A bilinear Orlicz–Pettis Theorem

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ABSTRACT

Suppose $E, F, G$ are locally convex spaces, $b : E \times F \to G$ is a bilinear operator and $\lambda$ is a scalar sequence space. A series $\sum_{j=1}^{\infty} x_j$ in $E$ is $\lambda b$ multiplier convergent if for every $t = \{t_j\} \in \lambda$ there exists $x_t \in E$ such that $\sum_{j=1}^{\infty} t_j b(x_j, y) = b(x_t, y)$ for every $y \in F$. Under continuity assumptions on the linear operators $b(x, \cdot)$, we establish several versions of the Orlicz–Pettis Theorem for multiplier convergent series. Applications to spaces of continuous linear operators are given.

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In [1], Blasco, Calabuig and Signes established an interesting subseries version of the Orlicz–Pettis Theorem with respect to a bilinear operator between Banach spaces and gave applications to Pettis integration and vector measures. In this note we extend the results of [1] to Orlicz–Pettis Theorems for multiplier convergent series and bilinear operators between locally convex spaces and give applications to multiplier convergent series of continuous linear operators.

The Orlicz–Pettis Theorem is one of the earliest abstract results of functional analysis (see [4,5] for the history of the Orlicz–Pettis Theorem). It has found numerous applications in the theory of vector measures and vector integration (see [2]). The classical version of the theorem asserts that any series $\sum_{j=1}^{\infty} x_j$ in a normed space which is subseries convergent in the weak topology of the space is actually norm subseries convergent [a series $\sum_{j=1}^{\infty} x_j$ in a topological vector space $(X, \tau)$ is subseries convergent if the subseries $\sum_{j=1}^{\infty} x_{n_j}$ is $\tau$ convergent for every subsequence $\{n_j\}$]. If $\lambda$ is a scalar sequence space which contains the space $c_{00}$ of all sequences which are eventually 0, then a series $\sum_{j=1}^{\infty} x_j$ in a topological vector space $(X, \tau)$ is $\lambda$ multiplier convergent with respect to $\tau$ if the series $\sum_{j=1}^{\infty} t_j x_j$ is $\tau$ convergent for every $t = \{t_j\} \in \lambda$; the elements $\{t_j\} \in \lambda$ are called multipliers. If $m_0 = \text{span}\{e^j : j \in \mathbb{N}\}$, where $e^j$ is the sequence with 1 in the $j$th coordinate and 0 in the other coordinates, is the sequence space of all sequences with finite range, then a series $\sum_{j=1}^{\infty} x_j$ is $m_0$ multiplier convergent iff it is subseries convergent. Thus, generalizations of the classical Orlicz–Pettis Theorem can be obtained by considering multiplier convergent series with respect to different spaces $\lambda$ of multipliers. In this note, we establish versions of the Orlicz–Pettis Theorem for multiplier convergent series and bilinear operators in the spirit of [1]. Even in the case of subseries convergent series and normed spaces our methods are quite different from those in [1].

We fix the notation which will be used. Let $E$ be a vector space, $(F, \tau_F)$, $(G, \tau_G)$ Hausdorff locally convex topological vector spaces with $b : E \times F \to G$ a bilinear operator. Let $\lambda$ be a scalar sequence space containing $c_{00}$.

Analogous to [1], we say that a series $\sum_{j=1}^{\infty} x_j$ in $E$ is $\lambda b$ multiplier convergent with respect to $\tau_G$ if for every $t = \{t_j\} \in \lambda$ there exists $x_t \in E$ such that $\sum_{j=1}^{\infty} t_j b(x_j, y) = b(x_t, y)$ for every $y \in F$ (convergence in $(G, \tau_G)$).

We now establish an analogue of the Orlicz–Pettis Theorem of [1] for multiplier convergent series. For this we must impose a condition on the multiplier space $\lambda$ (see Example 4.13 of [13]). An interval in $\mathbb{N}$ is a set of the form $[m, n] = \{j \in \mathbb{N} : m \leq j \leq n\}$ and a sequence of intervals $\{I_j\}$ is increasing if $\max I_j < \min I_{j+1}$. If $A \subseteq \mathbb{N}$, $\chi_A$ will denote the characteristic function of $A$ and if $t = \{t_j\}$ is any sequence, $\chi_{At}$ will denote the coordinate product of $\chi_A$ and $t$.

