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Prime Interchange Graphs of Classes of Matrices of Zeros and Ones

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Let $R = (r_1,..., r_m)$ and $S = (s_1,..., s_n)$ be nonnegative integral vectors, and let $\mathfrak{A}(R, S)$ denote the class of all $m \times n$ matrices of 0's and 1's having row sum vector R and column sum vector S. An invariant position of $\mathfrak{A}(R, S)$ is a position whose entry is the same for all matrices in $\mathfrak{A}(R, S)$. The interchange graph $G(R, S)$ is the graph where the vertices are the matrices in $\mathfrak{A}(R, S)$ and where two matrices are joined by an edge provided they differ by an interchange. We prove that when $1 \leqslant r_i \leqslant n-1$ $(i = 1,..., m)$ and $1 \leqslant s_j \leqslant m-1$ $(j = 1,..., n)$, $G(R, S)$ is prime if and only if $\mathfrak{A}(R, S)$ has no invariant positions.

1. INTRODUCTION

Let m and n be positive integers, and let $R = (r_1,..., r_m)$ and $S = (s_1,..., s_n)$ be nonnegative integral vectors. Denote by $\mathfrak{A}(R, S)$ the set of all $m \times n$ matrices of 0's and 1's with row sum vector R and column sum vector S. Thus an $m \times n$ matrix $A = [a_{ij}]$ belongs to $\mathfrak{A}(R, S)$ if and only if

$$a_{ij} = 0 \quad \text{or} \quad 1 \qquad (i = 1,..., m;\ j = 1,..., n),$$

$$\sum_{j=1}^{n} a_{ij} = r_i \qquad (i = 1,..., m),$$

$$\sum_{i=1}^{m} a_{ij} = s_j \qquad (j = 1,..., n).$$

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In 1957 Gale [3] and Ryser [7] independently determined necessary and sufficient conditions for $\mathfrak{A}(R, S)$ to be nonempty. Other criteria have been found by Moon [6]. Since then many interesting properties of $\mathfrak{A}(R, S)$ have been discovered. A survey of the results up to 1980 can be found in [1]. In this paper we assume $\mathfrak{A}(R, S)$ is nonempty.

Let A be an $m \times n$ matrix of 0's and 1's, and let $1 \leqslant i_1 < \cdots < i_p \leqslant m$ and $1 \leqslant j_1 < \cdots < j_q \leqslant n$. Then $A[i_1,..., i_p; j_1,..., j_q]$ denotes the $p \times q$ submatrix of A lying in rows $i_1,..., i_p$ and columns $j_1,..., j_q$. If $i'_1,..., i'_p$ is a permutation of $i_1,..., i_p$ and $j'_1,..., j'_q$ is a permutation of $j_1,..., j_q$, then we set $A[\{i'_1,..., i'_p\}; \{j'_1,..., j'_q\}] = A[i_1,..., i_p; j_1,..., j_q]$. Let $A \in \mathfrak{A}(R, S)$ and suppose there exist integers i, j, k, l such that $A[i, j; k, l]$ is one of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.1}$$

or

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.2}$$

Then an *interchange* at A is a transformation which replaces one of the submatrices (1.1) and (1.2) of A by the other. The resulting matrix clearly belongs to $\mathfrak{A}(R, S)$. Following [1], we define the *interchange graph* $G(R, S)$ of $\mathfrak{A}(R, S)$ as follows: The vertices are the matrices in $\mathfrak{A}(R, S)$ and two matrices A and B are joined by an edge if and only if one can be obtained from the other by a single interchange. Let $e = [A, B]$ be an edge, and suppose B is obtained from A by changing the 2×2 submatrix $A[i, j; k, l]$. Then A is obtained from B by changing the 2×2 submatrix $B[i, j; k, l]$. Depending on the context we identify the edge e with one of these 2×2 submatrices.

For example, let $R = (2, 1)$ and $S = (1, 1, 1)$. Then $\mathfrak{A}(R, S)$ consists of the three matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the interchange graph $G(R, S)$ is the triangle shown in Fig. 1.

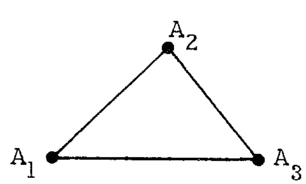

FIGURE 1

Ryser [7] proved that given A, $B \in \mathfrak{A}(R, S)$ there is a finite sequence of interchanges which transforms A into B, and hence $G(R, S)$ is a connected graph.

Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ and suppose R' and S' are obtained from R and S, respectively, by reordering coordinates. Then $\mathfrak{A}(R, S) \neq \varnothing$ if and only if $\mathfrak{A}(R', S') \neq \varnothing$, and the interchange graphs $G(R, S)$ and $G(R', S')$ are isomorphic. Hence there is no loss in generality in assuming that R and S are *monotone* in the sense that $r_1 \geqslant \cdots \geqslant r_m$ and $s_1 \geqslant \cdots \geqslant s_n$.

