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The singular limit of the Allen–Cahn equation and the FitzHugh–Nagumo system

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Abstract

We consider an Allen–Cahn type equation of the form $u_t = \Delta u + \varepsilon^{-2} f^\varepsilon(x,t,u)$, where ε is a small parameter and $f^\varepsilon(x,t,u) = f(u) - \varepsilon g^\varepsilon(x,t,u)$ a bistable nonlinearity associated with a double-well potential whose well-depths can be slightly unbalanced. Given a rather general initial data u_0 that is independent of ε, we perform a rigorous analysis of both the generation and the motion of interface. More precisely we show that the solution develops a steep transition layer within the time scale of order $\varepsilon^2|\ln \varepsilon|$, and that the layer obeys the law of motion that coincides with the formal asymptotic limit within an error margin of order ε. This is an optimal estimate that has not been known before for solutions with general initial data, even in the case where $g^\varepsilon \equiv 0$.

Next we consider systems of reaction–diffusion equations of the form

$$\begin{cases} u_t = \Delta u + \varepsilon^{-2} f^\varepsilon(u,v), \\ v_t = D\Delta v + h(u,v), \end{cases}$$

which include the FitzHugh–Nagumo system as a special case. Given a rather general initial data (u_0, v_0), we show that the component u develops a steep transition layer and that all the above-mentioned results remain true for the u-component of these systems.

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Keywords: Nonlinear PDE; Reaction–diffusion system; Singular perturbation; Allen–Cahn; FitzHugh–Nagumo; Interface motion

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1. Introduction

1.1. Perturbed Allen–Cahn equation

In some classes of nonlinear diffusion equations, solutions often develop internal transition layers—or "interfaces"—that separate the spatial domain into different phase regions. This happens, in particular, when the diffusion coefficient is very small or the reaction term is very large. The motion of such interfaces is often driven by their curvature. A typical example is the Allen–Cahn equation $u_t = \Delta u + \varepsilon^{-2} f(u)$, where $\varepsilon > 0$ is a small parameter and $f(u)$ is a bistable nonlinearity, whose meaning will be explained below. A usual strategy for studying such phenomena is to first derive the "sharp interface limit" as $\varepsilon \to 0$ by a formal analysis, then to check if this limit gives good approximation of the behavior of actual layers.

In this paper we study a perturbed Allen–Cahn type equation of the form

$$(P^\varepsilon) \quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big(f(u) - \varepsilon g^\varepsilon(x,t,u)\big) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

and study the behavior of layers near the sharp interface limit as $\varepsilon \to 0$. Here Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ and ν is the Euclidean unit normal vector exterior to $\partial\Omega$. The nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = \alpha_\pm$. More precisely we assume that f is C^2 and has exactly three zeros $\alpha_- < a < \alpha_+$ such that

$$f'(\alpha_\pm) < 0, \qquad f'(a) > 0 \quad \text{(bistable nonlinearity)}, \tag{1.1}$$

and that

$$\int_{\alpha_-}^{\alpha_+} f(u)\,du = 0. \tag{1.2}$$

The condition (1.1) implies that the potential $W(u)$ attains its local minima at $u = \alpha_-, \alpha_+$, and (1.2) implies that $W(\alpha_-) = W(\alpha_+)$. In other words, the two stable zeros of f, namely α_- and α_+, have "balanced" stability. A typical example is the cubic nonlinearity $f(u) = u(1-u^2)$.

The term $\varepsilon g^\varepsilon$ represents a small perturbation, where $g^\varepsilon(x,t,u)$ is a function defined on $\overline{\Omega} \times [0,+\infty) \times \mathbb{R}$. This has the role of breaking the balance of the two stable zeros slightly. In the special case where $g^\varepsilon \equiv 0$, problem (P^ε) reduces to the usual Allen–Cahn equation. As we will explain later, our main results are new even for this special case.

We assume that g^ε is C^2 in x and C^1 in t, u, and that, for any $T > 0$ there exist $\vartheta \in (0,1)$ and $C > 0$ such that, for all $(x,t,u) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$,

$$\big|\Delta_x g^\varepsilon(x,t,u)\big| \leqslant C\varepsilon^{-1} \quad \text{and} \quad \big|g_t^\varepsilon(x,t,u)\big| \leqslant C\varepsilon^{-1}, \tag{1.3}$$

$$\big|g_u^\varepsilon(x,t,u)\big| \leqslant C, \tag{1.4}$$

$$\big\|g^\varepsilon(\cdot,\cdot,u)\big\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leqslant C. \tag{1.5}$$

Moreover, we assume that there exists a function $g(x,t,u)$ and a constant, which we denote again by C, such that

$$\left|g^\varepsilon(x,t,u) - g(x,t,u)\right| \leqslant C\varepsilon, \tag{1.6}$$

for all small $\varepsilon > 0$. Note that the estimate (1.5) and the pointwise convergence $g^\varepsilon \to g$ (as $\varepsilon \to 0$) imply that g satisfies the same estimate as (1.5). For technical reasons we also assume that

$$\frac{\partial g^\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0,T] \times \mathbb{R}, \tag{1.7}$$

which, in turn, implies the same boundary condition for g. Apart from these bounds and regularity requirements, we do not make any specific assumptions on the perturbation term g^ε.

Remark 1.1. Since we will consider only bounded solutions in this paper, it is sufficient to assume (1.3)–(1.5) to hold in some bounded interval $-M \leqslant u \leqslant M$. Note that if g^ε does not depend on ε, then the assumptions (1.3)–(1.5) are automatically satisfied on any interval $-M \leqslant u \leqslant M$.

Remark 1.2. The reason why we do not assume more smoothness on g is that we will later apply our results to systems of equations including FitzHugh–Nagumo system, in which g^ε loses $C^{2,1}$-smoothness as $\varepsilon \to 0$.

As for the initial data $u_0(x)$, we assume $u_0 \in C^2(\overline{\Omega})$. Throughout the present paper the constant C_0 will stand for the following quantity:

$$C_0 := \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|\Delta u_0\|_{C^0(\overline{\Omega})}. \tag{1.8}$$

Furthermore we define the "initial interface" Γ_0 by

$$\Gamma_0 := \left\{x \in \Omega,\ u_0(x) = a\right\}, \tag{1.9}$$

and suppose that Γ_0 is a $C^{3+\vartheta}$ hypersurface without boundary such that, n being the outward unit normal vector to Γ_0,

$$\Gamma_0 \Subset \Omega \quad \text{and} \quad \nabla u_0(x)\cdot n(x) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{1.10}$$

$$u_0 > a \quad \text{in } \Omega_0^+, \qquad u_0 < a \quad \text{in } \Omega_0^-, \tag{1.11}$$

where Ω_0^- denotes the region enclosed by Γ_0 and Ω_0^+ the region enclosed between $\partial\Omega$ and Γ_0.

It is standard that problem (P^ε) has a unique smooth solution, which we denote by u^ε. As $\varepsilon \to 0$, a formal asymptotic analysis shows the following: in the very early stage, the diffusion term Δu is negligible compared with the reaction term $\varepsilon^{-2}(f(u) - \varepsilon g^\varepsilon(x,t,u))$ so that, in the rescaled time scale $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u) + O(\varepsilon)$. Hence, in view of the profile of f, the value of u^ε quickly becomes close to either α_+ or α_- in most part of Ω, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. Once such an interface develops, the diffusion term becomes large near the interface, and comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a much slower time scale.

To study such interfacial behavior, it is useful to consider a formal asymptotic limit of (P^ε) as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(x,t)$ will be a step function taking the value α_+ on one side of the interface, and α_- on the other side. This sharp interface, which we will denote by Γ_t, obeys a certain law of motion, which is expressed as follows (see Section 2 for details):

$$(P^0) \quad \begin{cases} V_n = -(N-1)\kappa + c_0\big(G(x,t,\alpha_+) - G(x,t,\alpha_-)\big) & \text{on } \Gamma_t, \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity of Γ_t in the exterior direction, κ the mean curvature at each point of Γ_t,

$$c_0 = \left[\sqrt{2}\int_{\alpha_-}^{\alpha_+} \big(W(s) - W(\alpha_-)\big)^{1/2}\, ds\right]^{-1},$$

$$W(s) = -\int_a^s f(r)\, dr, \qquad G(x,t,s) = \int_a^s g(x,t,r)\, dr. \tag{1.12}$$

It is well known that problem (P^0) possesses locally in time a unique smooth solution. Let $0 \leqslant t < T^{\max}$, $T^{\max} \in (0,+\infty]$, be the maximal time interval for the existence of the solution of (P^0) and denote this solution by $\Gamma = \bigcup_{0\leqslant t<T^{\max}} (\Gamma_t \times \{t\})$. Hereafter, we fix T such that $0 < T < T^{\max}$ and work on $[0,T]$. More precisely, so as $g(\cdot,\cdot,u)$, the function $G(\cdot,\cdot,u)$ is of class $C^{1+\vartheta, \frac{1+\vartheta}{2}}$, which implies, by the standard theory of parabolic equations, that Γ is of class $C^{3+\vartheta, \frac{3+\vartheta}{2}}$. For more details, we refer to [8, Lemma 2.1].

Next we set

$$Q_T := \Omega \times (0,T),$$

and, for each $t \in [0,T]$, we denote by Ω_t^- the region enclosed by the hypersurface Γ_t, and by Ω_t^+ the region enclosed between $\partial\Omega$ and Γ_t. We define a step function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) = \begin{cases} \alpha_+ & \text{in } \Omega_t^+, \\ \alpha_- & \text{in } \Omega_t^-, \end{cases} \quad \text{for } t \in [0,T], \tag{1.13}$$

which represents the formal asymptotic limit of u^ε (or the *sharp interface limit*) as $\varepsilon \to 0$.

The aim of the present paper is to make a detailed study of the limiting behavior of the solution u^ε of problem (P^ε) as $\varepsilon \to 0$. Our first main result, Theorem 1.3, describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial data u_0, the solution u^ε quickly becomes close to $\alpha_\pm$, except in a small neighborhood of the initial interface Γ_0, creating a steep transition layer around Γ_0 (*generation of interface*). The time needed to develop such a transition layer, which we will denote by t^ε, is of order $\varepsilon^2|\ln\varepsilon|$. The theorem then states that the solution u^ε remains close to the step function $\tilde{u}$ on the time interval $[t^\varepsilon, T]$ (*motion of interface*); in other words, the motion of the transition layer is well approximated by the limit interface equation (P^0).

Theorem 1.3 *(Generation and motion of interface). Let η be an arbitrary constant satisfying $0 < \eta < \min(a - \alpha_-, \alpha_+ - a)$ and set*

$$\mu = f'(a).$$

Then there exist positive constants ε_0 and C such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t^\varepsilon \leqslant t \leqslant T$, where $t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|$, we have

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{if } x \in \mathcal{N}_{C\varepsilon}(\Gamma_t), \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if } x \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if } x \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{1.14}$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega,\ \operatorname{dist}(x, \Gamma_t) < r\}$ denotes the r-neighborhood of Γ_t.

Corollary 1.4 *(Convergence). As $\varepsilon \to 0$, u^ε converges to $\tilde{u}$ everywhere in $\bigcup_{0<t\leqslant T}(\Omega_t^\pm \times \{t\})$.*

The next theorem is concerned with the relation between the actual interface $\Gamma_t^\varepsilon := \{x \in \Omega,\ u^\varepsilon(x,t) = a\}$ and the formal asymptotic limit Γ_t, which is given as the solution of (P^0).

Theorem 1.5 *(Error estimate). There exists $C > 0$ such that*

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{for } 0 \leqslant t \leqslant T. \tag{1.15}$$

Corollary 1.6 *(Convergence of interface). There exists $C > 0$ such that*

$$d_{\mathcal{H}}\big(\Gamma_t^\varepsilon, \Gamma_t\big) \leqslant C\varepsilon \quad \text{for } 0 \leqslant t \leqslant T, \tag{1.16}$$

where $d_{\mathcal{H}}(A,B) := \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$ denotes the Hausdorff distance between two compact sets A and B. Consequently, $\Gamma_t^\varepsilon \to \Gamma_t$ as $\varepsilon \to 0$ uniformly in $0 \leqslant t \leqslant T$, in the sense of Hausdorff distance.

Note that the estimates (1.15) and (1.16) follow from Theorem 1.3 in the range $t^\varepsilon \leqslant t \leqslant T$, but the range $0 \leqslant t \leqslant t^\varepsilon$ has to be treated by a separate argument since the behavior of the solution in this time range is quite different from that of the later stage.

The estimate (1.14) in our Theorem 1.3 implies that, once a transition layer is formed, its thickness remains within order ε for the rest of time. Here, by "thickness of interface" we mean the smallest $r > 0$ satisfying

$$\big\{x \in \Omega,\ u^\varepsilon(x,t) \notin [\alpha_- - \eta, \alpha_- + \eta] \cup [\alpha_+ - \eta, \alpha_+ + \eta]\big\} \subset \mathcal{N}_r\big(\Gamma_t^\varepsilon\big).$$

Naturally this quantity depends on η, but the estimates (1.14) and (1.16) assert that it is bounded by $2C\varepsilon$ (with the constant C depending on η) regardless of the choice of $\eta > 0$.