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**Definition 1.** The space $\lambda$ has the signed weak gliding hump property (signed-WGHP) if whenever $t = [t_i] \in \lambda$ and $\{I_j\}$ is an increasing sequence of intervals, there exist a sequence of signs $\{s_j\}$ and a subsequence $\{n_j\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} s_j x_{i_j} t \in \lambda$. If all signs $s_j$ can be chosen equal to 1 for every $t \in \lambda$, $\lambda$ has the weak gliding hump property (WGHP).

For example, any monotone space such as $m_0, l^p (0 < p \leq \infty)$ has WGHP while $bs$, the space of bounded series, has signed-WGHP but not WGHP [9,12,13].

We now prove our main result. For the proof we employ the Antosik–Mikusinski Matrix Theorem. For the convenience of the reader, we give a statement of a version of the theorem which will be used. For the proof (of more general versions), see [12, 2.14] or [13, D.3].

**Antosik–Mikusinski Matrix Theorem.** Let $M = [m_{ij}]$ be an infinite matrix of elements of $G$. Assume (1) $\lim m_{ij} = m_j$ exists for every $j$ and (2) for every increasing sequence of positive integers $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ and a sequence of signs $\{s_j\} \in \{-1, 1\}$ such that the series $\sum_{j=1}^{\infty} s_j m_{in_{ij}}$ converges and $\lim \sum_{j=1}^{\infty} s_j m_{in_{ij}}$ exists. Then $\lim m_{ii} = 0$.

**Theorem 2.** Assume $\lambda$ has signed-WGHP and

$$(\gamma) \quad b(x, \cdot) : F \to G \text{ is } \tau_F - \tau_G \text{ sequentially continuous for every } x \in E.$$ 

If the series $\sum x_j$ is a multiplier convergent with respect to $\tau_G$, then for every $t \in \lambda$ and every sequentially $\tau_F$ compact subset $K \subset F$ the series $\sum_{j=1}^{\infty} t_j b(x_j, y)$ converge uniformly in $\tau_G$ for $y \in K$.

**Proof.** For the uniform convergent statement, it suffices to show that the series $\sum_{j=1}^{\infty} t_j b(x_j, y)$ satisfy a Cauchy condition uniformly for $y \in K$ with respect to $\tau_G$. For this it suffices to show $\tau_G - \lim_{n} \sum_{j=1}^{\infty} t_j b(x_j, y) = 0$ uniformly for $y \in K$ when $\{l_k\}$ is an increasing sequence of intervals and for this it suffices to show

$$(*) \quad \tau_G - \lim_{k} \sum_{j \in l_k} t_j b(x_j, y_k) = 0 \quad \text{for every } \{y_k\} \subset K.$$ 

Since $K$ is $\tau_F$ sequentially compact, we may assume, by passing to a subsequence if necessary, $y_k \to y \in K$ in $\tau_F$. We establish $(\ast)$ by employing the version of the Antosik–Mikusinski Matrix Theorem stated above. For this consider the matrix

$$M = [m_{ij}] = \left[b \left( \sum_{l \in I_j} t(l x_l, y) \right) \right].$$

The columns of $M$ are $\tau_G$ convergent to $b(\sum_{l \in I_j} t(l x_l, y)$ by $(\gamma)$. Next, if $\{m_j\}$ is any increasing sequence of integers, by signed-WGHP there is a sequence of signs $\{s_j\}$ and a subsequence $\{n_j\}$ of $\{m_j\}$ such that the coordinate sum $u = [u_j] = \sum_{j=1}^{\infty} s_j x_{i_{n_j}} t \in \lambda$. Then