Now let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ and $\bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$, where $\bar{r}_i = n - r_i$ $(i = 1, \ldots, m)$ and $\bar{s}_j = m - s_j$ $(j = 1, \ldots, n)$. Each matrix $\bar{A}$ in $\mathfrak{A}(\bar{R}, \bar{S})$ can be obtained from a matrix A in $\mathfrak{A}(R, S)$ by replacing 0's with 1's and 1's with 0's. Each 2×2 submatrix (1.1), [respectively (1.2)] of A corresponds to a 2×2 submatrix (1.2) [respectively (1.1)] of $\bar{A}$. Hence it follows that $\mathfrak{A}(R, S) \neq \varnothing$ if and only if $\mathfrak{A}(\bar{R}, \bar{S}) \neq \varnothing$, and $G(R, S)$ and $G(\bar{R}, \bar{S})$ are isomorphic. We refer to $\mathfrak{A}(\bar{R}, \bar{S})$ as the *complementary class* of $\mathfrak{A}(R, S)$.

Let $1 \leqslant k \leqslant m$ and $1 \leqslant l \leqslant n$. The position (k, l) is an *invariant* 1-*position* of $\mathfrak{A}(R, S)$ provided each matrix in $\mathfrak{A}(R, S)$ has a 1 in the (k, l)-position. An *invariant* 0-*position* is defined similarly. The position (k, l) is an *invariant position* of $\mathfrak{A}(R, S)$ if it is either an invariant 1-position or an invariant 0-position. We denote by $J_{p,q}$ [respectively, $O_{p,q}$] the $p \times q$ matrix of all 1's [respectively, 0's]. If either p or q is 0, then these matrices are vacuous. From a result of Ryser [8] we obtain the following.

THEOREM 1. *Let R and S be monotone. $\mathfrak{A}(R, S)$ has an invariant* 1-*position if and only if there exist integers e and f with $1 \leqslant e \leqslant m$ and $1 \leqslant f \leqslant n$ such that one and hence every matrix $A \in \mathfrak{A}(R, S)$ has the form*

$$\begin{bmatrix} J_{e,f} & A_1 \\ A_2 & O_{m-e,n-f} \end{bmatrix}. \tag{1.3}$$

Similarly $\mathfrak{A}(R, S)$ has an invariant 0-*position if and only if there exist integers e and f with $0 \leqslant e \leqslant m-1$ and $0 \leqslant f \leqslant n-1$ such that one and hence every matrix A in $\mathfrak{A}(R, S)$ has the form* (1.3).

In (1.3) the positions occupied by $J_{e,f}$ are invariant 1-positions while those occupied by $O_{m-e,n-f}$ are invariant 0-positions. The form (1.3) is in general not unique and there may be other invariant positions.

Suppose in (1.3) that $1 \leqslant e \leqslant m-1$ and $1 \leqslant f \leqslant n-1$, and let $A_i \in \mathfrak{A}(R_i, S_i)$ for $i = 1, 2$. Then the graph $G(R, S)$ is isomorphic to the *cartesian product* $G(R_1, S_1) \times G(R_2, S_2)$ of the graphs $G(R_1, S_1)$ and $G(R_2, S_2)$. [A formal definition of Cartesian product is given in Section 2.] If one of the factors, $G(R_1, S_1)$ or $G(R_2, S_2)$, consists of a single vertex, then we say that the product $G(R_1, S_1) \times G(R_2, S_2)$ is *trivial*. When $0 < r_i < n$ $(i = 1, \ldots, m)$ and $0 < s_j < m$ $(j = 1, \ldots, n)$, both of the factors $G(R_1, S_1)$ and $G(R_2, S_2)$

contain at least two vertices and the factorization $G(R_1, S_1) \times G(R_2, S_2)$ of $G(R, S)$ is nontrivial (Theorem 9). Our main result (Theorem 8) states that conversely when $\mathfrak{A}(R, S)$ has no invariant positions, $G(R, S)$ is *prime* in the sense that every factorization of $G(R, S)$ into a cartesian product is trivial.

For example, let $R = S = (3, 3, 1, 1)$. Then $\mathfrak{A}(R, S)$ contains the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{1.4}$$

and $G(R, S)$ is a rectangle without diagonals. The matrix (1.4), and hence every matrix in $\mathfrak{A}(R, S)$, has the form (1.3) with $e = f = 2$. Thus with $R_i = S_i = (1, 1)$ $(i = 1, 2)$, $G(R, S)$ is isomorphic to $G(R_1, S_1) \times G(R_2, S_2)$, the cartesian product of two edges.