Remark 1.7 *(Optimality of the thickness estimate).* The above $O(\varepsilon)$ estimate is optimal, i.e. the interface cannot be thinner than this order. In fact, rescaling time and space as $\tau := t/\varepsilon^2$, $y := x/\varepsilon$, we get

$$u_\tau = \Delta_y u + f(u) - \varepsilon g^\varepsilon.$$

Thus, by the uniform boundedness of u and by standard parabolic estimates, we have $|\nabla_y u| \leqslant M$ for some constant $M > 0$, which implies

$$\left|\nabla_x u(x,t)\right| \leqslant \frac{M}{\varepsilon}.$$

From this bound it is clear that the thickness of interface cannot be smaller than $M^{-1}(\alpha_+ - \alpha_-)\varepsilon$, hence, by (1.14), it has to be exactly of order ε. Intuitively, this $O(\varepsilon)$ estimate follows also from the formal asymptotic expansion (2.3), but the validity of such an expansion is far from obvious for solutions with arbitrary initial data.

Our $O(\varepsilon)$ estimate is new, even in the special case where $g^\varepsilon \equiv 0$, provided that $N \geqslant 2$. Previously, the best thickness estimate in the literature was of order $\varepsilon|\ln \varepsilon|$ (see [6]), except that X. Chen has recently obtained an $O(\varepsilon)$ estimate for the case $N = 1$ by a different argument (private communication). We also refer to the forthcoming papers [17] and [16], in which the same $O(\varepsilon)$ estimate is established for different but related problems. The paper [17] is concerned with a "balanced type" Allen–Cahn equation with large spatial inhomogeneity, namely an equation of the form $u_t = \nabla(k(x)\nabla u) + \varepsilon^{-2}h(x)f(u)$, and [16] is concerned with a Lotka–Volterra competition-diffusion system with large spatial inhomogeneity whose nonlinearity is of the balanced bistable type.

Remark 1.8 *(Optimality of the generation time).* The estimate (1.14) also implies that the generation of interface takes place within the time span of t^ε. This estimate is optimal. In other words, a well-developed interface cannot appear much earlier; see Proposition 3.10 for details.

The singular limit of the Allen–Cahn equation was first studied in the pioneering work of Allen and Cahn [2] and, slightly later, in Kawasaki and Ohta [20] from the point of view of physicists. They derived the interface equation by formal asymptotic analysis, thereby revealing that the interface moves by the mean curvature. These early observations triggered a flow of mathematical studies aiming at rigorous justification of the above limiting procedure; see, for example, [5–7] and [21,22] for results in the framework of classical solutions, and [3,4,13] and [18] for the case where Γ_t is a viscosity solution of the interface equation.

As for problem (P^ε), whose nonlinearity is slightly unbalanced, the limit interface equation involves a pressure term as well as the curvature term as indicated in (P^0). This fact has been long known on a formal level; see e.g. [24]. Ei, Iida and Yanagida [12] proved rigorously that the motion of the layers of (P^ε) is well approximated by the limit interface equation (P^0), on the condition that the initial data has already a well-developed transition layer. In other words, they studied the motion of interface, but not the generation of interface.

1.2. Singular limit of reaction–diffusion systems

Our results can be extended to reaction–diffusion systems of the form

$$
(RD^\varepsilon) \quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f^\varepsilon(u, v) & \text{in } \Omega \times (0, +\infty), \\ v_t = D\Delta v + h(u, v) & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ v(x, 0) = v_0(x) & \text{in } \Omega, \end{cases}
$$

where D is a positive constant, and f^ε, h are C^2 functions such that

(F) there exist C^2 functions $f_1(u, v)$, $f_2^\varepsilon(u, v)$ such that

$$
f^\varepsilon(u, v) = f(u) + \varepsilon f_1(u, v) + \varepsilon^2 f_2^\varepsilon(u, v), \tag{1.17}
$$

where $f(u)$ is a bistable nonlinearity satisfying (1.1), (1.2), and f_2^ε, along with its derivatives in u, v, remain bounded as $\varepsilon \to 0$;

(H) for any constants $L, M > 0$ there exists a constant $M_1 \geqslant M$ such that

$$
h(u, -M_1) \geqslant 0 \geqslant h(u, M_1) \quad \text{for } |u| \leqslant L. \tag{1.18}
$$

The conditions (F) and (H) imply that the ODE system

$$
\dot{u} = \frac{1}{\varepsilon^2} f^\varepsilon(u, v), \qquad \dot{v} = h(u, v)
$$

has a family of invariant rectangles of the form $\{|u| \leqslant L,\ |v| \leqslant M\}$, provided that ε is sufficiently small. The maximum principle and standard parabolic estimates then guarantee that every solution of (RD^ε) exists globally for $t \geqslant 0$ and remains bounded as $t \to \infty$ (see Proposition 7.1). Apart from (1.18), we do not make any specific assumptions on the function h.

Problem (RD^ε) represents a large class of important reaction–diffusion systems including the FitzHugh–Nagumo system

$$
\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big(f(u) - \varepsilon v\big), \\ v_t = D\Delta v + \alpha u - \beta v, \end{cases} \tag{1.19}
$$

which is a simplified model for nervous transmission, and the following type of prey–predator system:

$$
\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big((1-u)(u-1/2) - \varepsilon v\big)u, \\ v_t = D\Delta v + (\alpha u - \beta v)v. \end{cases} \tag{1.20}
$$

Remark 1.9. In some equations such as the prey–predator system (1.20), only nonnegative solutions are to be considered. In such a case, we replace the condition (1.18) by

$$
h(u, 0) \geqslant 0 \geqslant h(u, M_1) \quad \text{for } 0 \leqslant u \leqslant L,
$$

and assume $f^\varepsilon(0, v) \geqslant 0$. The rest of the argument remains the same.

Now the same formal analysis as is used to derive (P^0) shows that the singular limit of (RD^ε) as $\varepsilon \to 0$ is the following moving boundary problem:

$$(RD^0) \quad \begin{cases} V_n = -(N-1)\kappa - c_0 F_1\big(\tilde{v}(x,t)\big) & \text{on } \Gamma_t, \\ \tilde{v}_t = D\Delta\tilde{v} + h(\tilde{u},\tilde{v}) & \text{in } \Omega\times(0,+\infty), \\ \Gamma_t|_{t=0} = \Gamma_0, & \\ \dfrac{\partial\tilde{v}}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ \tilde{v}(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $\tilde{u}$ is the step function defined in (1.13) and

$$F_1(v) = \int_{\alpha_-}^{\alpha_+} f_1(r,v)\,dr.$$

This is a system consisting of an equation of surface motion and a partial differential equation. Since $\tilde{u}$ is determined straightforwardly from Γ_t, in what follows, by a solution of (RD^0) we mean the pair $(\Gamma,\tilde{v}) := (\Gamma_t, \tilde{v}(x,t))$. In the case of the FitzHugh–Nagumo system (1.19), (RD^0) reduces to

$$\begin{cases} V_n = -(N-1)\kappa + c_0(\alpha_+ - \alpha_-)\tilde{v}(x,t), \\ \tilde{v}_t = D\Delta\tilde{v} + \alpha\tilde{u} - \beta\tilde{v}, \end{cases}$$

while in the prey–predator system (1.20), (RD^0) reduces to

$$\begin{cases} V_n = -(N-1)\kappa + c_0\tilde{v}(x,t)/2, \\ \tilde{v}_t = D\Delta\tilde{v} + (\alpha\tilde{u} - \beta\tilde{v})\tilde{u}. \end{cases}$$

Note that the positive sign in front of the term $c_0\tilde{v}(x,t)$ in the interface equation implies an inhibitory effect on $\tilde{u}$, since the velocity V_n is measured in the exterior normal direction, toward which $\tilde{u}$ decreases.

Lemma 1.10 *(Local existence). Assume that $v_0 \in C^2(\overline{\Omega})$ and that Γ_0 is a C^2 hypersurface which is the boundary of a domain $D_0 \Subset \Omega$. Then there exists $T^{\max} \in (0,+\infty]$ such that the limit free boundary problem (RD^0) has a unique solution $(\Gamma,\tilde{v})$ in the interval $[0,T^{\max})$.*

This existence result was established in [9]. The uniqueness can be obtained by using the estimates in [7].

Hereafter, we fix T such that $0 < T < T^{\max}$ and work on $[0,T]$. Our main results for the system (RD^ε) are the following:

Theorem 1.11 *(Thickness of interface). Let* (1.17) *and* (1.18) *hold* (*or let the assumptions in Remark* 1.9 *hold*). *Assume also that u_0 satisfies* (1.10) *and* (1.11). *Then the same conclusion as in Theorem* 1.3 *holds for* (RD^ε).

Corollary 1.12 *(Convergence). Under the assumptions of Theorem* 1.11, *the same conclusion as in Corollary* 1.4 *holds for* (RD^ε).

Theorem 1.13 *(Error estimate). Let the assumptions of Theorem* 1.11 *hold. Then the same conclusion as in Theorem* 1.5 *holds for* (RD^ε). *Moreover, there exists a constant* $C > 0$ *such that*

$$\left\| v^\varepsilon - \tilde{v} \right\|_{L^\infty(\Omega \times (0,T))} \leqslant C\varepsilon.$$

Corollary 1.14 *(Convergence of interface). Under the assumptions of Theorem* 1.11*, the same conclusion as in Corollary* 1.6 *holds for* (RD^ε).

The organization of this paper is as follows. In Section 2, we derive the interface equation (P^0) from (P^ε) by formal asymptotic expansions which involve the so-called signed distance function.

In Sections 3 and 4, we present basic estimates concerning the generation of interface for (P^ε). For the clarity of underlying ideas, we first consider the special case where $g^\varepsilon \equiv 0$ in Section 3, and deal with the general case in Section 4.

In Section 5 we prove a preliminary result on the motion of interface (Lemma 5.1), which implies that if the initial data has already a well-developed transition layer, then the layer remains to exist for $0 \leqslant t \leqslant T$ and its motion is well approximated by the interface equation (P^0).

Our approach in Sections 3 to 5 is based on the sub- and super-solution method, but we use two completely different sets of sub- and super-solutions. More precisely, the sub- and super-solutions for the motion of interface are constructed by using the first two terms of the formal asymptotic expansion (2.3), while those for the generation of interface are constructed by modifying the solution of the equation in the absence of diffusion: $u_t = \varepsilon^{-2} f(u)$.

In Section 6, we prove our main results for (P^ε): Theorems 1.3, 1.5 and their respective corollaries.

In the final section, we study the reaction–diffusion system (RD^ε) and prove Theorems 1.11, 1.13 and their corollaries. These results are obtained by applying a slightly modified version of the results for (P^ε). The strategy is to regard $f^\varepsilon(u, v)$ as a perturbation of $f(u)$. Indeed, the equation for u in (RD^ε) is identical to (P^ε) if we set $g^\varepsilon = -f_1 - \varepsilon f_2^\varepsilon$. However, what makes the analysis difficult is the fact that g^ε is no longer a given function but a quantity that depends on the unknown function v^ε. In particular, the existence of the limit $g^\varepsilon \to g(\varepsilon \to 0)$ is not a priori guaranteed, and the estimate (1.6) is far from obvious. As it turns out, the standard L^p or Schauder estimates for v^ε would not yield (1.6), because of the fact that u^ε converges to a discontinuous function as $\varepsilon \to 0$. In order to overcome this difficulty, we derive a fine estimate of v^ε that is based on estimates of the heat kernel and the fact that u^ε remains uniformly smooth outside of an $O(\varepsilon)$ neighborhood of the smooth hypersurface Γ_t.

2. Formal derivation of the interface motion equation

In this section we derive the equation of interface motion corresponding to problem (P^ε) by using a formal asymptotic expansion. The resulting interface equation can be regarded as the singular limit of (P^ε) as $\varepsilon \to 0$. Our argument is basically along the same lines with the formal derivation given by Nakamura, Matano, Hilhorst and Schätzle [23], who studied a similar but slightly different type of spatially inhomogeneous equations by formal analysis. Let us also mention some earlier papers [1,15] and [24] involving the method of matched asymptotic expansions for problems that are related to ours.

As in [23], the first two terms of the asymptotic expansion determine the interface equation. Though our analysis in this section is for the most part formal, the observations we make here will help the rigorous analysis in later sections.