$$\sum_{j=1}^{\infty} s_j m_{in_j} = \sum_{j=1}^{\infty} s_j b \left( \sum_{l \in I_{n_j}} t(l x_{l}, y) \right) = \sum_{j=1}^{\infty} s_j b (t(x_j, y_l)) = \sum_{j=1}^{\infty} u_j b(x_j, y_l) = b(x_u, y_l)$$

and

$$(\ast\ast) \quad \tau_G - \lim_{i} \sum_{j=1}^{\infty} s_j m_{in_j} = b(x_u, y)$$

by $(\gamma)$. We have shown that $M$ satisfies conditions (1) and (2) of the Antosik–Mikusinski Matrix Theorem stated above (with respect to $(G, \tau_G)$) so by the Antosik–Mikusinski Matrix Theorem, the diagonal of $M$ is $\tau_G$ convergent to 0 giving $(\ast)$.

**Remark 3.** The proof of Theorem 2 can be used to obtain a similar result under different hypotheses. A subset $K \subset F$ is conditionally $\tau_F$ sequentially compact if every sequence $\{y_j\} \subset K$ has a $\tau_F$ Cauchy subsequence (this terminology is essentially that of Dinculeanu [3]). We can then replace the assumption that $K$ is sequentially $\tau_F$ compact with the assumption that $K$ is conditionally $\tau_F$ sequentially compact and that $\tau_G$ is sequentially complete in Theorem 2. Under these assumptions the columns of $M$ will be $\tau_G$ convergent and the limit in $(\ast\ast)$ will exist so the Antosik–Mikusinski Theorem is again applicable.

**Remark 4.** There have been other Orlicz–Pettis Theorems relative to bilinear operators established; for example, see [7], [13, Chapter 4]. The continuity assumption $(\gamma)$ essentially introduced in [1] is new and allows for the uniform convergence conclusion. It is also this condition which allows one to obtain Orlicz–Pettis results for continuous linear operators which will be derived later.
We next observe that Theorem 2 gives a locally convex version of the Orlicz–Pettis Theorem. Suppose \( X, X' \) is a dual pair and consider the bilinear map \( b : X \times X' \to \mathbb{R} \) given by \( b(x, x') = x'(x) \). Then a series \( \sum_j x_j \) is \( \lambda \) multiplier convergent with respect to \( \sigma(X, X') \) if \( \sum_j x_j \) is \( \lambda \) \( b \) multiplier convergent with respect to \( \sigma(X, X') \) and \( b(x, \cdot) : X' \to \mathbb{R} \) is \( \sigma(X', X) - \mathbb{R} \) continuous. Let \( \nu(X, X') \) [resp. \( \gamma(X, X') \)] be the polar topology of uniform convergence on the \( \sigma(X', X) \) sequentially compact subsets of \( X' \) [resp. topology of uniform convergence on conditionally \( \sigma(X', X) \) sequentially compact subsets of \( X' \)]. From Theorem 2 and Remark 3, we then have

**Corollary 5.** Let \( \lambda \) have signed-WGHP. If the series \( \sum_j x_j \) is \( \lambda \) multiplier convergent with respect to \( \sigma(X, X') \), then the series is \( \lambda \) multiplier convergent with respect to \( \nu(X, X') \) and \( \gamma(X, X') \).

The result in Corollary 5 can be used to derive other, perhaps, more familiar versions of the Orlicz–Pettis Theorem for locally convex spaces. For example, the methods in either [10, Theorem 4], [12, 10.3.2] or [7, Theorem 2] along with Corollary 5 can be used to show that a series which is \( \lambda \) multiplier convergent in \( \sigma(X, X') \) is \( \lambda \) multiplier convergent in the topology \( \lambda(X, X') \) of uniform convergence on \( \sigma(X', X) \) compact subsets of \( X' \) (provided \( \lambda \) has signed-WGHP). The topology \( \lambda(X, X') \) is obviously stronger than the Mackey topology \( \tau(X, X') \) of uniform convergence on absolutely \( \sigma(X', X) \) subsets of \( X' \) and this is the topology often used in statements of the Orlicz–Pettis Theorem for locally convex spaces ([12, 10.3.3], [13, 4.10]).

