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Power linear keller maps of rank two are linearly triangularizable

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Abstract

Let R be a field with characteristic zero. In this paper it is proved that power linear Keller maps $R^n \to R^n$ with rank at most two are linearly triangularizable.

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Let R be a field. A *polynomial map* F over R in dimension n is an n-tuple $(F_1, F_2, \ldots, F_n)$ of polynomials in $R[X_1, \ldots, X_n]$. The *degree* of F is defined by $\deg F = \max_{i=1}^{n} \deg F_i$. If G is another polynomial map of the same dimension then the composition $F \circ G$ is defined to be the polynomial map $(F_1(G_1, G_2, \ldots, G_n), \ldots, F_n(G_1, G_2, \ldots, G_n))$. We say that F is *invertible* if there exists a polynomial map G such that $F \circ G = G \circ F = (X_1, \ldots, X_n)$. It is said to be a *Keller map* if the determinant of its Jacobian $JF = (\partial F_i / \partial X_j)$ is a nonzero element in R. By the chain rule for Jacobians, invertible polynomial maps are Keller maps. The famous Jacobian Conjecture states that if $\operatorname{char} R = 0$ then any Keller map is invertible (See [1,7] or [8]).

A polynomial map of the form $(X_1 + H_1, \ldots, X_n + H_n)$ is said to be *homogeneous* of degree d if each H_i is homogeneous in $X_1, \ldots, X_n$ of degree d. It is shown in [1] that in case $\operatorname{char} R = 0$ if homogeneous Keller maps of degree 3 are invertible then the Jacobian Conjecture would be true.

A polynomial map F is *power linear* of degree d if it is of the form $(X_1 + A_1^d, \ldots, X_n + A_n^d)$ where A_i is a linear form in $X_1, X_2, \ldots, X_n$. When $d = 3$, it is said to be *cubic linear*.

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Druzkowski [5] showed that in case char $R=0$ if cubic linear Keller maps are invertible then the Jacobian Conjecture would be true.

A polynomial map is *elementary* if it is of the form $(X_1,\dots,X_{i-1},X_i+p,X_{i+1},\dots,X_n)$ where $p\in R[X_1,\dots,X_{i-1},\hat{X}_i,X_{i+1},\dots,X_n]$. It is *tame* if it can be written as a composition of invertible linear maps and elementary maps. A polynomial map is *upper triangular* if it is of the form $(X_1+p_1,X_2+p_2,\dots,X_n+p_n)$ where p_i is a polynomial in $R[X_{i+1},X_{i+2},\dots,X_n]$. It is *lower triangular* if p_i is a polynomial in $R[X_1,\dots,X_{i-1}]$. We shall say a map is *triangular* if it is either upper or lower triangular. It is *linearly triangularizable* if it becomes triangular after conjugated by some linear invertible map.

It is not hard to see that linearly triangularizable maps are tame and that tame maps are invertible. The tame generators conjecture asserts that an invertible polynomial map is tame (see [7] or [8]). This is proved in dimension two by Jung [10] and van der Kulk [11]. Recently Shestakov and Umirbaev [14,15] proved that the conjecture is false in dimension three. Note that Rusek [13] conjectures that a quadratic Keller map in any dimension is tame.

In this paper we investigate the structure of power linear Keller maps of degree d with rank two. We prove that if $\mathrm{char}(R)=0$ they are linearly triangularizable for any degree $d\geqslant 2$. Hence these maps are also tame and invertible. More explicitly we prove the following:

Theorem 1. *Let R be a field of characteristic zero and let $d\geqslant 2$ be a positive integer. Suppose $F=(X_1+A_1^d,\dots,X_n+A_n^d):R^n\to R^n$ is a Keller polynomial map where $A_i=\sum_{j=1}^n a_{ij}X_j$ and $a_{ij}\in R$. Suppose also that* rank $A=2$ *where $A=(a_{ij})$. Then F is linearly triangularizable.*

When $R=\mathbf{C}$ and $d=2$ or 3 this gives the main results of [2,4]. The proof of [6] can be carried out verbatim to show that power linear Keller maps of corank one are linearly triangularizable. This, together with Theorem 1, implies that all power linear Keller maps of dimension four are linearly triangularizable giving rise to the main result of [3].

To prove Theorem 1 we need the following theorem which is essentially Proposition 2.9 of [7].