Let u^ε be the solution of (P^ε). We recall that $\Gamma_t^\varepsilon := \{x \in \Omega,\ u^\varepsilon(x,t) = a\}$ is the interface at time t and call $\Gamma^\varepsilon := \bigcup_{t\geqslant 0}(\Gamma_t^\varepsilon \times \{t\})$ the interface. Let $\Gamma = \bigcup_{0\leqslant t\leqslant T}(\Gamma_t \times \{t\})$ be the unique solution of the limit geometric motion problem (P^0) and let $\tilde d$ be the signed distance function to Γ defined by

$$\tilde d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+, \\ -\operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{2.1}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from x to the hypersurface Γ_t in Ω. We remark that $\tilde d = 0$ on Γ and that $|\nabla \tilde d| = 1$ in a neighborhood of Γ. We then define

$$Q_T^+ = \bigcup_{0<t\leqslant T} \big(\Omega_t^+ \times \{t\}\big), \qquad Q_T^- = \bigcup_{0<t\leqslant T} \big(\Omega_t^- \times \{t\}\big).$$

We also assume that the solution u^ε has the expansions

$$u^\varepsilon(x,t) = \alpha_\pm + \varepsilon u_1^\pm(x,t) + \varepsilon^2 u_2^\pm(x,t) + \cdots \quad \text{in } Q_T^\pm \tag{2.2}$$

away from the interface Γ (the outer expansion), and

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots \tag{2.3}$$

near Γ (the inner expansion), where $U_j(x,t,z)$, $j = 0,1,2,\ldots$, are defined for $x \in \overline{\Omega}$, $t \geqslant 0$, $z \in \mathbb{R}$ and $\xi := d^\varepsilon(x,t)/\varepsilon$, where $d^\varepsilon(x,t)$ is of the form $d^\varepsilon(x,t) = \tilde d(x,t) + \varepsilon d_1(x,t) + \varepsilon^2 d_2(x,t) + \cdots$. The stretched space variable ξ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. We normalize U_0 in such a way that

$$U_0(x,t,0) = a$$

(normalization conditions). To make the inner and outer expansions consistent, we require that

$$\begin{aligned} &U_0(x,t,+\infty) = \alpha_+, \qquad U_k(x,t,+\infty) = u_k^+(x,t), \\ &U_0(x,t,-\infty) = \alpha_-, \qquad U_k(x,t,-\infty) = u_k^-(x,t), \end{aligned} \tag{2.4}$$

for all $k \geqslant 1$ (matching conditions). As we will see later, the normalization condition and the matching condition for $k = 0$ will determine U_0 uniquely, which will then determine U_1.

In what follows we will substitute the inner expansion (2.3) into the parabolic equation of problem (P^ε) and collect the ε^{-2} and ε^{-1} terms. To that purpose we compute the needed terms and get

$$u_t^\varepsilon = U_{0t} + U_{0z}\frac{\tilde d_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\tilde d_t + \cdots,$$

$$\nabla u^\varepsilon = \nabla U_0 + U_{0z}\frac{\nabla \tilde d}{\varepsilon} + \varepsilon \nabla U_1 + U_{1z}\nabla \tilde d + \cdots,$$

$$\begin{aligned}\Delta u^\varepsilon &= \Delta U_0 + 2\frac{\nabla\tilde d}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\tilde d}{\varepsilon} + U_{0zz}\frac{|\nabla\tilde d|^2}{\varepsilon^2} + \varepsilon\Delta U_1\\ &\quad + 2\nabla\tilde d\cdot\nabla U_{1z} + U_{1z}\Delta\tilde d + U_{1zz}\frac{|\nabla\tilde d|^2}{\varepsilon} + \cdots,\\ f\big(u^\varepsilon\big) &= f(U_0) + \varepsilon f'(U_0)U_1 + O\big(\varepsilon^2\big),\\ g^\varepsilon\big(x,t,u^\varepsilon\big) &= g\big(x,t,u^\varepsilon\big) + O(\varepsilon) \quad \big(\leftarrow \text{in view of (1.6)}\big)\\ &= g(x,t,U_0) + O(\varepsilon),\end{aligned}$$

where the functions U_i $(i = 0,1)$, as well as their derivatives, are taken at the point $(x,t,\tilde d(x,t)/\varepsilon)$. Note also that ∇ and Δ stand for ∇_x and Δ_x, respectively. Collecting the ε^{-2} terms yields

$$U_{0zz} + f(U_0) = 0.$$

In view of the normalization and matching conditions, we can now assert that $U_0(x,t,z) = U_0(z)$, where $U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0,\\ U_0(-\infty) = \alpha_-, \quad U_0(0) = a, \quad U_0(+\infty) = \alpha_+. \end{cases} \tag{2.5}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. Note that the integral condition (1.2) guarantees the existence of a solution of (2.5). For example, in the simple case where $f(u) = u(1-u^2)$, we have $U_0(z) = \tanh(z/\sqrt{2})$. In the general case, the following standard estimates hold:

Lemma 2.1. *There exist positive constants C and λ such that*

$$\begin{aligned} 0 < \alpha_+ - U_0(z) \leqslant Ce^{-\lambda|z|} \quad \textit{for } z \geqslant 0,\\ 0 < U_0(z) - \alpha_- \leqslant Ce^{-\lambda|z|} \quad \textit{for } z \leqslant 0. \end{aligned}$$

In addition, U_0 is a strictly increasing function and, for $j = 1, 2$,

$$\big|D^j U_0(z)\big| \leqslant Ce^{-\lambda|z|} \quad \textit{for } z \in \mathbb{R}. \tag{2.6}$$

Proof. We only give an outline. Rewriting the equation in (2.5) as

$$\dot u = v, \qquad \dot v = -f(u),$$

we see that $(U_0(z), U_0'(z))$ is a heteroclinic orbit of the above system connecting the equilibria $(\alpha_-, 0)$ and $(\alpha_+, 0)$. These equilibria are saddle points, with the linearized eigenvalues $\{\lambda_-, -\lambda_-\}$ and $\{\lambda_+, -\lambda_+\}$, respectively, where

$$\lambda_- = \sqrt{-f'(\alpha_-)}, \qquad \lambda_+ = \sqrt{-f'(\alpha_+)}.$$

Consequently, we have

$$U_0(z) = \begin{cases} \alpha_- + C_1 e^{\lambda_- z} + o(e^{\lambda_- z}) & \text{as } z \to -\infty, \\ \alpha_+ + C_2 e^{-\lambda_+ z} + o(e^{-\lambda_+ z}) & \text{as } z \to +\infty, \end{cases} \tag{2.7}$$

for some constants C_1, C_2. The desired estimates now follow by setting $\lambda = \min(\lambda_+, \lambda_-)$. □

Next we collect the ε^{-1} terms. Recalling that $\nabla U_{0z} = 0$ and that $|\nabla \tilde{d}| = 1$ near Γ_t, we get

$$U_{1zz} + f'(U_0)U_1 = U_0'(\tilde{d}_t - \Delta \tilde{d}) + g(x, t, U_0). \tag{2.8}$$

This equation can be seen as a linearized problem for (2.5) with an inhomogeneous term. As is well known (see, for instance, [23]), the solvability condition for the above equation plays the key role in determining the equation of interface motion. The following lemma is rather standard, but we give an outline of the proof for the convenience of the reader.

Lemma 2.2 *(Solvability condition). Let $A(z)$ be a bounded function on $-\infty < z < \infty$. Then the problem*

$$\begin{cases} \psi_{zz} + f'(U_0(z))\psi = A(z), & z \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi \in L^\infty(\mathbb{R}), \end{cases} \tag{2.9}$$

has a solution if and only if

$$\int_{\mathbb{R}} A(z) U_0'(z)\, dz = 0. \tag{2.10}$$

Moreover the solution, if it exists, is unique and satisfies

$$\left|\psi(z)\right| \leqslant C \|A\|_{L^\infty} \quad \text{for } z \in \mathbb{R}, \tag{2.11}$$

for some constant $C > 0$.

Proof. Multiplying the equation by U_0' and integrating it by parts, we easily see that the condition (2.10) is necessary. Conversely, suppose that this condition is satisfied. Then, since U_0' is a bounded positive solution to the homogeneous equation $\psi_{zz} + f'(U_0(z))\psi = 0$, one can use the method of variation of constants to find the above solution ψ explicitly. More precisely,

$$\begin{aligned} \psi(z) &= \varphi(z) \int_0^z \left(\varphi^{-2}(\zeta) \int_{-\infty}^{\zeta} A(\xi)\varphi(\xi)\, d\xi \right) d\zeta \\ &= -\varphi(z) \int_0^z \left(\varphi^{-2}(\zeta) \int_{\zeta}^{\infty} A(\xi)\varphi(\xi)\, d\xi \right) d\zeta, \end{aligned} \tag{2.12}$$

where $\varphi := U_0'$. The estimate (2.11) now follows from the above expression and (2.7). □

From the above lemma, the solvability condition for (2.8) is given by

$$\int_{\mathbb{R}} \big[U_0'^2(z)(\tilde{d}_t - \Delta\tilde{d})(x,t) + g\big(x,t,U_0(z)\big)U_0'(z)\big]\,dz = 0,$$

for all $(x,t) \in Q_T$. Hence we get

$$\tilde{d}_t - \Delta\tilde{d} = -\frac{\int_{\mathbb{R}} g(x,t,U_0(z))U_0'(z)\,dz}{\int_{\mathbb{R}} U_0'^2(z)\,dz},$$

which gives

$$\tilde{d}_t = \Delta\tilde{d} - \frac{G(x,t,\alpha_+) - G(x,t,\alpha_-)}{\int_{\mathbb{R}} U_0'^2(z)\,dz}.$$

Moreover, multiplying Eq. (2.5) by U_0' and integrating it from $-\infty$ to z, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{z} \big(U_0''U_0' + f(U_0)U_0'\big)(s)\,ds \\ &= \frac{1}{2}{U_0'}^2(z) - W\big(U_0(z)\big) + W(\alpha_-), \end{aligned}$$

where we have also used the fact that $U_0(-\infty) = \alpha_-$ and $U_0'(-\infty) = 0$. This implies that

$$U_0'(z) = \sqrt{2}\big(W\big(U_0(z)\big) - W(\alpha_-)\big)^{1/2},$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}} U_0'^2(z)\,dz &= \int_{\mathbb{R}} U_0'(z)\sqrt{2}\big(W\big(U_0(z)\big) - W(\alpha_-)\big)^{1/2}\,dz \\ &= \sqrt{2}\int_{\alpha_-}^{\alpha_+} \big(W(s) - W(\alpha_-)\big)^{1/2}\,ds. \end{aligned}$$

It then follows, in view of the definition of c_0 in (1.12), that

$$\tilde{d}_t = \Delta\tilde{d} - c_0\big(G(x,t,\alpha_+) - G(x,t,\alpha_-)\big). \tag{2.13}$$

We are now ready to derive the equation of interface motion. Since $\nabla\tilde{d}$ $(= \nabla_x \tilde{d}(x,t))$ coincides with the outward normal unit vector to the hypersurface Γ_t, we have $\tilde{d}_t(x,t) = -V_n$, where V_n is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is equal to $\Delta\tilde{d}/(N-1)$. Thus the equation of interface motion is given by

$$V_n = -(N-1)\kappa + c_0\big(G(x,t,\alpha_+) - G(x,t,\alpha_-)\big) \quad \text{on } \Gamma_t. \tag{2.14}$$

Summarizing, under the assumption that the solution u^ε of problem (P^ε) satisfies

$$u^\varepsilon \to \begin{cases} \alpha_+ & \text{in } Q_T^+, \\ \alpha_- & \text{in } Q_T^-, \end{cases} \quad \text{as } \varepsilon \to 0,$$

we have formally proved that the boundary $\varGamma_t$ between $\varOmega_t^-$ and $\varOmega_t^+$ moves according to the law (2.14).

To conclude this section, we give basic estimates for $U_1(x,t,z)$, which we will need in Section 5 to study the motion of interface. Substituting (2.13) into (2.8) gives

$$\begin{cases} U_{1zz} + f'\big(U_0(z)\big)U_1 = g\big(x,t,U_0(z)\big) - \gamma(x,t)U_0'(z), \\ U_1(x,t,0) = 0, \quad U_1(x,t,\cdot) \in L^\infty(\mathbb{R}), \end{cases} \tag{2.15}$$

where

$$\gamma(x,t) = c_0\big(G(x,t,\alpha_+) - G(x,t,\alpha_-)\big). \tag{2.16}$$

Thus $U_1(x,t,z)$ is a solution of (2.9) with

$$A = A_0(x,t,z) := g\big(x,t,U_0(z)\big) - \gamma(x,t)U_0'(z), \tag{2.17}$$

where the variables x, t are considered parameters. The problem (2.15) has a unique solution by virtue of Lemma 2.2. Moreover, since $A_0(x,t,z)$ remains bounded as (x,t,z) varies in $\overline{\varOmega} \times [0,T] \times \mathbb{R}$, the estimate (2.11) implies

$$\big|U_1(x,t,z)\big| \leqslant M \quad \text{for } x \in \overline{\varOmega},\ t \in [0,T],\ z \in \mathbb{R}, \tag{2.18}$$

for some constant $M > 0$. Similarly, since ∇U_1 is a solution of (2.9) with

$$A = \nabla_x A_0(x,t,z) \quad \Big(= \nabla_x\big(g\big(x,t,U_0(z)\big) - \gamma(x,t)U_0'(z)\big)\Big),$$

and since g is assumed to be C^1 in x, we obtain

$$\big|\nabla_x U_1(x,t,z)\big| \leqslant M \quad \text{for } x \in \overline{\varOmega},\ t \in [0,T],\ z \in \mathbb{R}, \tag{2.19}$$

for some constant $M > 0$.