We can use Corollary 5 and the observations above to obtain a bilinear version of the Orlicz–Pettis Theorem more in line with the version in [1].

**Corollary 6.** Let \( \lambda \) have signed-WGHP and

\[
(\gamma') \quad b(x, \cdot) : F \to G \text{ is } \sigma(F, F') - \tau_G \text{ sequentially continuous for every } x \in E.
\]

If \( \sum_j x_j \) is \( \lambda \) \( b \) multiplier convergent with respect to \( \sigma(G, G') \), then for every \( t \in \lambda \) and every sequentially \( \sigma(F, F') \) compact subset \( K \subseteq F \) the series

\[
\sum_{j=1}^{\infty} t_j b(x_j, y) = b(x_t, y)
\]

converge uniformly in \( \tau_G \) for \( y \in K \).

**Proof.** From Corollary 5 the series \( \sum_j x_j \) is \( \lambda \) \( b \) multiplier convergent with respect to \( \tau_G \) so the result follows from Theorem 2. \( \square \)

**Remark 7.** As in Remark 3 the same result holds if we assume \( K \) is conditionally \( \sigma(F, F') \) sequentially compact and \( \tau_G \) is sequentially complete.

In [1] the authors impose a condition on the bilinear operator \( b \) which allows one to conclude convergence in the space \( E \) when all spaces \( E, F \) and \( G \) are normed spaces. We now describe their condition. Let \( X, Y \) and \( Z \) be normed spaces and \( B : X \times Y' \to Z \) a separately continuous bilinear operator. The operator \( B \) satisfies condition \( (\alpha) \) if:

\[
(\alpha) \quad \text{there exists } k > 0 \text{ such that } \|x\| \leq \|B(x, \cdot)\| \text{ for all } x \in X,
\]

where \( \|B(x, \cdot)\| \) is the operator norm from \( L(Y', Z) \), the space of continuous linear operators from \( Y' \) into \( Z \). From Corollary 6, we have

**Theorem 8.** Let \( \lambda \) have signed-WGHP,

\[
(\gamma') \quad B(x, \cdot) : Y' \to Z \text{ is weak*-norm sequentially continuous}
\]

and the closed unit ball \( B_{Y'} \) of \( Y' \) be weak* sequentially compact. If \( \sum_j x_j \) is \( \lambda \) \( B \) multiplier convergent with respect to \( \sigma(Z, Z') \), then for every \( t_j \in \lambda \) the series \( \sum_{j=1}^{\infty} t_j B(x_j, \cdot) \) converges in operator norm. Thus, if \( (\alpha) \) is satisfied and \( X \) is complete, the series \( \sum_j x_j \) is \( \lambda \) multiplier convergent with respect to the norm of \( X \).

This is Theorem 1 of [1].

Similarly, if \( B : X \times Y \to Z \) is a separately continuous bilinear operator, the operator \( B \) satisfies condition \( (\alpha') \) if

\[
(\alpha') \quad \text{there exists } k > 0 \text{ such that } \|x\| \leq \|B(x, \cdot)\| \text{ for all } x \in X.
\]

From Corollary 6, we have
Then a series which states that if weak operator topology, then $X$ from SC sc topology and than the assumptions in Kalton’s theorem. See also [15] for a locally convex version of Kalton’s theorem.

From Remark 3 the same conclusion holds if we assume the closed unit ball of $X$ is sequentially compact. This condition is satisfied, for example, if $X'$ is separable.

It should be noted that from Corollary 5 a series of operators is $\lambda$ multiplier convergent in the weak operator topology when $\lambda$ has signed-WGHP.