In the proof of the main result we consider monotone row and column sum vectors R and S, and we use induction on a matrix $\tilde{A}$ in $\mathfrak{A}(R, S)$ which is constructed as follows [8, pp. 63–65; 1, p. 165]: For each $k = 1, \ldots, n$, the leading $m \times k$ submatrix $\tilde{A}_k$ of $\tilde{A}$ has a monotone row sum vector, and the 1's in column k of $\tilde{A}_k$ are in those rows with the largest row sums, preference given to the bottom most positions in case of ties. We refer to $\tilde{A}$ as the *special matrix* of $\mathfrak{A}(R, S)$ (or the canonical representative of $\mathfrak{A}(R, S)$). In the previous example, the matrix (1.4) is the special matrix $\tilde{A}$ of $\mathfrak{A}(R, S)$. The following result [2, pp. 182–183] is used in the inductive step.

THEOREM 2. *Let $n > 2$ and let R and S be monotone, and suppose that $\mathfrak{A}(R, S)$ has no invariant positions. Let $\tilde{A}_{n-1}$ be the matrix obtained from $\tilde{A} \in \mathfrak{A}(R, S)$ by eliminating its last column. Suppose $\tilde{A}_{n-1} \in \mathfrak{A}(R_{n-1}, S_{n-1})$. Then $\mathfrak{A}(R_{n-1}, S_{n-1})$ does not have both invariant 1-positions and invariant 0-positions.*

Note that in view of Theorem 1, the class $\mathfrak{A}(R_{n-1}, S_{n-1})$ above has invariant positions if and only if some coordinate of R_{n-1} is 0 or $n - 1$.

2. EDGE EQUIVALENCE AND PRIME GRAPHS

A graph in this paper is assumed to be finite. We start with a formal definition of the cartesian product of two graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then the *cartesian product* $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ and with an edge joining (x_1, x_2) and (y_1, y_2) if and only if $x_1 = y_1$ and $[x_2, y_2] \in E_2$ or $x_2 = y_2$ and $[x_1, y_1] \in E_1$. Following Sabidussi [6], we say that a graph G is *prime* if whenever G is isomorphic to $G_1 \times G_2$, G_1 or G_2 consists of a single vertex. Sabidussi [9] proves that every connected graph can be uniquely factored into prime factors. Other proofs of this fact has been given by Imrich [4], Miller [5], and Vizing [10].

In [9] Sabidussi defines an equivalence relation on the edges of a graph G and proved that G is prime if and only if all its edges are equivalent. In defining this equivalence relation, Sabidussi introduces a hierarchy of equivalence relations the first of which we denote by $\sim_G$. The relation $\sim_G$ is the smallest equivalence relation on the edges of G containing the following two relations:

(2.1) Let e and e' be two nonincident edges of G. Then $e \sim_G^{(0)} e'$ if and only if e and e' are opposite edges of a 4-cycle of G.

(2.2) Let e and e' be incident edges of G. Then $e \sim_G^{(1)} e'$ if and only if every 4-cycle containing e and e' has a diagonal.

It follows that if e and e' are edges of a triangle, then $e \sim_G^{(1)} e'$.

Henceforth we say that two edges e and e' are *equivalent* if and only if $e \sim_G e'$. From Sabidussi [9] we obtain the following.

LEMMA 3. *If all edges of a graph G are equivalent, then G is prime.*

Thus to prove a graph is prime, it suffices to show all edges are equivalent. The next result shows that for connected graphs it is enough to show all edges incident at a single vertex are equivalent.

LEMMA 4. *Let G be a connected graph such that all edges at a given vertex v are equivalent. Then all edges of G are equivalent.*

Proof. Let $e = [v, w]$ and $e' = [w, z]$ where $z \neq v$. If there is no 4-cycle containing both e and e', then $e \sim_G^{(1)} e'$. Otherwise, $e \sim_G^{(0)} e''$, where e'' is incident at v. Hence all edges incident at w are equivalent to all edges incident at v. Since G is connected a similar argument shows that all edges are equivalent.

We conclude this section by noting that if H is an induced subgraph of G, then $\sim_H$ is not in general the restriction of $\sim_G$ to the edges of H. More precisely, we have the following.

LEMMA 5. *Let H be an induced subgraph of G, and let e_1 and e_2 be edges of H. Then*

(i) $e_1 \sim_G^{(0)} e_2$ *if and only if* $e_1 \sim_H^{(0)} e_2$;

(ii) $e_1 \sim_G^{(1)} e_2$ *implies* $e_1 \sim_H^{(1)} e_2$, *but in general not conversely.*

Proof. Since H is an induced subgraph of G, e_1 and e_2 are opposite edges of a rectangle in G if and only if they are opposite edges of a rectangle in H. Hence (i) holds. Now suppose we have $e_1 \sim_G^{(1)} e_2$ but not $e_1 \sim_H^{(1)} e_2$. Then e_1

and e_2 are incident and there is a rectangle without diagonals in H containing e_1 and e_2. Since H is an induced subgraph of G, this rectangle has no diagonals in G contradicting the fact that $e_1 \sim_G^{(1)} e_2$. To show the converse need not hold, we take G to be a 4-cycle without diagonals and H to be the subgraph consisting of two incident edges e_1 and e_2 of G. Then $e_1 \sim_H^{(1)} e_2$ holds but $e_1 \sim_G^{(1)} e_2$ does not.