To obtain estimates as $z \to \pm\infty$, we first observe that (2.7) implies

$$A_0(x,t,z) - g(x,t,\alpha_\pm) = O\big(e^{-\lambda|z|}\big) \quad \text{as } z \to \pm\infty, \tag{2.20}$$

uniformly in $x \in \overline{\varOmega}$, $t \in [0,T]$. We then apply the following general estimates:

Lemma 2.3. *Let the assumptions of Lemma 2.2 hold, and assume further that $A(z) - A^\pm = O(e^{-\delta|z|})$ as $z \to \pm\infty$ for some constants A^+, A^- and $\delta > 0$. Then there exists a constant $\lambda > 0$ such that*

$$\psi(z) - \frac{A^\pm}{f'(\alpha_\pm)} = O\big(e^{-\lambda|z|}\big), \qquad \big|\psi'(z)\big| + \big|\psi''(z)\big| = O\big(e^{-\lambda|z|}\big) \tag{2.21}$$

as $z \to \pm\infty$.

Proof. We only state the outline. To derive the former estimate, we need a slightly more elaborate version of (2.7). Since $f(u)$ is C^2, we have $f(u) = (u-\alpha_\pm)f'(\alpha_\pm) + O((u-\alpha_\pm)^2)$. Consequently,

$$U_0(z) = \begin{cases} \alpha_- + C_1 e^{\lambda_- z} + O(e^{2\lambda_- z}) & \text{as } z \to -\infty, \\ \alpha_+ + C_2 e^{-\lambda_+ z} + O(e^{-2\lambda_+ z}) & \text{as } z \to +\infty. \end{cases} \tag{2.22}$$

Using the expression (2.12) along with the estimate $A(z) - A^\pm = O(e^{-\delta|z|})$ and (2.22), we see that

$$\psi(z) = -\frac{A^\pm}{(\lambda_\pm)^2} + O\big(|z|e^{-\lambda_\pm|z|}\big) + O\big(e^{-\min(\delta,\lambda_\pm)|z|}\big) \quad \text{as } z \to \pm\infty.$$

This implies the former estimate in (2.21), where λ can be any constant satisfying $0 < \lambda < \min(\lambda_-, \lambda_+, \delta)$. Substituting this into Eq. (2.9) gives the estimate for ψ_{zz}. Finally, the estimate for ψ_z follows by integrating ψ_{zz} from $\pm\infty$ to z. □

From the above lemma and (2.20) we obtain the estimate

$$\big|U_{1z}(x,t,z)\big| + \big|U_{1zz}(x,t,z)\big| \leqslant Ce^{-\lambda|z|}, \tag{2.23}$$

for $x \in \overline{\Omega}$, $t \in [0,T]$, $z \in \mathbb{R}$. Similarly, since (2.6) implies

$$(\nabla_x A_0)(x,t,z) - (\nabla_x g)(x,t,\alpha_\pm) = O\big(e^{-\lambda|z|}\big) \quad \text{as } z \to \pm\infty,$$

we can apply Lemma 2.3 to $\psi = \nabla_x U_1$, to obtain

$$\big|\nabla_x U_{1z}(x,t,z)\big| + \big|\nabla_x U_{1zz}(x,t,z)\big| \leqslant Ce^{-\lambda|z|}$$

for $x \in \overline{\Omega}$, $t \in [0,T]$, $z \in \mathbb{R}$. As a consequence, there is a constant, which we denote again by M, such that

$$\big|\nabla_x U_{1z}(x,t,z)\big| \leqslant M. \tag{2.24}$$

Next we consider the boundary condition. Note that (1.7) implies

$$\frac{\partial}{\partial\nu} A_0 = \frac{\partial}{\partial\nu}\big[g\big(x,t,U_0(z)\big) - \gamma(x,t)U_0'(z)\big] = 0 \quad \text{on } \partial\Omega. \tag{2.25}$$

Consequently, from the expression (2.12), or equivalently the expression

$$U_1(x,t,z) = U_0'(z)\int_0^z \left(\big(U_0'(\zeta)\big)^{-2}\int_{-\infty}^{\zeta} A_0(x,t,\xi)U_0'(\xi)\,d\xi\right)d\zeta,$$

we see that

$$\frac{\partial U_1}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{2.26}$$

3. Generation of interface: The case $g^\varepsilon \equiv 0$

This section deals with the generation of interface, namely the rapid formation of internal layers that takes place in a neighborhood of $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ within the time span of order $\varepsilon^2|\ln\varepsilon|$. For the time being we focus on the special case where $g^\varepsilon \equiv 0$. We will discuss the general case in Section 4. In the sequel, η_0 will stand for the following quantity:

$$\eta_0 := \min(a - \alpha_-, \alpha_+ - a).$$

Our main result in this section is the following:

Theorem 3.1. *Let $\eta \in (0, \eta_0)$ be arbitrary and define μ as the derivative of $f(u)$ at the unstable zero $u = a$, that is*

$$\mu = f'(a). \tag{3.1}$$

Then there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,

(i) *for all $x \in \Omega$,*

$$\alpha_- - \eta \leqslant u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \leqslant \alpha_+ + \eta; \tag{3.2}$$

(ii) *for all $x \in \Omega$ such that $|u_0(x) - a| \geqslant M_0\varepsilon$, we have that*

$$\text{if } u_0(x) \geqslant a + M_0\varepsilon \quad \text{then } u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \geqslant \alpha_+ - \eta, \tag{3.3}$$

$$\text{if } u_0(x) \leqslant a - M_0\varepsilon \quad \text{then } u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \leqslant \alpha_- + \eta. \tag{3.4}$$

The above theorem will be proved by constructing a suitable pair of sub- and super-solutions. Note that we do not need condition (1.2) in proving this theorem.

3.1. The bistable ordinary differential equation

Let us first consider the problem without diffusion:

$$\bar{u}_t = \frac{1}{\varepsilon^2} f(\bar{u}), \qquad \bar{u}(x, 0) = u_0(x).$$

This solution is written in the form

$$\bar{u}(x, t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x)\right),$$

where $Y(\tau, \xi)$ denotes the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau, \xi) = f\big(Y(\tau, \xi)\big) & \text{for } \tau > 0, \\ Y(0, \xi) = \xi. \end{cases} \tag{3.5}$$

Here ξ ranges over the interval $(-2C_0, 2C_0)$, with C_0 being the constant defined in (1.8). We first study basic properties of Y.

Lemma 3.2. *We have $Y_\xi > 0$, for all $\xi \notin \{\alpha_-, a, \alpha_+\}$, $\tau > 0$. Furthermore,*

$$Y_\xi(\tau,\xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}.$$

Proof. First, differentiating Eq. (3.5) by ξ, we obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0,\xi) = 1, \end{cases}$$

which is integrated as follows:

$$Y_\xi(\tau,\xi) = \exp\left[\int_0^\tau f'\big(Y(s,\xi)\big)\,ds\right] > 0. \tag{3.6}$$

We then differentiate Eq. (3.5) by τ and obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0,\xi) = f(\xi), \end{cases}$$

which in turn implies

$$\begin{aligned} Y_\tau(\tau,\xi) &= f(\xi)\exp\left[\int_0^\tau f'\big(Y(s,\xi)\big)\,ds\right] \\ &= f(\xi)Y_\xi(\tau,\xi). \end{aligned}$$

This last equality, in view of (3.5), completes the proof of Lemma 3.2. □

For $\xi \notin \{\alpha_-, a, \alpha_+\}$, we define a function $A(\tau,\xi)$ by

$$A(\tau,\xi) = \frac{f'(Y(\tau,\xi)) - f'(\xi)}{f(\xi)}. \tag{3.7}$$

Lemma 3.3. *We have, for all $\xi \notin \{\alpha_-, a, \alpha_+\}$, $\tau > 0$,*

$$A(\tau,\xi) = \int_0^\tau f''\big(Y(s,\xi)\big)Y_\xi(s,\xi)\,ds.$$

Proof. Differentiating by ξ the equality of Lemma 3.2 leads to

$$Y_{\xi\xi} = A(\tau,\xi)Y_\xi, \tag{3.8}$$

whereas differentiating (3.6) by ξ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''\big(Y(s,\xi)\big) Y_\xi(s,\xi)\,ds.$$

These two last results complete the proof of Lemma 3.3. $\square$

Next we need some estimates on Y and its derivatives. First, we estimate the speed of the evolution of Y when the initial value ξ lies between $\alpha_- + \eta$ and $\alpha_+ - \eta$.

Lemma 3.4. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\tilde{C}_1 = \tilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that*

(i) *if $\xi \in (a, \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$, we have*

$$\tilde{C}_1 e^{\mu\tau} \leqslant Y_\xi(\tau,\xi) \leqslant \tilde{C}_2 e^{\mu\tau}, \tag{3.9}$$

$$\big|A(\tau,\xi)\big| \leqslant C_3\big(e^{\mu\tau} - 1\big), \tag{3.10}$$

where μ is the constant defined in (3.1);

(ii) *if $\xi \in (\alpha_- + \eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha_- + \eta, a)$, we have* (3.9) *and* (3.10).

Proof. We take $\xi \in (a, \alpha_+ - \eta)$ and suppose that, for $s \in (0,\tau)$, $Y(s,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$. Integrating the equality

$$\frac{Y_\tau(s,\xi)}{f(Y(s,\xi))} = 1$$

from 0 to τ yields

$$\int_0^\tau \frac{Y_\tau(s,\xi)}{f(Y(s,\xi))}\,ds = \tau. \tag{3.11}$$

Hence by the change of variable $q = Y(s,\xi)$ we get

$$\int_\xi^{Y(\tau,\xi)} \frac{dq}{f(q)} = \tau. \tag{3.12}$$

Moreover, the equality of Lemma 3.2 leads to

$$\begin{aligned}\ln Y_\xi(\tau,\xi) &= \int_\xi^{Y(\tau,\xi)} \frac{f'(q)}{f(q)}\,dq\\ &= \int_\xi^{Y(\tau,\xi)} \left[\frac{f'(a)}{f(q)} + \frac{f'(q)-f'(a)}{f(q)}\right]dq\\ &= \mu\tau + \int_\xi^{Y(\tau,\xi)} h(q)\,dq,\end{aligned} \tag{3.13}$$

where $h(q) = (f'(q)-\mu)/f(q)$. As $h(q)$ tends to $f''(a)/f'(a)$ when q tends to a, h is continuous on $[a, \alpha_+ - \eta]$. Hence we can define

$$H = H(\eta) := \|h\|_{L^\infty(a,\alpha_+-\eta)}.$$

Since $|Y(\tau,\xi) - \xi|$ takes its value in the interval $[0, \alpha_+ - a - \eta] \subset [0, \alpha_+ - a]$, it follows from (3.13) that

$$\mu\tau - H(\alpha_+ - a) \leqslant \ln Y_\xi(\tau,\xi) \leqslant \mu\tau + H(\alpha_+ - a),$$

which, in turn, proves (3.9). Next Lemma 3.3 and (3.9) yield

$$\begin{aligned}\bigl|A(\tau,\xi)\bigr| &\leqslant \|f''\|_{L^\infty(\alpha_-,\alpha_+)} \int_0^\tau \tilde{C}_2 e^{\mu s}\,ds\\ &\leqslant C_3\bigl(e^{\mu\tau}-1\bigr),\end{aligned}$$

which completes the proof of (3.10). The case where ξ and $Y(\tau,\xi)$ are in $(\alpha_- + \eta, a)$ is similar and omitted. □

Corollary 3.5. *Let $\eta \in (0,\eta_0)$ be arbitrary. Then there exist positive constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that*

(i) *if $\xi \in (a, \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$, we have*

$$C_1 e^{\mu\tau}(\xi - a) \leqslant Y(\tau,\xi) - a \leqslant C_2 e^{\mu\tau}(\xi - a); \tag{3.14}$$

(ii) *if $\xi \in (\alpha_- + \eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha_- + \eta, a)$, we have*

$$C_2 e^{\mu\tau}(\xi - a) \leqslant Y(\tau,\xi) - a \leqslant C_1 e^{\mu\tau}(\xi - a). \tag{3.15}$$

Proof. We can find $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (a, \alpha_+ - \eta)$,

$$B_1(q-a) \leqslant f(q) \leqslant B_2(q-a). \tag{3.16}$$

We use this inequality for $a < Y(\tau,\xi) < \alpha_+ - \eta$ to obtain

$$B_1\big(Y(\tau,\xi)-a\big) \leqslant f\big(Y(\tau,\xi)\big) \leqslant B_2\big(Y(\tau,\xi)-a\big).$$

We also use this inequality for $a < \xi < \alpha_+ - \eta$ to obtain

$$B_1(\xi-a) \leqslant f(\xi) \leqslant B_2(\xi-a).$$

Next we use the equality $Y_\xi = f(Y)/f(\xi)$ of Lemma 3.2 to deduce that

$$\frac{B_1}{B_2}\big(Y(\tau,\xi)-a\big) \leqslant (\xi-a)Y_\xi(\tau,\xi) \leqslant \frac{B_2}{B_1}\big(Y(\tau,\xi)-a\big),$$

which, in view of (3.9), implies that

$$\frac{B_1}{B_2}\tilde{C}_1 e^{\mu\tau}(\xi-a) \leqslant Y(\tau,\xi)-a \leqslant \frac{B_2}{B_1}\tilde{C}_2 e^{\mu\tau}(\xi-a).$$

This proves (3.14). The proof of (3.15) is similar and is omitted. □

We now present estimates in the case where the initial value ξ is smaller than $\alpha_- + \eta$ or larger than $\alpha_+ - \eta$.