We now give an application of Theorem 2 to spaces of continuous linear operators. Let $SC(F, G)$ be the space of linear operators from $F$ into $G$ which are $\tau_F - \tau_C$ sequentially continuous. Let $SC_s(F, G)$ be $SC(F, G)$ with the topology of pointwise convergence, i.e., the strong operator topology, and let $SC_{sc}(F, G)$ be $SC(F, G)$ with the topology of uniform convergence on sequentially $\tau_F$ compact subsets of $F$. Consider the bilinear map $b : SC(F, G) \times F \rightarrow G$ defined by $b(T, x) = Tx$. Then a series $\sum_j T_j$ in $SC(F, G)$ is $\lambda \ b$ multiplier convergent with respect to $\tau_C$ iff the series is $\lambda$ multiplier convergent in $SC(F, G)$. From Theorem 2 we have

**Theorem 11.** Let $\lambda$ have signed-WGHP. If $\sum_j T_j$ is $\lambda$ multiplier convergent in $SC_s(F, G)$, then $\sum_j T_j$ is $\lambda$ multiplier convergent in $SC_{sc}(F, G)$.

We consider some special cases of linear operators which were employed in [1]. Let $X, Y$ be normed spaces. Let $W^*(X', Y)$ be the space of all linear operators from $X'$ into $Y$ which are weak$^*$-norm sequentially continuous (these operators are bounded and have been studied extensively by Mohsen [8]). Theorem 11 is then applicable when $X'$ has the weak$^*$ topology and $SC_{sc}(X', Y)$ is $W^*(X', Y)$ with the topology of uniform convergence on weak$^*$ sequentially compact subsets of $X'$.

Another condition imposed in [1] is

$$\beta$$ the unit ball of $X'$ is weak$^*$ sequentially compact.

If condition $\beta$ is imposed in Theorem 11, we have convergence in the uniform operator topology. Let $W^*_b(X', Y)$ be $W^*(X', Y)$ equipped with the operator norm or the uniform operator topology.

**Corollary 12.** Let $\lambda$ have signed-WGHP and assume $\beta$. If $\sum_j T_j$ is $\lambda$ multiplier convergent in the weak operator topology of $W^*(X', Y)$, then $\sum_j T_j$ is $\lambda$ multiplier convergent in $W^*_b(X', Y)$.

We consider the space of completely continuous linear operators. Let $CC(X, Y)$ be the space of all linear operators from $X$ into $Y$ which are sequentially weak-norm continuous. In this case Theorem 11 is applicable when $X$ has the weak topology and $SC_{sc}(X, Y)$ is $CC(X, Y)$ with the topology of uniform convergence on weakly compact subsets of $X$. If $X$ is reflexive, then $CC(X, Y)$ is the space of all compact operators $K(X, Y)$ from $X$ into $Y$ [11, 28.1.2]; let $K_b(X, Y)$ be $K(X, Y)$ with the operator norm. Then Theorem 11 in this case becomes

**Corollary 13.** Let $\lambda$ have signed-WGHP and assume $X$ is reflexive. If $\sum_j T_j$ is $\lambda$ multiplier convergent in $K(X, Y)$ with respect to the weak operator topology, then $\sum_j T_j$ is $\lambda$ multiplier convergent in $K_b(X, Y)$.

**Remark 14.** Results with conclusions concerning multiplier convergence in the operator norm as given in Corollaries 12 and 13 are difficult to come by. They can sometimes be obtained by imposing strong conditions on the multiplier space $\lambda$; see [14], [13, 6.12, 6.13]. The strongest result for subseries convergent series of compact operators is the result of Kalton [6] which states that if $X'$ does not contain a copy of $l^\infty$, then any series $\sum_j T_j$ in $K(X, Y)$ which is subseries convergent in the weak operator topology is subseries convergent in the operator norm $K_b(X, Y)$. There are several known proofs of this result but none carry forward to the case of multiplier convergent series since the proofs use the fact that if $X'$ does not contain a copy of $l^\infty$, then a series $\sum_j x_j'$ in $X'$ is subseries convergent in the weak$^*$ topology iff the series is subseries convergent in the norm topology and there is no analogue of this result for multiplier convergent series. Corollary 13 can be regarded as a version of Kalton’s theorem for multiplier convergent series; note that the reflexivity assumption in Corollary 13 is stronger than the assumptions in Kalton’s theorem. See also [15] for a locally convex version of Kalton’s theorem.
If the multiplier space $\lambda$ satisfies stronger gliding hump properties, the conclusion of Theorem 2 can be improved to include uniform convergence over certain subsets of the multiplier space $\lambda$. We now indicate such a property.