3. Invariant Positions and Prime Interchange Graphs

Consider the interchange graph $G = G(R, S)$ of the class $\mathfrak{A}(R, S)$ where $R = (r_1, ..., r_m)$ and $S = (s_1, ..., s_n)$. Let $A \in \mathfrak{A}(R, S)$ and let $A' = A[\alpha; \beta]$ be a proper submatrix of A where $\alpha = i_1, ..., i_p$ and $\beta = j_1, ..., j_q$. Let

$$\mathfrak{A}' = \{B = [b_{ij}] \in \mathfrak{A}(R, S): b_{ij} = a_{ij} \text{ for all } (i, j) \notin \alpha \times \beta\},$$

the subclass of all matrices in $\mathfrak{A}(R, S)$ which agree with A outside of A'. So if R' and S' are the row and column sum vectors of A', then $\mathfrak{A}'$ can be identified with the class $\mathfrak{A}(R', S')$. Moreover, the subgraph G' of $G(R, S)$ induced by the vertices in $\mathfrak{A}'$ is isomorphic to $G(R', S')$. The next result shows that $\sim_{G'}$ is the restriction of $\sim_G$ to the edges of G', a fact which as shown in Lemma 5 does not generally hold for induced subgraphs.

Lemma 6. *Let $A \in \mathfrak{A}(R, S)$, A', G, and G' be as above, and let e_1 and e_2 be edges of G'. Then $e_1 \sim_G e_2$ if and only if $e_1 \sim_{G'} e_2$.*

Proof. In view of Lemma 5 along with the definitions of $\sim_G$ and $\sim_{G'}$, it suffices to show that when e_1 and e_2 are distinct; then $e_1 \sim_{G'}^{(1)} e_2$ implies $e_1 \sim_G^{(1)} e_2$. Assume to the contrary that $e_1 \sim_{G'}^{(1)} e_2$ holds but $e_1 \sim_G^{(1)} e_2$ does not. Then e_1 and e_2 are incident and there is a rectangle μ without diagonals in G containing them. Let the vertices of μ be B, B_1, B_2, and B_3 where $e_1 = [B, B_1]$, $e_2 = [B, B_2]$ and B, B_1, $B_2 \in \mathfrak{A}'$. Using the identification of edges with 2×2 submatrices of the forms (1.1) and (1.2), let

$$e_1 = B[k_1, k_2; l_1, l_2], \qquad e_2 = B[u_1, u_2; v_1, v_2].$$

Since e_1 and e_2 are distinct edges,

$$|(\{k_1, k_2\} \times \{l_1, l_2\}) \cap (\{u_1, u_2\} \times \{v_1, v_2\})| = 0, 1, \text{or } 2.$$

Hence it follows that B_1 and B_2 differ in 8, 6, or 4 positions, respectively. We consider two cases.

Case 1. B_1 and B_2 differ in exactly 4 positions. It follows that $\{k_1, k_2\} = \{u_1, u_2\}$ and $|\{l_1, l_2\} \cap \{v_1, v_2\}| = 1$, or $\{l_1, l_2\} = \{v_1, v_2\}$ and $|\{k_1, k_2\} \cap$

$\{u_1, u_2\}| = 1$. It suffices to consider the former possibility. Without loss of generality we may assume that $l_1 = v_1$ and

$$B[k_1, k_2; \{l_1, l_2, v_2\}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hence

$$B_1[k_1, k_2; \{l_1, l_2, v_2\}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$B_2[k_1, k_2; \{l_1, l_2, v_2\}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Since B_2 can be obtained from B_1 by a single interchange, $[B_1, B_2]$ is a diagonal of μ, contradicting the assumption on μ.

Case 2. B_1 and B_2 differ in 6 or 8 positions. The fact that $e_1 \sim_{G'}^{(1)} e_2$ implies that μ is not a rectangle in G', and hence B_3 is not a vertex of G'. Thus the interchange defining the edge $[B_1, B_3]$ of μ changes a position outside of A' and hence changes at least two such positions. This implies that B_2 and B_3 differ in at least 6 positions (at least two of these positions are outside of A'). Hence no single interchange can transform B_2 to B_3, contradicting the fact that $[B_2, B_3]$ is an edge of G.

This completes the proof.

Before stating the main result, we prove the following.