Lemma 3.6. *Let $\eta \in (0,\eta_0)$ and $M > 0$ be arbitrary. Then there exists a positive constant $C_4 = C_4(\eta, M)$ such that*

(i) *if $\xi \in [\alpha_+ - \eta, \alpha_+ + M]$, then, for all $\tau > 0$, $Y(\tau,\xi)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ and*

$$\big|A(\tau,\xi)\big| \leqslant C_4\tau \quad \textit{for } \tau > 0; \tag{3.17}$$

(ii) *if $\xi \in [\alpha_- - M, \alpha_- + \eta]$, then, for all $\tau > 0$, $Y(\tau,\xi)$ remains in the interval $[\alpha_- - M, \alpha_- + \eta]$ and* (3.17) *holds.*

Proof. Since statement (i) and statement (ii) can be treated in the same way, we will only prove the former. The fact that $Y(\tau,\xi)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ directly follows from the bistable properties of f, or, more precisely, from the sign conditions $f(\alpha_+ - \eta) > 0$, $f(\alpha_+ + M) < 0$.

To prove (3.17), suppose first that $\xi \in [\alpha_+, \alpha_+ + M]$. In view of (1.1), f' is strictly negative in an interval of the form $[\alpha_+, \alpha_+ + c]$ and f is negative in $[\alpha_+, \infty)$. We denote by $-m < 0$ the maximum of f on $[\alpha_+ + c, M]$. Then, as long as $Y(\tau,\xi)$ remains in the interval $[\alpha_+ + c, M]$, the ordinary differential equation (3.5) implies

$$Y_\tau \leqslant -m.$$

This means that, for any $\xi \in [\alpha_+, \alpha_+ + M]$, we have

$$Y(\tau,\xi) \in [\alpha_+, \alpha_+ + c] \quad \text{for } \tau \geqslant \bar{\tau} := \frac{M-c}{m}.$$

In view of this, and considering that $f'(Y) < 0$ for $Y \in [\alpha_+, \alpha_+ + c]$, we see from the expression (3.6) that

$$\begin{aligned} Y_\xi(\tau,\xi) &= \exp\left[\int_0^{\bar{\tau}} f'\big(Y(s,\xi)\big)\,ds\right] \exp\left[\int_{\bar{\tau}}^{\tau} f'\big(Y(s,\xi)\big)\,ds\right] \\ &\leqslant \exp\left[\int_0^{\bar{\tau}} f'\big(Y(s,\xi)\big)\,ds\right] \\ &\leqslant \exp\left[\int_0^{\bar{\tau}} \sup_{z\in[\alpha_- - M, \alpha_+ + M]} \big|f'(z)\big|\,ds\right] =: \tilde{C}_4, \end{aligned}$$

for all $\tau \geqslant \bar{\tau}$. It is clear from the same estimate (3.6) that $Y_\xi \leqslant \tilde{C}_4$ holds also for $0 \leqslant \tau \leqslant \bar{\tau}$. We can then use Lemma 3.3 to deduce that

$$\big|A(\tau,\xi)\big| \leqslant \tilde{C}_4 \int_0^{\tau} \big|f''\big(Y(s,\xi)\big)\big|\,ds \leqslant C_4\tau.$$

The case $\xi \in [\alpha_+ - \eta, \alpha_+]$ can be treated in the same way. This completes the proof of the lemma. □

Now we choose the constant M in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [\alpha_- - M, \alpha_+ + M]$, and fix M hereafter. Then C_4 only depends on η. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from (3.10) and (3.17) the following general estimate.

Lemma 3.7. *Let $\eta \in (0, \eta_0)$ be arbitrary and let C_0 be the constant defined in* (1.8). *Then there exists a positive constant $C_5 = C_5(\eta)$ such that, for all $\tau > 0$ and all $\xi \in (-2C_0, 2C_0)$,*

$$\big|A(\tau,\xi)\big| \leqslant C_5\big(e^{\mu\tau} - 1\big).$$

3.2. Construction of sub- and super-solutions

We are now ready to construct the sub- and super-solutions for the study of generation of interface. For simplicity, we first consider the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{3.18}$$

In this case, our sub- and super-solutions are given by

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C_6\left(e^{\mu t/\varepsilon^2} - 1\right)\right). \tag{3.19}$$

In the general case where (3.18) does not necessarily hold, we have to slightly modify $w_\varepsilon^\pm(x,t)$ near the boundary $\partial\Omega$. This will be discussed later.

Lemma 3.8. *Assume* (3.18). *Then there exist positive constants ε_0 and C_6 such that, for all $\varepsilon \in (0,\varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for problem (P^ε), in the domain $\overline{\Omega} \times [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, satisfying $w_\varepsilon^-(x,0) = w^+(x,0) = u_0(x)$. Consequently*

$$w_\varepsilon^-(x,t) \leqslant u^\varepsilon(x,t) \leqslant w_\varepsilon^+(x,t) \quad \textit{for } x \in \overline{\Omega},\ 0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|. \tag{3.20}$$

Proof. The assumption (3.18) implies

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,+\infty).$$

Now we define an operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 u := u_t - \Delta u - \frac{1}{\varepsilon^2} f(u),$$

and prove that $\mathcal{L}_0 w_\varepsilon^+ \geqslant 0$. Straightforward computations yield

$$\mathcal{L}_0 w_\varepsilon^+ = \frac{1}{\varepsilon^2} Y_\tau + C_6 \mu e^{\mu t/\varepsilon^2} Y_\xi - \Delta u_0 Y_\xi - |\nabla u_0|^2 Y_{\xi\xi} - \frac{1}{\varepsilon^2} f(Y),$$

therefore, in view of the ordinary differential equation (3.5),

$$\mathcal{L}_0 w_\varepsilon^+ = \left[C_6 \mu e^{\mu t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\right] Y_\xi.$$

We note that, in the range $0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|$, we have, for ε_0 sufficiently small,

$$0 \leqslant \varepsilon^2 C_6\left(e^{\mu t/\varepsilon^2} - 1\right) \leqslant \varepsilon^2 C_6\left(\varepsilon^{-1} - 1\right) \leqslant C_0,$$

where C_0 is the constant defined in (1.8). Hence

$$\xi := u_0(x) \pm \varepsilon^2 C_6\left(e^{\mu t/\varepsilon^2} - 1\right) \in (-2C_0, 2C_0),$$

and it follows from the estimate of $A = Y_{\xi\xi}/Y_\xi$ in Lemma 3.7, with the choice $\tau := t/\varepsilon^2$, that

$$\begin{aligned}\mathcal{L}_0 w_\varepsilon^+ &\geqslant \left[C_6 \mu e^{\mu t/\varepsilon^2} - |\Delta u_0| - C_5\left(e^{\mu t/\varepsilon^2} - 1\right)|\nabla u_0|^2\right] Y_\xi \\ &\geqslant \left[\left(C_6\mu - C_5|\nabla u_0|^2\right)e^{\mu t/\varepsilon^2} - |\Delta u_0| + C_5|\nabla u_0|^2\right] Y_\xi.\end{aligned}$$

Since $Y_\xi > 0$, this inequality implies that, for C_6 large enough,

$$\mathcal{L}_0 w_\varepsilon^+ \geqslant \left[C_6\mu - C_5 C_0^2 - C_0\right] Y_\xi \geqslant 0.$$

Hence w_ε^+ is a super-solution for problem (P^ε). Similarly w_ε^- is a sub-solution. Obviously $w_\varepsilon^-(x,0) = w^+(x,0) = u_0(x)$. Lemma 3.8 is proved. □

In the more general case where (3.18) is not necessarily valid, one can proceed as follows: in view of (1.10) and (1.11) there exist positive constants d_1, ρ such that $u_0(x) \geqslant a + \rho$ if $d(x, \partial\Omega) \leqslant d_1$. Let χ be a smooth cut-off function defined on $[0, +\infty)$ such that $0 \leqslant \chi \leqslant 1$, $\chi(0) = \chi'(0) = 0$ and $\chi(z) = 1$ for $z \geqslant d_1$. Then we define

$$u_0^+(x) = \chi\big(d(x, \partial\Omega)\big) u_0(x) + \big[1 - \chi\big(d(x, \partial\Omega)\big)\big] \max_{x \in \overline{\Omega}} u_0(x),$$

$$u_0^-(x) = \chi\big(d(x, \partial\Omega)\big) u_0(x) + \big[1 - \chi\big(d(x, \partial\Omega)\big)\big](a + \rho).$$

Clearly, $u_0^- \leqslant u_0 \leqslant u_0^+$, and both u_0^+ and u_0^+ satisfy (3.18). Now we set

$$\tilde{w}_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0^\pm(x) \pm \varepsilon^2 C_6\big(e^{\mu t/\varepsilon^2} - 1\big)\right).$$

Then the same argument as in Lemma 3.8 shows that $(\tilde{w}_\varepsilon^-, \tilde{w}_\varepsilon^+)$ is a pair of sub- and super-solutions for problem (P^ε). Furthermore, since $\tilde{w}_\varepsilon^-(x,0) = u_0^-(x) \leqslant u_0(x) \leqslant u_0^+(x) = \tilde{w}_\varepsilon^+(x,0)$, the comparison principle asserts that

$$\tilde{w}_\varepsilon^-(x,t) \leqslant u^\varepsilon(x,t) \leqslant \tilde{w}_\varepsilon^+(x,t) \quad \text{for } x \in \overline{\Omega},\ 0 \leqslant t \leqslant \mu^{-1}\varepsilon^2 |\ln\varepsilon|. \tag{3.21}$$

3.3. Proof of Theorem 3.1

In order to prove Theorem 3.1 we first present basic estimates of the function Y after a time of order $\tau \sim |\ln\varepsilon|$.

Lemma 3.9. *Let $\eta \in (0, \eta_0)$ be arbitrary; there exist positive constants ε_0 and C_7 such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

(i) *for all $\xi \in (-2C_0, 2C_0)$,*

$$\alpha_- - \eta \leqslant Y\big(\mu^{-1}|\ln\varepsilon|, \xi\big) \leqslant \alpha_+ + \eta; \tag{3.22}$$

(ii) *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \geqslant C_7\varepsilon$, we have that*

$$\text{if } \xi \geqslant a + C_7\varepsilon \quad \text{then } Y\big(\mu^{-1}|\ln\varepsilon|, \xi\big) \geqslant \alpha_+ - \eta, \tag{3.23}$$

$$\text{if } \xi \leqslant a - C_7\varepsilon \quad \text{then } Y\big(\mu^{-1}|\ln\varepsilon|, \xi\big) \leqslant \alpha_- + \eta. \tag{3.24}$$

Proof. We first prove (3.23). For $\xi \geqslant a + C_7\varepsilon$, as long as $Y(\tau,\xi)$ has not reached $\alpha_+ - \eta$, we can use (3.14) to deduce that

$$\begin{aligned} Y(\tau,\xi) &\geqslant a + C_1 e^{\mu\tau}(\xi - a) \\ &\geqslant a + C_1 C_7 e^{\mu\tau}\varepsilon \\ &\geqslant \alpha_+ - \eta, \end{aligned}$$

provided that τ satisfies

$$\tau \geqslant \tau^\varepsilon =: \mu^{-1} \ln \frac{\alpha_+ - a - \eta}{C_1 C_7 \varepsilon}.$$

Choosing

$$C_7 = \frac{\max(a - \alpha_-, \alpha_+ - a) - \eta}{C_1},$$

we see that $\mu^{-1}|\ln\varepsilon| \geqslant \tau^\varepsilon$, which completes the proof of (3.23). Using (3.15), one easily proves (3.24).

