Let $\Sigma_1 = (X, \mathcal{B}_1)$ be a *simply* [resp. *strongly*] balanced $P_6(6)$-design. Let $\Sigma_2 = (Y, \mathcal{B}_2)$ be a strongly balanced $P_6(v)$-design, where $v = 5k + 1$, $k \geq 1$ and $X \cap Y = \{0\}$.

Define a family $\mathcal{F}$ of P_6-paths, containing all the pairs $\{x, y\}$ with $x \in X - \{0\}$ and $y \in Y - \{0\}$, as follows.

Since $v = 5k + 1$, $|Y - \{0\}| = 5k$ and it is possible to partition the set $Y - \{0\}$ into k classes as follows:

$$\Gamma_i = \{a_{i1}, a_{i2}, \dots, a_{i5}\},$$

for every $i = 1, 2, \dots, k$.

So, if $X = \{0, A, B, C, D, E\}$, let $\mathcal{F}$ be the family containing the following blocks:

$$(A, a_{i1}, B, a_{i5}, C, a_{i4}),\ (B, a_{i2}, C, a_{i1}, D, a_{i5}),\ (C, a_{i3}, D, a_{i2}, E, a_{i1}),$$
$$(D, a_{i4}, E, a_{i3}, A, a_{i2}),\ (E, a_{i5}, A, a_{i4}, B, a_{i3}),$$

for every $i = 1, 2, \dots, k$.

Since Σ_1 is a *simply* [resp. *strongly*] balanced $P_6(6)$-design and Σ_2 is a strongly balanced $P_6(v)$-design in which all vertices have degree k in any position of P_6, if $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}$, it is possible to verify that $\Sigma = (X \cup Y, \mathcal{B})$ is a *simply* balanced $P_6(v + 9)$-design [resp. *strongly balanced* $P_6(v + 9)$-design with all vertices having degree $k + 1$ in any position of P_6]. □

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