LEMMA 7. *If $m + n \leqslant 7$, then all edges of $G(R, S)$ are equivalent.*

Proof. Without loss of generality we may assume $m \leqslant n$. Since $G(R, S)$ is connected, it suffices by Lemma 4 to show that each pair of distinct incident edges are equivalent. Let $A \in \mathfrak{A}(R, S)$ and let

$$e_1 = A[k_1, k_2; l_1, l_2], \qquad e_2 = A[u_1, u_2; v_1, v_2]$$

be two distinct edges incident at A. We consider three cases.

Case 1. $m = 2$. In this case, $n > 2$ and e_1 and e_2 either belong to a triangle or to a rectangle with a diagonal. Hence $e_1 \sim_G e_2$.

Case 2. $m = 3$ and $n = 3$. If $\{k_1, k_2\} = \{u_1, u_2\}$ or $\{l_1, l_2\} = \{v_1, v_2\}$, then e_1 and e_2 are edges of a triangle and hence $e_1 \sim_G e_2$. Otherwise, $|\{k_1, k_2\} \cap \{u_1, u_2\}| = |\{l_1, l_2\} \cap \{v_1, v_2\}| = 1$. Using the fact that complementary classes have isomorphic interchange graphs, we may assume that $R = S = (1, 1, 1)$ or $R = S = (2, 2, 1)$. We consider the case where the entry at the common position of the 2×2 submatrices e_1 and e_2 is 1; a similar argument holds when this entry is 0. After row and column permutations, we may suppose that

$$A = \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad a = 0 \text{ or } 1,$$

where $e_1 = A[1, 3; 1, 3]$ and $e_2 = A[2, 3; 2, 3]$. It now follows that $G(R, S)$ contains the graph of Fig. 2 as a subgraph, where

$$B = \begin{bmatrix} 0 & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$a = 0\colon D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$a = 1\colon D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence $e_1 = [A, B] \sim_G [C, E] \sim_G [B, D] \sim_G [A, C] = e_2$. We note that when the entry at the common position of the 2×2 submatrices e_1 and e_2 is 0, $G(R, S)$ also contains a subgraph isomorphic to the graph of Fig. 2.

Case 3. $m = 3$ and $n = 4$. If $|\{k_1, k_2\} \cap \{u_1, u_2\}| = 2$ or $|\{l_1, l_2\} \cap \{v_1, v_2\}| = 2$, then as in Case 1, e_1 and e_2 belong either to a triangle in $G(R, S)$ or to a rectangle with a diagonal and hence $e_1 \sim_G e_2$. If $|\{k_1, k_2\} \cap \{u_1, u_2\}| = |\{l_1, l_2\} \cap \{v_1, v_2\}| = 1$, then the 2×2 matrices e_1 and e_2 are contained in a 3×3 submatrix of A; hence as in Case 2, $G(R, S)$ contains a subgraph isomorphic to the graph of Fig. 2 and $e_1 \sim_G e_2$. Finally, suppose that $|\{k_1, k_2\} \cap \{u_1, u_2\}| = 1$ and $|\{l_1, l_2\} \cap \{v_1, v_2\}| = 0$. After row and column permutations we may assume that

$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix},$$

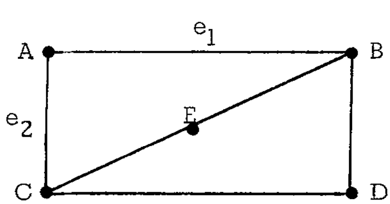

FIGURE 2

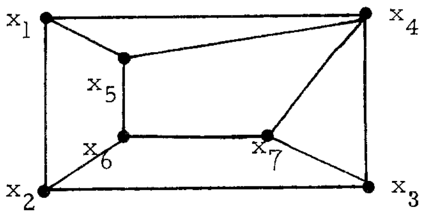

FIGURE 3

where a, b, c, and d are 0 or 1, $e_1 = A[1, 2; 1, 2]$ and $e_2 = A[2, 3; 2, 3]$. First consider the case where $d = 0$ and $b = 1$. Then $G(R, S)$ contains the graph of Fig. 3 as a subgraph, where $A = A_1$ and with the A_i's as below, $x_i = A_i$:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ c & d & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ c & d & 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ c & d & 0 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ c & d & 0 & 1 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ c & d & 0 & 1 \end{bmatrix}.$$

Using the definition of equivalence, one easily verifies that all edges of the graph of Fig. 3 are equivalent and thus $e_1 \sim_G e_2$.