Next we prove (3.22). First, in view of the profile of f, if we leave from $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau,\xi)$ will remain in $[\alpha_- - \eta, \alpha_+ + \eta]$. Now suppose that $\alpha_+ + \eta \leqslant \xi \leqslant 2C_0$. We check below that $Y(\mu^{-1}|\ln\varepsilon|, \xi) \leqslant \alpha_+ + \eta$. First, in view of (1.1), we can find $p > 0$ such that

$$\begin{aligned} &\text{if } \alpha_+ \leqslant u \leqslant 2C_0 \quad \text{then } f(u) \leqslant p(\alpha_+ - u), \\ &\text{if } -2C_0 \leqslant u \leqslant \alpha_- \quad \text{then } f(u) \geqslant -p(u - \alpha_-). \end{aligned} \tag{3.25}$$

We then use the ordinary differential equation to obtain, as long as $\alpha_+ + \eta \leqslant Y \leqslant 2C_0$, the inequality $Y_\tau \leqslant p(\alpha_+ - Y)$. It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \leqslant -p.$$

Integrating this inequality from 0 to τ leads to

$$\begin{aligned} Y(\tau,\xi) &\leqslant \alpha_+ + (\xi - \alpha_+)e^{-p\tau} \\ &\leqslant \alpha_+ + (2C_0 - \alpha_+)e^{-p\tau}. \end{aligned}$$

One easily checks that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ small enough, we have $Y(\mu^{-1}|\ln\varepsilon|, \xi) \leqslant \alpha_+ + \eta$, which completes the proof of (3.22). □

We are now ready to prove Theorem 3.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (3.21), we obtain

$$\begin{aligned} &Y\big(\mu^{-1}|\ln\varepsilon|, u_0^-(x) - \big(C_6\varepsilon - C_6\varepsilon^2\big)\big) \\ &\quad \leqslant u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \leqslant Y\big(\mu^{-1}|\ln\varepsilon|, u_0^+(x) + C_6\varepsilon - C_6\varepsilon^2\big). \end{aligned} \tag{3.26}$$

Furthermore, by the definition of C_0 in (1.8), we have, for ε_0 small enough,

$$-2C_0 \leqslant u_0^{\pm}(x) \pm \left(C_6\varepsilon - C_6\varepsilon^2\right) \leqslant 2C_0 \quad \text{for } x \in \Omega.$$

Thus the assertion (3.2) of Theorem 3.1 is a direct consequence of (3.22) and (3.26).

Next we prove (3.3). We choose M_0 large enough so that $M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geqslant C_7\varepsilon$. Then, for any $x \in \Omega$ such that $u_0^-(x) \geqslant a + M_0\varepsilon$, we have

$$u_0^-(x) - \left(C_6\varepsilon - C_6\varepsilon^2\right) \geqslant a + M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geqslant a + C_7\varepsilon.$$

Combining this, (3.26) and (3.23), we see that

$$u^\varepsilon\left(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\right) \geqslant \alpha_+ - \eta,$$

for any $x \in \Omega$ with $u_0^-(x) \geqslant a + M_0\varepsilon$. From the definition of u_0^- it is clear that $u_0^-(x) \geqslant a + M_0\varepsilon$ if and only if $u_0(x) \geqslant a + M_0\varepsilon$, provided that ε is small enough. This proves (3.3). The inequality (3.4) can be shown in the same way. This completes the proof of Theorem 3.1.

3.4. Optimality of the generation time

To conclude this section we show that the generation time $t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|$ that appears in Theorem 3.1 is optimal. In other words, the interface will not be fully developed until t comes close to t^ε.

Proposition 3.10. *Denote by $t^\varepsilon_{\min}$ the smallest time such that* (1.14) *holds for all $t \in [t^\varepsilon_{\min}, T]$. Then there exists a constant $b = b(C)$ such that*

$$t^\varepsilon_{\min} \geqslant \mu^{-1}\varepsilon^2\left(|\ln\varepsilon| - b\right),$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. For simplicity, we deal with the case where (3.18) is valid. In that case, (3.20) holds for all small $\varepsilon > 0$. For each $b > 0$, we put

$$t^\varepsilon(b) := \mu^{-1}\varepsilon^2\left(|\ln\varepsilon| - b\right),$$

and evaluate $u^\varepsilon(x, t^\varepsilon(b))$ at a point $x \in \Omega_0^+$ where $\operatorname{dist}(x, \Gamma_0) = C\varepsilon$. Since $u_0 = a$ on Γ_t and since $|\nabla u_0| \leqslant C_0$ by (1.8), we have

$$u_0(x) \leqslant a + C_0 C\varepsilon. \tag{3.27}$$

It follows from this and (3.14) that

$$\begin{aligned}
w_\varepsilon^+\left(x, t^\varepsilon(b)\right) &= Y\left(\mu^{-1}\left(|\ln\varepsilon| - b\right), u_0(x) + \varepsilon C_6 e^{-b} - \varepsilon^2 C_6\right) \\
&\leqslant a + C_2 e^{|\ln\varepsilon| - b}\left(u_0(x) + \varepsilon C_6 e^{-b} - \varepsilon^2 C_6 - a\right) \\
&\leqslant a + C_2\varepsilon^{-1}e^{-b}\left(C_0 C\varepsilon + \varepsilon C_6 e^{-b}\right) \\
&= a + C_2 e^{-b}\left(C_0 C + C_6 e^{-b}\right).
\end{aligned}$$

Now we choose b to be sufficiently large, so that

$$a + C_2 e^{-b}\big(C_0 C + C_6 e^{-b}\big) < \alpha_+ - \eta.$$

Then the above estimate and (3.20) yield

$$u^\varepsilon\big(x, t^\varepsilon(b)\big) \leqslant w_\varepsilon^+\big(x, t^\varepsilon(b)\big) < \alpha_+ - \eta.$$

This implies that (1.14) does not hold at $t = t^\varepsilon(b)$, hence $t^\varepsilon(b) < t^\varepsilon_{\min}$. The lemma is proved. □

4. Generation of interface in the general case

In this section we extend Theorem 3.1 to the case where $g^\varepsilon \not\equiv 0$. The proof is more technical than the case $g^\varepsilon \equiv 0$, but the underlying ideas are the same. Hence we will basically follow the argument of Section 3, simply pointing out the main differences.

4.1. The perturbed ordinary differential equation

We first consider a slightly perturbed nonlinearity:

$$f_\delta(u) = f(u) + \delta,$$

where δ is any constant. For $|\delta|$ small enough, this function is still bistable. More precisely, f_δ has the following properties, whose proof is omitted:

Lemma 4.1. *Let δ_0 be small enough. Then for any $\delta \in (-\delta_0, \delta_0)$,*

(i) *f_δ has exactly three zeros, namely $\alpha_-(\delta) < a(\delta) < \alpha_+(\delta)$, and there exists a positive constant C such that*

$$\big|\alpha_-(\delta) - \alpha_-\big| + \big|a(\delta) - a\big| + \big|\alpha_+(\delta) - \alpha_+\big| \leqslant C|\delta|; \tag{4.1}$$

(ii) *we have*

$$\begin{aligned} f_\delta > 0 \quad &\text{in } \big(-\infty, \alpha_-(\delta)\big) \cup \big(a(\delta), \alpha_+(\delta)\big),\\ f_\delta < 0 \quad &\text{in } \big(\alpha_-(\delta), a(\delta)\big) \cup \big(\alpha_+(\delta), +\infty\big); \end{aligned} \tag{4.2}$$

(iii) *there exists a positive constant, denoted again by C, such that*

$$\big|\mu(\delta) - \mu\big| \leqslant C|\delta|, \tag{4.3}$$

where

$$\mu(\delta) := f_\delta'\big(a(\delta)\big) = f'\big(a(\delta)\big).$$

Now for each $\delta \in (-\delta_0, \delta_0)$, we define $Y(\tau, \xi; \delta)$ as the solution of the following ordinary differential equation:

$$\begin{cases} Y_\tau(\tau, \xi; \delta) = f_\delta\big(Y(\tau, \xi; \delta)\big) & \text{for } \tau > 0, \\ Y(0, \xi; \delta) = \xi, \end{cases} \tag{4.4}$$

where ξ varies in $(-2C_0, 2C_0)$, with C_0 being the constant defined in (1.8).

To prove Theorem 3.1, we will construct a pair of sub- and super-solutions for (P^ε) by simply replacing the function $Y(\tau, \xi)$ in (3.19) by $Y(\tau, \xi; \delta)$, with an appropriate choice of δ. For this strategy to work, we have to check that the basic properties of $Y(\tau, \xi)$ in Section 3.1 carry over to $Y(\tau, \xi; \delta)$.

First, it is clear that all the differential and integral identities in Section 3.1 that follow directly from (3.5) are still valid for (4.4). In particular, Lemmas 3.2 and 3.3 remain to hold if we replace $Y(\tau, \xi)$ by $Y(\tau, \xi; \delta)$, f by f_δ and $A(\tau, \xi)$ by $A(\tau, \xi; \delta)$, where

$$A(\tau, \xi, \delta) = \frac{f'_\delta(Y(\tau, \xi; \delta)) - f'_\delta(\xi)}{f_\delta(\xi)}.$$

Next let us show that the basic estimates which we have established in Section 3.1 are also valid for $Y(\tau, \xi; \delta)$. The following lemma, which is an analogue of Lemma 3.4, is fundamental.

Lemma 4.2. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $\tilde{C}_1 = \tilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for any $\delta \in [-\delta_0, \delta_0]$,*

(i) *if $\xi \in (a(\delta), \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(a(\delta), \alpha_+ - \eta)$, we have*

$$\tilde{C}_1 e^{\mu(\delta)\tau} \leqslant Y_\xi(\tau, \xi; \delta) \leqslant \tilde{C}_2 e^{\mu(\delta)\tau}, \tag{4.5}$$

$$\big|A(\tau, \xi; \delta)\big| \leqslant C_3\big(e^{\mu(\delta)\tau} - 1\big); \tag{4.6}$$

(ii) *the same estimates as above hold if the interval $(a(\delta), \alpha_+ - \eta)$ is replaced by $(\alpha_- + \eta, a(\delta))$.*

Proof. In view of (4.1), we can choose a small constant $\delta_0 = \delta_0(\eta) > 0$ such that $(a(\delta), \alpha_+ - \eta) \subset (a(\delta), \alpha_+(\delta))$, for every $\delta \in [-\delta_0, \delta_0]$. Therefore $f_\delta(q)$ does not change sign in the interval $(a(\delta), \alpha_+ - \eta)$. Thus, in order to prove the lemma, we just have to write again the proof of Lemma 3.4, simply replacing $Y(\tau, \xi)$ by $Y(\tau, \xi; \delta)$. We do not repeat the entire proof here. Instead, let us explain why $\tilde{C}_1$, $\tilde{C}_2$ and C_3 can be chosen independent of δ. In view of the proof of Lemma 3.4, it is sufficient to estimate, for $q \in (a(\delta), \alpha_+ - \eta]$, the modulus of the quantity

$$h_\delta(q) := \frac{f'(q) - f'(a(\delta))}{f_\delta(q)}$$

by a constant depending on η, but not on $\delta \in [-\delta_0, \delta_0]$. Since

$$h_\delta(q) \to \frac{f''_\delta(a(\delta))}{f'_\delta(a(\delta))} = \frac{f''(a(\delta))}{f'(a(\delta))} \quad \text{as } q \to a(\delta),$$

we see that the function $(q,\delta)\mapsto h_\delta(q)$ is continuous in the compact region $\{|\delta|\leqslant\delta_0,\ a(\delta)\leqslant q\leqslant\alpha_+-\eta\}$. It follows that $|h_\delta(q)|$ is bounded as (q,δ) varies in this region. This completes the proof of Lemma 4.2. □

Corollary 4.3. *Let $\eta\in(0,\eta_0)$ be arbitrary. Then there exist positive constants $\delta_0=\delta_0(\eta)$, $C_1=C_1(\eta)$ and $C_2=C_2(\eta)$ such that, for any $\delta\in[-\delta_0,\delta_0]$,*

(i) *if $\xi\in(a(\delta),\alpha_+-\eta)$ then, for every $\tau>0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(a(\delta),\alpha_+-\eta)$, we have*

$$C_1e^{\mu(\delta)\tau}\big(\xi-a(\delta)\big)\leqslant Y(\tau,\xi;\delta)-a(\delta)\leqslant C_2e^{\mu(\delta)\tau}\big(\xi-a(\delta)\big);\tag{4.7}$$

(ii) *if $\xi\in(\alpha_-+\eta,a(\delta))$ then, for every $\tau>0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(\alpha_-+\eta,a(\delta))$, we have*

$$C_2e^{\mu(\delta)\tau}\big(\xi-a(\delta)\big)\leqslant Y(\tau,\xi;\delta)-a(\delta)\leqslant C_1e^{\mu(\delta)\tau}\big(\xi-a(\delta)\big).\tag{4.8}$$