**Definition 15.** Suppose $\lambda$ is a K-space under a Hausdorff locally convex topology (i.e., the coordinate functionals $t = \{t_j\} \rightarrow t_j$ are continuous from $\lambda$ into the scalar field). The space $\lambda$ has the signed strong gliding hump property (signed-SGHP) if whenever $B$ is a bounded subset of $\lambda$, $\{\|t_j\|\} \subset B$ and $\{l_j\}$ is an increasing sequence of intervals, then there exist a sequence of signs $\{s_j\}$ and a subsequence $\{n_j\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} s_j x_{n_j} t_j^{\infty} \in \lambda$. If all signs can be chosen equal to one for all $B, \{l_j\}$, $\lambda$ has the strong gliding hump property (SGHP).

For example, $l^\infty$ has SGHP and $bs$ has signed-SGHP but not SGHP. For further examples, see [12,13].

**Theorem 16.** Assume $\lambda$ has signed-SGHP and

$$(\gamma) \quad b(x, \cdot : F \rightarrow G \text{ is } \tau_F \rightarrow \tau_G \text{ sequentially continuous for every } x \in E.$$ 

If the series $\sum_{j=1}^{\infty} x_j$ is $\lambda$ b multiplier convergent with respect to $\tau_G$, then for every bounded subset $B$ of $\lambda$ and every sequentially $\tau_F$ compact subset $K \subset F$ the series $\sum_{j=1}^{\infty} t_j b(x_j, y) = b(x_t, y)$ converge uniformly in $\tau_G$ for $y \in K, t = \{t_j\} \subset B$.

We indicate how the proof of Theorem 2 would have to be altered. As in that proof it suffices to show

$$(*) \quad \tau_G - \lim_{k \rightarrow \infty} \sum_{j \in I_k} t_j b(x_j, y_k) = 0 \quad \text{for every } \{y_k\} \subset K \text{ and } \{t_k\} \subset B,$$

where $B$ is a bounded subset of $\lambda$. The matrix $M$ is then defined to be

$$M = [m_{ij}] = \left[ b \left( \sum_{l \in I_j} t_l x_l, y_j \right) \right].$$

Then signed-SGHP is used to check the conditions in the Antosik–Mikusinski Matrix Theorem as the signed-WGHP was used in the proof of Theorem 2.

The statements in the results following Theorem 2 can then be strengthened if signed-SGHP is assumed.

There is another gliding hump property which allows a conclusion as in Theorem 16.

**Definition 17.** Suppose $\lambda$ is a K-space. The space $\lambda$ has the zero gliding hump property (0-GHP) if whenever $\{t_j\}$ is a null sequence in $\lambda$ and $\{l_j\}$ is an increasing sequence of intervals, there is a subsequence $\{n_j\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} X_{n_j} t_j^{\infty} \in \lambda$.

For example, $c, c_0, l^p$ ($0 < p \leq \infty$) have 0-GHP; see [12,13] for further examples. As in Theorem 16, we have

**Theorem 18.** Assume $\lambda$ has 0-GHP and

$$(\gamma) \quad b(x, \cdot : F \rightarrow G \text{ is } \tau_F \rightarrow \tau_G \text{ sequentially continuous for every } x \in E.$$ 

If the series $\sum_{j=1}^{\infty} x_j$ is $\lambda$ b multiplier convergent with respect to $\tau_G$, then for every null sequence $\{t_k\} \subset \lambda$ and every sequentially $\tau_F$ compact subset $K \subset F$ the series $\sum_{j=1}^{\infty} t_j b(x_j, y) = b(x_t, y)$ converge uniformly in $\tau_G$ for $y \in K, k \in \mathbb{N}$.

**Remark 19.** As in Remark 3 we may replace the sequential compactness hypothesis in Theorems 16 and 18 by the assumption that $K$ is conditionally $\tau_F$ sequentially compact and $\tau_G$ is sequentially complete.

**References**