Next consider the case where $a = 1$ and $b = 0$. Then A can be obtained from A_2 by permuting columns 3 and 4. It follows that $G(R, S)$ contains the graph of Fig. 3 as a subgraph where $x_i = A_i'$ $(i = 1, \ldots, 7)$ and A_i' is obtained

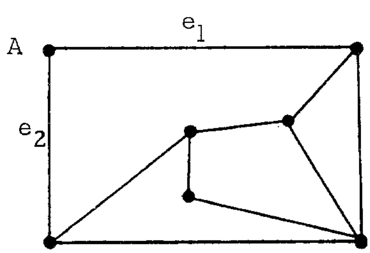

FIGURE 4

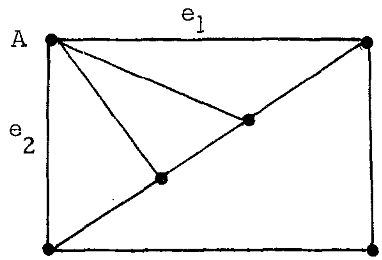

FIGURE 5

from A_i by permuting columns 3 and 4. Since e_1 and e_2 are edges of this subgraph incident at $A = A_2'$, as before $e_1 \sim_G e_2$. This completes the argument when $\{a, b\} = \{0, 1\}$. Since complementary classes have isomorphic interchange graphs, it remains to consider the case where $a = b = c = d = 0$ and the case where $a = b = 1$ and $c = d = 0$. In the former case, $G(R, S)$ contains the graph of Fig. 4 as a subgraph; since all edges of this graph are equivalent, $e_1 \sim_G e_2$. In the latter case, $G(R, S)$ contains the graph of Fig. 5 as a subgraph and it follows that $e_1 \sim_G e_2$. This completes the proof of the lemma.

THEOREM 8. *Let $R = (r_1,..., r_m)$ and $S = (s_1,..., s_n)$. If $\mathfrak{A}(R, S)$ has no invariant positions, then $G(R, S)$ is a prime graph.*

Proof. Without loss of generality we assume that R and S are monotone. By Lemmas 3 and 4 it suffices to prove that all edges of $G = G(R, S)$ incident at the special matrix $\tilde{A} \in \mathfrak{A}(R, S)$ are equivalent. We do this by induction on $m + n$. Since $\mathfrak{A}(R, S)$ has no invariant positions, $m \geqslant 2$ and $n \geqslant 2$. When $m = n = 2$, we have $R = S = (1, 1)$ and $G(R, S)$ consists of two vertices joined by an edge and the assertion holds.

Now let $m + n > 4$ and assume $\mathfrak{A}(R, S)$ has no invariant positions. Let e_1 and e_2 be two distinct edges incident at $\tilde{A}$ where

$$e_1 = \tilde{A}[k_1, k_2; l_1, l_2] \qquad \text{and} \qquad e_2 = \tilde{A}[u_1, u_2; v_1, v_2].$$

When either $\{k_1, k_2\} \cap \{u_1, u_2\} \neq \varnothing$ or $\{l_1, l_2\} \cap \{v_1, v_2\} \neq \varnothing$, the 2×2 submatrices e_1 and e_2 are contained in a $p \times q$ submatrix of $\tilde{A}$ with $p + q \leqslant 7$, and by Lemmas 6 and 7, $e_1 \sim_G e_2$.

Otherwise, $\{k_1, k_2\} \cap \{u_1, u_2\} = \{l_1, l_2\} \cap \{v_1, v_2\} = \varnothing$. Let

$$B = \tilde{A}[\{k_1, k_2, u_1, u_2\}; \{l_1, l_2, v_1, v_2\}]$$

and

$$B_1 = \tilde{A}[k_1, k_2; v_1, v_2], \qquad B_2 = \tilde{A}[l_1, l_2; u_1, u_2].$$

We consider first the case where $B_i \neq O$ or J $(i=1,2)$, where O and J are matrices of all 0's and all 1's, respectively. For instance, let row 1 of B_1 be [1 0], and

$$e_1 = e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.1}$$

Hence after row and column permutations B has the form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & * & * \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}.$$

Let

$$e_3 = \tilde{A}[\{k_1, u_2\}; v_1, v_2].$$

Then e_2 and e_3 are edges of a triangle in G and hence $e_2 \sim_G e_3$. The submatrices e_1 and e_3 are contained in the 3×4 submatrix $A[\{k_1, k_2, u_2\}; \{l_1, l_2, v_1, v_2\}]$, and hence by Lemmas 6 and 7, $e_3 \sim_G e_1$. Thus $e_1 \sim_G e_2$. A similar argument shows that whenever B_1 or B_2 contains a row or column with distinct entries, $e_1 \sim_G e_2$.

Now consider the case where $B_1 = B_2 = O$. Without loss of generality suppose (3.1) holds. Let

$$e_4 = \tilde{A}[\{k_2, u_1\}; \{l_2, v_1\}].$$

By considering the 3×3 submatrices $\tilde{A}[\{k_1, k_2, u_1\}; \{l_1, l_2, v_1\}]$ and $\tilde{A}[\{k_2, u_1, u_2\}; \{l_2, v_1, v_2\}]$ and applying Lemmas 6 and 7 we conclude that $e_1 \sim_G e_2$. When $B_1 = B_2 = J$, a similar argument establishes that $e_1 \sim_G e_2$. It remains to consider the cases $B_1 = O$, $B_2 = J$ and $B_1 = J$, $B_2 = O$. By symmetry we need only consider one of these two cases.