Proof. We can simply follow the proof of Corollary 3.5. In order to prove that C_1 and C_2 are independent of δ, all we have to do is to find constants $B_1=B_1(\eta)>0$ and $B_2=B_2(\eta)>0$ such that, for all $\delta\in[-\delta_0,\delta_0]$ and all $q\in(a(\delta),\alpha_+-\eta)$,

$$B_1\big(q-a(\delta)\big)\leqslant f_\delta(q)\leqslant B_2\big(q-a(\delta)\big).\tag{4.9}$$

This can be easily done, since $(q,\delta)\mapsto f_\delta(q)/(q-a(\delta))$ is a positive continuous function on the compact region $\{|\delta|\leqslant\delta_0,\ a(\delta)\leqslant q\leqslant\alpha_+-\eta\}$. □

Now, it is no trouble to establish an analogue of Lemmas 3.6 and 3.7 with constants independent of δ. We claim, without proof, that:

Lemma 4.4. *Let $\eta\in(0,\eta_0)$ and $M>0$ be arbitrary. Then there exist positive constants $\delta_0=\delta_0(\eta,M)$ and $C_4=C_4(\eta,M)$ such that, for any $\delta\in[-\delta_0,\delta_0]$,*

(i) *if $\xi\in[\alpha_+-\eta,\alpha_++M]$, then, for all $\tau>0$, $Y(\tau,\xi;\delta)$ remains in the interval $[\alpha_+-\eta,\alpha_++M]$ and*

$$\big|A(\tau,\xi;\delta)\big|\leqslant C_4\tau\quad\textit{for }\tau>0;\tag{4.10}$$

(ii) *if $\xi\in[\alpha_--M,\alpha_-+\eta]$, then, for all $\tau>0$, $Y(\tau,\xi;\delta)$ remains in the interval $[\alpha_--M,\alpha_-+\eta]$ and* (4.10) *holds.*

Lemma 4.5. *Let $\eta\in(0,\eta_0)$ be arbitrary and let C_0 be the constant defined in* (1.8)*. Then there exist positive constants $\delta_0=\delta_0(\eta)$, $C_5=C_5(\eta)$ such that, for all $\delta\in[-\delta_0,\delta_0]$, for all $\tau>0$ and all $\xi\in(-2C_0,2C_0)$,*

$$\big|A(\tau,\xi;\delta)\big|\leqslant C_5\big(e^{\mu(\delta)\tau}-1\big).$$

4.2. Construction of sub- and super-solutions

We now construct a pair of sub- and super-solutions by modifying the definition (3.19). We set

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r\left(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}\right); \pm\varepsilon\mathcal{G}\right),$$

where the function $r(\delta,\tau)$ is given by

$$r(\delta,\tau) = C_6\big(e^{\mu(\delta)\tau} - 1\big),$$

and the constant $\mathcal{G}$ is chosen such that, for all small $\varepsilon > 0$,

$$\big|g^\varepsilon(x,t,u)\big| \leqslant \mathcal{G} \quad \text{for } (x,t,u) \in \overline{\Omega} \times [0,T] \times \mathbb{R},$$

which, in view of (1.5), is clearly possible.

Lemma 4.6. *There exist positive constants ε_0 and C_6 such that for all $\varepsilon \in (0,\varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for problem (P^ε), in the domain $\overline{\Omega} \times [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, satisfying $w_\varepsilon^-(x,0) = w^+(x,0) = u_0(x)$.*

Proof. First, the same cut-off argument as in Section 3.2 enables us to assume (3.18) for simplicity. Hence $w_\varepsilon^\pm$ satisfy the Neumann boundary conditions. We define an operator $\mathcal{L}$ by

$$\mathcal{L}u := u_t - \Delta u - \varepsilon^{-2}\big(f(u) - g^\varepsilon(x,t,u)\big),$$

and prove below that $\mathcal{L}w_\varepsilon^+ \geqslant 0$ by slightly modifying the argument which we have used to prove $\mathcal{L}_0 w_\varepsilon^+ \geqslant 0$ in Section 3. A straightforward calculation yields

$$\mathcal{L}w_\varepsilon^+ = \frac{1}{\varepsilon^2}\big[Y_\tau - f(Y) + \varepsilon g^\varepsilon(x,t,Y)\big] + Y_\xi\left[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})\frac{t}{\varepsilon^2}} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\right].$$

If ε_0 is sufficiently small, we note that $\pm\varepsilon\mathcal{G} \in (-\delta_0,\delta_0)$ and that, in the range $0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\big|\varepsilon^2 C_6\big(e^{\mu(\pm\varepsilon\mathcal{G})t/\varepsilon^2} - 1\big)\big| \leqslant \varepsilon^2 C_6\big(\varepsilon^{-\mu(\pm\varepsilon\mathcal{G})/\mu} - 1\big) \leqslant C_0,$$

which implies that

$$u_0(x) \pm \varepsilon^2 r\left(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}\right) \in (-2C_0, 2C_0).$$

These observations allow us to use the results of the previous subsection with the choices $\tau := t/\varepsilon^2$, $\xi := u_0(x) + \varepsilon^2 r(\varepsilon\mathcal{G}, t/\varepsilon^2)$ and $\delta := \varepsilon\mathcal{G}$. In particular, the ordinary differential equation (4.4) yields $Y_\tau = f(Y) + \varepsilon\mathcal{G}$, which implies that

$$\mathcal{L}w_\varepsilon^+ = \frac{1}{\varepsilon}\big[\mathcal{G} + g^\varepsilon(x,t,Y)\big] + Y_\xi\left[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\right].$$

By the choice of $\mathcal{G}$ the first term of the right-hand side member is positive. Using the estimate of $A = Y_{\xi\xi}/Y_\xi$ in Lemma 4.5, we obtain, for a constant C_5 that is independent of ε,

$$\begin{aligned}\mathcal{L}w_\varepsilon^+ &\geqslant Y_\xi\big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| - C_5\big(e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - 1\big)|\nabla u_0|^2\big]\\ &\geqslant Y_\xi\big[\big(C_6\mu(\varepsilon\mathcal{G}) - C_5|\nabla u_0|^2\big)e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| + C_5|\nabla u_0|^2\big].\end{aligned}$$

In view of (4.3), this inequality implies that, for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small enough, and for C_6 large enough,

$$\mathcal{L}w_\varepsilon^+ \geqslant \left[C_6\frac{1}{2}\mu - C_5C_0^2 - C_0\right] \geqslant 0.$$

This completes the proof of the lemma. □

Hence, as in Section 3, the comparison principle can be applied to deduce

$$w_\varepsilon^-(x,t) \leqslant u^\varepsilon(x,t) \leqslant w_\varepsilon^+(x,t) \quad \text{for } x \in \overline{\Omega},\ 0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|. \tag{4.11}$$

4.3. *Proof of Theorem 3.1 for the general case*

As in Section 3.3, we first present a key estimate of the function Y after a time interval of order $\tau \sim |\ln\varepsilon|$. Roughly speaking, a perturbation δ of order ε does not affect the result of Lemma 3.9.

Lemma 4.7. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants ε_0 and C_7 such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

(i) *for all $\xi \in (-2C_0, 2C_0)$,*

$$\alpha_- - \eta \leqslant Y\big(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}\big) \leqslant \alpha_+ + \eta; \tag{4.12}$$

(ii) *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \geqslant C_7\varepsilon$, we have that*

$$\text{if } \xi \geqslant a + C_7\varepsilon \quad \text{then } Y\big(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}\big) \geqslant \alpha_+ - \eta, \tag{4.13}$$

$$\text{if } \xi \leqslant a - C_7\varepsilon \quad \text{then } Y\big(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}\big) \leqslant \alpha_- + \eta. \tag{4.14}$$

Proof. In the sequel, by ε we always mean $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ small enough. In view of (4.1), we have, for C_7 large enough, $a + C_7\varepsilon \geqslant a(\pm\varepsilon\mathcal{G}) + \frac{1}{2}C_7\varepsilon$. Hence for $\xi \geqslant a + C_7\varepsilon$, as long as $Y(\tau, \xi; \pm\varepsilon\mathcal{G})$ has not reached $\alpha_+ - \eta$, we can use (4.7) to deduce, as in Section 3, that (4.13) is valid provided that

$$\tau \geqslant \frac{1}{\mu(\pm\varepsilon\mathcal{G})}\ln\frac{m_0 - \eta + C\mathcal{G}\varepsilon}{\frac{1}{2}C_1C_7\varepsilon} =: \mu^{-1}(\varepsilon)|\ln\varepsilon|,$$

where $m_0 = \max(a - \alpha_-, \alpha_+ - a)$. To complete the proof of (4.13) we must choose C_7 so that $\mu^{-1}|\ln\varepsilon| - \mu^{-1}(\varepsilon)|\ln\varepsilon| \geqslant 0$. A simple computation shows that

$$\mu^{-1}|\ln\varepsilon| - \mu^{-1}(\varepsilon)|\ln\varepsilon| = \frac{\mu(\pm\varepsilon\mathcal{G}) - \mu}{\mu(\pm\varepsilon\mathcal{G})\mu}|\ln\varepsilon| - \frac{1}{\mu(\pm\varepsilon\mathcal{G})}\ln\frac{m_0 - \eta + C\mathcal{G}\varepsilon}{\frac{1}{2}C_1C_7}.$$

The first term, thanks to (4.3), is of order $\varepsilon|\ln\varepsilon|$. Hence, for C_7 large enough, the upper quantity can be made positive for all ε. The proof of (4.14) is similar and omitted.

Next we prove (4.12). First, we can assume that the stable zeros of $f_{\pm\varepsilon\mathcal{G}}$, $\alpha_-(\pm\varepsilon\mathcal{G})$ and $\alpha_+(\pm\varepsilon\mathcal{G})$, are in $[\alpha_- - \eta, \alpha_+ + \eta]$. Hence, in view of the profile of $f_{\pm\varepsilon\mathcal{G}}$, if we leave from $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau, \xi; \pm\varepsilon\mathcal{G})$ will remain in $[\alpha_- - \eta, \alpha_+ + \eta]$. Now suppose that $\alpha_+ + \eta \leqslant \xi \leqslant 2C_0$. We check below that $Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \leqslant \alpha_+ + \eta$. As in Section 3, as long as $\alpha_+ + \eta \leqslant Y \leqslant 2C_0$, (3.25) leads to the inequality $Y_\tau \leqslant p(\alpha_+ - Y) + \varepsilon\mathcal{G}$. It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \leqslant -p + \varepsilon\frac{\mathcal{G}}{\eta},$$

which implies, by integration from 0 to τ, that

$$Y(\tau, \xi; \pm\varepsilon\mathcal{G}) \leqslant \alpha_+ + (2C_0 - \alpha_+)e^{(-p+\varepsilon\frac{\mathcal{G}}{\eta})\tau}.$$

One easily checks that, for ε, we have $Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \leqslant \alpha_+ + \eta$, which completes the proof of (4.12). □

We are now ready to prove Theorem 3.1 in the general case. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (4.11), we get

$$\begin{aligned} &Y\big(\mu^{-1}|\ln\varepsilon|, u_0(x) - \varepsilon^2 r\big(-\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|\big); -\varepsilon\mathcal{G}\big) \\ &\quad\leqslant u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \leqslant Y\big(\mu^{-1}|\ln\varepsilon|, u_0(x) + \varepsilon^2 r\big(\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|\big); +\varepsilon\mathcal{G}\big). \end{aligned} \tag{4.15}$$

The point will be that, in view of (4.3),

$$\lim_{\varepsilon\to 0}\frac{\mu - \mu(\pm\varepsilon\mathcal{G})}{\mu}\ln\varepsilon = 0. \tag{4.16}$$

It follows that

$$\varepsilon^2 r\big(\pm\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|\big) = C_6\varepsilon\big(\varepsilon^{(\mu-\mu(\pm\varepsilon\mathcal{G}))/\mu} - \varepsilon\big) \in \left(\frac{1}{2}C_6\varepsilon, \frac{3}{2}C_6\varepsilon\right).$$

Hence, as in Section 3, the result (3.2) of Theorem 3.1 is a direct consequence of (4.12) and (4.15).

Next we prove (3.3). We take $x \in \Omega$ such that $u_0(x) \geqslant a + M_0\varepsilon$; then

$$\begin{aligned} u_0(x) - \varepsilon^2 r\big(-\varepsilon\mathcal{G}, \mu^{-1}(\varepsilon)|\ln\varepsilon|\big) &\geqslant a + M_0\varepsilon - \frac{3}{2}C_6\varepsilon \\ &\geqslant a + C_7\varepsilon, \end{aligned}$$

if we choose M_0 large enough. Using (4.15) and (4.13) we obtain (3.3) which completes the proof of Theorem 3.1.

5. Motion of interface

In Sections 3 and 4, we have proved that the solution u^ε develops a clear transition layer within a very short time. The aim of the present section is to show that, once such a clear transition layer is formed, it persists for the rest of time and that its law of motion is well approximated by the interface equation (P^0).