Theorem 2. *Let R be a field and let $F=(X_1+A_1^d,X_2+A_2^d,\dots,X_n+A_n^d)$: $R^n\to R^n$ be a Keller polynomial map where $A_i=\sum_{j=1}^n a_{ij}X_j$ and $a_{ij}\in R$. Suppose* rank $A=r$ *where $A=(a_{ij})$ and all homogeneous Keller maps of degree d with dimension r are linearly triangularizable. Then F is linearly triangularizable.*

Proof. There exists an invertible matrix $\bar{T}=(t_{ij})$ such that $A\bar{T}$ is in column echelon form with its last $n-r$ columns equal to zero. Let T be the polynomial map whose ith component is $t_{i1}X_1+\cdots+t_{in}X_n$. Then

$$(A_1^d,\dots,A_n^d)\circ T=(b_1(X_1,\dots,X_r)^d,\dots,b_n(X_1,\dots,X_r)^d),$$

where $b_i(X_1,\dots,X_r)$ is a linear form in $X_1,\dots,X_r$.

Note that T is invertible since $\bar{T}$ is an invertible matrix. Let $F_1 = T^{-1} \circ F \circ T$. Then

$$\begin{aligned} F_1 &= X + T^{-1} \circ (A_1^d, \ldots, A_n^d) \circ T \\ &= X + T^{-1} \circ (b_1(X_1, \ldots, X_r)^d, \ldots, b_n(X_1, \ldots, X_r)^d) \\ &= X + (h_1(X_1, \ldots, X_r), \ldots, h_n(X_1, \ldots, X_r)), \end{aligned}$$

where $h_i(X_1, \ldots, X_r)$ is a linear combination of $b_1(X_1, \ldots, X_r)^d, b_2(X_1, \ldots, X_r)^d, \ldots, b_n(X_1, \ldots, X_r)^d$ and thus is homogeneous of degree d. Since $\det JF_1 = \det JF = 1$, by assumption, there exists an invertible linear polynomial map L such that $L^{-1} \circ F_1 \circ L$ is triangular. Hence F is linearly triangularizable. □

Remark. Since all cubic homogeneous Keller maps of dimension three are linearly triangularizable [16], the above theorem implies that all cubic linear Keller maps of rank three are linearly triangularizable. Since all cubic linear maps of rank or corank at most two are linearly triangularizable [2,6], this implies that all cubic linear Keller maps of dimension six are linearly triangularizable. Similarly, as quadratic homogeneous Keller maps of dimension four are linearly triangularizable [9,12], all quadratic linear Keller maps of rank four are linearly triangularizable. Using the fact that quadratic linear Keller maps of rank at most two or corank at most one are linearly triangularizable [4], we deduce that all quadratic linear Keller maps of dimension six are linearly triangularizable.

Using Theorem 2, Theorem 1 can be derived from Lemma 3 below.

Lemma 3. *Let $d \geqslant 2$ and let R be a field with characteristic zero. Suppose $F = (X + H, Y + K)$: $R^2 \to R^2$ is a Keller polynomial map where H, K are homogeneous in X, Y of degree d. Then F is linearly triangularizable.*

Proof. Since (H, K) is homogeneous of degree $d \geqslant 2$ it is well-known [8, 6.2.11] that $\operatorname{tr} J(H, K) = 0$, i.e. $H_X + K_Y = 0$ and $\det J(H, K) = 0$. The last equality implies that H and K are algebraically dependent over R [8, 1.1.3]. So $P(H, K) = 0$ for some non-zero polynomial $P \in R[t_1, t_2]$, which we may assume to be homogeneous since H and K are homogeneous of the same degree. Writing P as a product of linear factors over $\bar{R}$, the algebraic closure of R, it follows that H and K are linearly dependent over $\bar{R}$ and hence over R, say $c_1 H + c_2 K = 0$ with $c_2 \neq 0$. Let T be the 2×2 matrix with first row $(1, 0)$ and second row (c_1, c_2). Then T is invertible and the second component of $T \circ (H, K)$ equals zero. So $T \circ (H, K) \circ T^{-1} = (a, 0)$ where $a \in R[X, Y]$. Finally observe that since $J(H, K)$ is nilpotent, the same holds for $J(T \circ (H, K) \circ T^{-1}) = TJ(H, K)(T^{-1}(X, Y))T^{-1}$. So $J(a, 0)$ is nilpotent. In particular $\operatorname{tr} J(a, 0) = 0$, i.e. $a_X = 0$. So $a = a(Y)$ and hence $T \circ F \circ T^{-1} = (X + a(Y), Y)$, which completes the proof. □

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