Let $B_1 = O$ and $B_2 = J$. Consider the matrix $C = \tilde{A}_{n-1} \in \mathfrak{A}(R_{n-1}, S_{n-1})$ obtained from $\tilde{A}$ by eliminating column n. Although the class $\mathfrak{A}(R_{n-1}, S_{n-1})$ may have invariant positions, it follows from Theorem 2 that they appear only in rows of C consisting of all 0's or all 1's. Form the matrix C' from C by deleting all rows containing only 1's or only 0's, and let $C' \in \mathfrak{A}(R', S')$. By Theorems 1 and 2, $\mathfrak{A}(R', S')$ has no invariant positions. We also observe that C' is the special matrix of $\mathfrak{A}(R', S')$. Hence we may apply the inductive assumption to the graph $G(R', S')$ at the vertex C'. Because of the nature of the invariant positions of $\mathfrak{A}(R_{n-1}, S_{n-1})$ the graphs $G(R_{n-1}, S_{n-1})$ and $G(R', S')$ are isomorphic, and hence we may apply the inductive assumption to the graph $G^* = G(R_{n-1}, S_{n-1})$ at the vertex C. This we do for simplicity of exposition.

If $n \notin \{l_1, l_2, v_1, v_2\}$, it follows from the inductive assumption that $e_1 \sim_{G^*} e_2$ and hence from Lemma 6 that $e_1 \sim_G e_2$. Otherwise, $n \in \{l_1, l_2, v_1, v_2\}$ so that B meets the last column of $\tilde{A}$. We suppose that it is a column of B with exactly one 1 that is contained in the last column of $\tilde{A}$ (thus $n = v_2$). An almost identical argument carries through when it is a column of B with exactly one 0. There are two cases to consider.

Case (i). $\tilde{A}[u_1, u_2; n] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. If there is an integer $p < n$ such that

$$\tilde{A}[u_1, u_2; p] = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then let $e = \tilde{A}[u_1, u_2; \{v_1, p\}]$. Then e and e_2 are edges of a triangle in G and hence $e \sim_G e_2$.

By the inductive assumption, $e \sim_{G^*} e_1$ and hence by Lemma 6, $e \sim_G e_1$. Thus $e_1 \sim_G e_2$. But such an integer p must exist, for otherwise the monotonicity of the row sum vector R implies $r_{u_1} = r_{u_2}$, contradicting the tie-breaking rule in the construction of $\tilde{A}$.

Case (ii). $A[u_1, u_2; n] = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. If there is an integer $p < n$ such that

$$\tilde{A}[u_1, u_2; p] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.2}$$

then an argument similar to that used in Case (i) shows that $e_1 \sim_G e_2$. Hence we assume that (3.2) does not hold for any $p < n$. But now there can be no integer $q \neq v_1$ such that

$$\tilde{A}[u_1, u_2; q] = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

for otherwise $r_{u_1} > r_{u_2}$ and the u_1 and u_2 entries of the last column of $\tilde{A}$ are inconsistent with its construction. Hence for each integer j with $j \neq v_1, n$, we assume that

$$\tilde{A}[u_1, u_2; j] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $\alpha = \{i: \tilde{A}[i; v_1, n] = [0\ 1] \text{ or } [1\ 0]\}$, and let $W = \tilde{A}[\alpha; v_1, n]$. By permuting the rows of $\tilde{A}$ and its first $n-1$ columns we obtain the matrix

$$M = \left[\begin{array}{c|c|c} J & O & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline U & T & J \\ \hline Y & V & O \\ \hline X & Z & W \end{array}\right].$$

The 2×2 submatrix, e_2', of M in the upper right corner corresponds to the edge e_2 of G and some 2×2 submatrix, e_1', of Y corresponds to the edge e_1. With R'' and S'' the row and column sum vectors of M, $\mathfrak{A}(R'', S'')$ has no invariant positions. Let $G'' = G(R'', S'')$. We prove that $e_1' \sim_{G''} e_2'$, and since G'' is isomorphic to G, this will imply that $e_1 \sim_G e_2$. We consider the matrix M_{n-1} obtained from M by eliminating column n. Let $M_{n-1} \in \mathfrak{A}(R''_{n-1}, S''_{n-1})$ and let $G^{**} = G(R''_{n-1}, S''_{n-1})$. By the nature of the invariant positions of $\mathfrak{A}(R_{n-1}, S_{n-1})$ we may as before apply the inductive assumption to $G(R_{n-1}, S_{n-1})$ at the vertex $\tilde{A}_{n-1}$. Since M_{n-1} is obtained from $\tilde{A}_{n-1}$ by permuting rows and columns, we may apply the inductive assumption to G^{**} at the vertex M_{n-1}.

First suppose that U contains a 0 entry. Then M has a 3×3 submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{3.3}$$

containing e_2' in its upper right corner. Let f be the edge of G'' corresponding to the lower left 2×2 submatrix of (3.3). Then f, e_1', and e_2' are all incident at M. By the inductive assumption and Lemma 6 $f \sim_{G''} e_1'$ and by Lemmas 6 and 7, $f \sim_{G''} e_2'$. Hence $e_1' \sim_{G''} e_2'$. In a similar way one can show that when V contains a 1, $e_1' \sim_{G''} e_2'$.

We now suppose that $U = J$ and $V = O$. Since $\mathfrak{A}(R'', S'')$ has no invariant positions, it follows from Theorem 1 that X is nonvacuous and $X \neq J$. Let row i of X contain a 0 entry. Consider first the case when row i of W is $[1 \; 0]$. Then M has a 3×3 submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.4}$$

containing e_2' in its upper right corner. It follows by an argument similar to that used when (3.3) was considered that $e_1' \sim_{G''} e_2'$. It remains to consider the case where row i of W is $[0 \; 1]$ whenever row i of X contains a 0. Using row permutations we may assume that

$$[X \;\; Z \;\; W] = \left[\begin{array}{c|c|c} \overbrace{X_1 \quad\; Z_1}^{n-1} & & O \;\; J \\ \hline \underbrace{J \quad\; Z_2}_{n-2} & & W_2 \end{array}\right],$$

where X_1 is nonvacuous and each of its rows contains a 0 entry. If Z_1 is nonempty and $Z_1 \neq O$, then M has a 3×3 submatrix of the form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

containing e_2' in its upper right corner and as before, using an intermediate edge, we obtain $e_1' \sim_{G''} e_2'$. Finally let Z_1 be empty or $Z_1 = O$. Then

$$M = \left[\begin{array}{c|c|c} J & O & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline J & T & J \\ \hline Y & O & O \\ \hline X_1 & O & O \quad J \\ \hline J & Z_2 & W_2 \end{array}\right]. \tag{3.5}$$

Now (3.5) and Theorem 1 imply that the class $\mathfrak{A}(R''_{n-1}, S''_{n-1})$ has both invariant 0-positions and invariant 1-positions. Hence the same is true of the class $\mathfrak{A}(R_{n-1}, S_{n-1})$. This contradiction completes the proof of the theorem.

If the matrices in $\mathfrak{A}(R, S)$ have no rows or columns consisting of all 0's or all 1's, then the converse of Theorem 8 holds.

THEOREM 9. *Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ where $1 \leqslant r_i \leqslant n - 1$ $(i = 1, \ldots, m)$ and $1 \leqslant s_j \leqslant m - 1$ $(j = 1, \ldots, n)$. Then $G(R, S)$ is a prime graph if and only if $\mathfrak{A}(R, S)$ has no invariant positions.*

Proof. By Theorem 8, if $\mathfrak{A}(R, S)$ has no invariant positions, then $G(R, S)$ is prime. Suppose $\mathfrak{A}(R, S)$ has invariant positions, and without loss of generality assume R and S are monotone. By Theorem 1 and the fact that $1 \leqslant r_i \leqslant n - 1$ $(i = 1, \ldots, m)$ and $1 \leqslant s_j \leqslant m - 1$ $(j = 1, \ldots, n)$, there exist integers e and f with $1 \leqslant e \leqslant m - 1$ and $1 \leqslant f \leqslant n - 1$ such that every matrix A in $\mathfrak{A}(R, S)$ has the form

$$\begin{bmatrix} J_{e,f} & A_1 \\ A_2 & O_{m-e,n-f} \end{bmatrix}.$$

Let $A_k \in \mathfrak{A}(R_k, S_k)$ and $G_k = G(R_k, S_k)$ $(k = 1, 2)$. Then $G(R, S)$ is isomorphic to $G_1 \times G_2$. We show that $G_1 \times G_2$ is a nontrivial product. Suppose $\mathfrak{A}(R_k, S_k)$ consists of the single matrix A_k, namely, A_k does not contain a submatrix of the form (1.1) or (1.2). The monotonicity of R and S now implies that neither $[0\ 1]$ nor $[0\ 1]^t$ can be a submatrix of A_k. It now follows that if A_1 (resp. A_2) contains a 1 in its upper right (resp. lower left) corner, then the first row (resp. first column) of A contains only 1's contradicting $r_1 \leqslant n-1$ (resp. $s_1 \leqslant m-1$). Similarly, if either of the corners has a 0 entry, we contradict $r_m \geqslant 1$ or $s_n \geqslant 1$. Thus neither G_1 nor G_2 consists of a single vertex and G is not prime.

If $\mathfrak{A}(R, S)$ has invariant positions, then the prime factors of $G(R, S)$ correspond to certain classes with no invariant positions, which can be obtained by successive application of Theorem 1. More details of this procedure can be found in [1, pp. 184–191].

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