Let us formulate the above assertion more clearly. By taking the first two terms of the formal asymptotic expansion (2.3), we get a formal approximation of a solution up to order ε:

$$u^\varepsilon(x,t) \approx \tilde{u}^\varepsilon(x,t) := U_0\left(\frac{\tilde{d}(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{\tilde{d}(x,t)}{\varepsilon}\right). \tag{5.1}$$

Here U_0, U_1 are as defined in (2.5) and (2.15). The right-hand side has a clear transition layer which lies exactly on Γ_t. Our goal is to show that this function is a good approximation of a real solution; more precisely:

If u^ε becomes close to $\tilde{u}^\varepsilon$ at some $t = t_0$, then it stays close to $\tilde{u}^\varepsilon$ for the rest of time. Consequently, Γ_t^ε evolves roughly like Γ_t.

In order to prove this assertion, we will construct a pair of sub- and super-solutions u_ε^- and u_ε^+ for problem (P^ε) by slightly modifying the above function $\tilde{u}^\varepsilon$. It then follows that, if the solution u^ε satisfies

$$u_\varepsilon^-(x,t_0) \leqslant u^\varepsilon(x,t_0) \leqslant u_\varepsilon^+(x,t_0),$$

for some $t_0 \geqslant 0$, then

$$u_\varepsilon^-(x,t) \leqslant u^\varepsilon(x,t) \leqslant u_\varepsilon^+(x,t),$$

for $t_0 \leqslant t \leqslant T$, which implies that the solution u^ε stays close to $\tilde{u}^\varepsilon$.

The rest of this section is devoted to the construction of these sub- and super-solutions. We begin with some preparations.

5.1. A modified signed distance function

For our later analysis, it is convenient to introduce a "cut-off signed distance function" d, which is defined as follows. First, choose $d_0 > 0$ small enough so that the signed distance function $\tilde{d}$ defined in (2.1) is smooth in the following tubular neighborhood of Γ:

$$\left\{(x,t) \in \overline{Q_T},\ \left|\tilde{d}(x,t)\right| < 3d_0\right\},$$

and that

$$\operatorname{dist}(\Gamma_t, \partial\Omega) \geqslant 3d_0 \quad \text{for all } t \in [0,T]. \tag{5.2}$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leqslant d_0, \\ -2d_0 & \text{if } s \leqslant -2d_0, \\ 2d_0 & \text{if } s \geqslant 2d_0. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(x,t) = \zeta\big(\tilde{d}(x,t)\big). \tag{5.3}$$

Note that $|\nabla d| = 1$ in the region $\{(x,t) \in \overline{Q_T},\ |\tilde{d}(x,t)| < d_0\}$ and that, in view of (5.2), $\nabla d = 0$ in a neighborhood of $\partial\Omega$. Note also that the equation of motion (P^0), which is equivalent to (2.13), is now written as

$$d_t = \Delta d - \gamma(x,t) \quad \text{on } \Gamma_t, \tag{5.4}$$

where $\gamma(x,t)$ is the function defined in (2.16).

5.2. *Construction of sub- and super-solutions*

As we stated earlier, we now construct sub- and super-solutions by modifying the function $\tilde{u}^\varepsilon$ in (5.1). Concerning the second term U_1, which is defined in (2.15), the terms ΔU_1 and U_{1t} do not make sense as we only assume that $g(\cdot,\cdot,u) \in C^{1+\vartheta,\frac{1+\vartheta}{2}}$. In order to cope with this lack of smoothness, we replace U_1 by a smooth function U_1^ε, which is defined by

$$\begin{cases} U_{1zz}^\varepsilon + f'\big(U_0(z)\big)U_1^\varepsilon = g^\varepsilon\big(x,t,U_0(z)\big) - \gamma^\varepsilon(x,t)U_0'(z), \\ U_1^\varepsilon(x,t,0) = 0, \quad U_1^\varepsilon(x,t,\cdot) \in L^\infty(\mathbb{R}), \end{cases} \tag{5.5}$$

where

$$\gamma^\varepsilon(x,t) = c_0\big(G^\varepsilon(x,t,\alpha_+) - G^\varepsilon(x,t,\alpha_-)\big), \tag{5.6}$$

with $G^\varepsilon(x,t,s) = \int_a^s g^\varepsilon(x,t,r)\,dr$. Thus $U_1^\varepsilon(x,t,z)$ is a solution of (2.9) with

$$A = A_0^\varepsilon(x,t,z) := g^\varepsilon\big(x,t,U_0(z)\big) - \gamma^\varepsilon(x,t)U_0'(z), \tag{5.7}$$

where the variables x, t, ε are considered parameters. Using (1.5) and the same arguments as in the end of Section 2, we obtain estimates analogous to (2.18) and (2.19), with a constant M independent of ε:

$$\big|U_1^\varepsilon(x,t,z)\big| \leqslant M, \qquad \big|\nabla_x U_1^\varepsilon(x,t,z)\big| \leqslant M. \tag{5.8}$$

Moreover, g^ε being C^2 in x and C^1 in t, $\Delta_x U_1^\varepsilon$ and U_{1t}^ε are solutions of (2.9) with $A = \Delta_x A_0^\varepsilon$ and $A = A_{0t}^\varepsilon$, respectively. Thus, in view of (1.3), we obtain

$$\big|\Delta_x U_1^\varepsilon(x,t,z)\big| \leqslant C/\varepsilon, \qquad \big|U_{1t}^\varepsilon(x,t,z)\big| \leqslant C/\varepsilon, \tag{5.9}$$

with some constant C independent of ε. Similarly, (1.5) and Lemma 2.3 yield estimates analogous to (2.23) and (2.24) for U_1^ε, with C and M independent of ε:

$$\left|U_{1z}^\varepsilon(x,t,z)\right| + \left|U_{1zz}^\varepsilon(x,t,z)\right| \leqslant Ce^{-\lambda|z|}, \tag{5.10}$$

$$\left|\nabla_x U_{1z}^\varepsilon(x,t,z)\right| \leqslant M. \tag{5.11}$$

In the rest of this section, C and M will stand for the constants that appear in inequalities (5.8)–(5.11). Note also that (1.7) implies the Neumann boundary conditions (2.26) for U_1^ε.

We look for a pair of sub- and super-solutions $u_\varepsilon^\pm$ for (P^ε) of the form

$$u_\varepsilon^\pm(x,t) = U_0\left(\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) + \varepsilon U_1^\varepsilon\left(x,t,\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) \pm q(t), \tag{5.12}$$

where

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K,$$
$$q(t) = \sigma\left(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\right).$$

Note that $q = \sigma\varepsilon^2 p_t$. It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = \begin{cases} \alpha_+ & \text{for all } (x,t) \in Q_T^+, \\ \alpha_- & \text{for all } (x,t) \in Q_T^-. \end{cases} \tag{5.13}$$

The main result of this section is the following:

Lemma 5.1. *Choose $\beta, \sigma > 0$ appropriately. Then for any $K > 1$, there exist constants $\varepsilon_0, L > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions $(u_\varepsilon^-, u_\varepsilon^+)$ are a pair of sub- and super-solutions for (P^ε) in the domain $\overline{\Omega} \times [0, T]$.*

5.3. Proof of Lemma 5.1

By virtue of (2.26) and the fact that $\nabla d = 0$ near $\partial\Omega$, we have

$$\frac{\partial u_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0, T].$$

What we have to show is

$$\mathcal{L}u_\varepsilon^+ := \left(u_\varepsilon^+\right)_t - \Delta u_\varepsilon^+ - \frac{1}{\varepsilon^2}\left(f\left(u_\varepsilon^+\right) - \varepsilon g^\varepsilon\left(x,t,u_\varepsilon^+\right)\right) \geqslant 0,$$

and that $\mathcal{L}u_\varepsilon^- \leqslant 0$. We will prove only the former inequality for u_ε^+, since the latter follows by the same argument.

5.3.1. Computation of $\mathcal{L}u_\varepsilon^+$

Straightforward computations yield

$$\begin{aligned}
(u_\varepsilon^+)_t &= U_0'\left(\frac{d_t}{\varepsilon} + p_t\right) + \varepsilon U_{1t}^\varepsilon + U_{1z}^\varepsilon(d_t + \varepsilon p_t) + q_t,\\
\nabla u_\varepsilon^+ &= U_0'\frac{\nabla d}{\varepsilon} + \varepsilon \nabla U_1^\varepsilon + U_{1z}^\varepsilon \nabla d,\\
\Delta u_\varepsilon^+ &= U_0''\frac{|\nabla d|^2}{\varepsilon^2} + U_0'\frac{\Delta d}{\varepsilon} + \varepsilon\Delta U_1^\varepsilon + 2\nabla U_{1z}^\varepsilon\cdot\nabla d + U_{1zz}^\varepsilon\frac{|\nabla d|^2}{\varepsilon} + U_{1z}^\varepsilon\Delta d,
\end{aligned}$$

where the function U_0, as well as its derivatives, are evaluated at $z = (d(x,t)+\varepsilon p(t))/\varepsilon$, whereas the function U_1^ε, as well as its derivatives, are evaluated at $(x,t,(d(x,t)+\varepsilon p(t))/\varepsilon)$. Note that ∇ and Δ stand for ∇_x and Δ_x, respectively. We also have

$$\begin{aligned}
f(u_\varepsilon^+) &= f(U_0) + (\varepsilon U_1^\varepsilon + q) f'(U_0) + \frac{1}{2}(\varepsilon U_1^\varepsilon + q)^2 f''(\theta),\\
g(x,t,u_\varepsilon^+) &= g(x,t,U_0) + (\varepsilon U_1^\varepsilon + q) g_u(x,t,\omega),
\end{aligned}$$

where $\theta(x,t)$ and $\omega(x,t)$ are some functions satisfying $U_0 < \theta < u_\varepsilon^+$, $U_0 < \omega < u_\varepsilon^+$. Writing $g^\varepsilon = g + g^\varepsilon - g$ and combining the above expressions with (2.5) and (5.5), we obtain

$$\mathcal{L}u_\varepsilon^+ = E_1 + \cdots + E_7,$$

where

$$\begin{aligned}
E_1 &= -\frac{1}{\varepsilon^2}q\left(f'(U_0) + \frac{1}{2}qf''(\theta)\right) + U_0' p_t + q_t,\\
E_2 &= \left(\frac{U_0''}{\varepsilon^2} + \frac{U_{1zz}^\varepsilon}{\varepsilon}\right)(1 - |\nabla d|^2),\\
E_3 &= \left(\frac{U_0'}{\varepsilon} + U_{1z}^\varepsilon\right)(d_t - \Delta d + \gamma),\\
E_4 &= \varepsilon U_{1z}^\varepsilon p_t + \frac{1}{\varepsilon}q\big(g_u(x,t,\omega) - U_1^\varepsilon f''(\theta)\big),\\
E_5 &= -\gamma U_{1z}^\varepsilon - \frac{1}{2}(U_1^\varepsilon)^2 f''(\theta) + U_1^\varepsilon g_u(x,t,\omega) - 2\nabla U_{1z}^\varepsilon\cdot\nabla d,\\
E_6 &= \varepsilon U_{1t}^\varepsilon - \varepsilon\Delta U_1^\varepsilon,\\
E_7 &= \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,u_\varepsilon^+) - \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,U_0) + \frac{1}{\varepsilon}(\gamma^\varepsilon - \gamma)(x,t)U_0'.
\end{aligned}$$

Before starting to estimate each of the above terms, let us present some useful inequalities. First, by assumption (1.1), there exist positive constants b, m such that

$$f'(U_0(z)) \leqslant -m \quad \text{if } U_0(z) \in [\alpha_-, \alpha_- + b] \cup [\alpha_+ - b, \alpha_+]. \tag{5.14}$$

On the other hand, since the region $\{z \in \mathbb{R} \mid U_0(z) \in [\alpha_- + b, \alpha_+ - b]\}$ is compact and since $U_0' > 0$ on $\mathbb{R}$, there exists a constant $a_1 > 0$ such that

$$U_0'(z) \geqslant a_1 \quad \text{if } U_0(z) \in [\alpha_- + b, \alpha_+ - b]. \tag{5.15}$$

We set

$$\beta = \frac{m}{4}, \tag{5.16}$$

and choose σ that satisfies

$$0 < \sigma \leqslant \min(\sigma_0, \sigma_1, \sigma_2), \tag{5.17}$$

where

$$\sigma_0 := \frac{a_1}{m + F_1}, \qquad \sigma_1 := \frac{1}{\beta + 1}, \qquad \sigma_2 := \frac{4\beta}{F_2(\beta + 1)},$$
$$F_1 := \|f'\|_{L^\infty(\alpha_-, \alpha_+)}, \qquad F_2 := \|f''\|_{L^\infty(\alpha_- - 2, \alpha_+ + 2)}.$$

Combining (5.14) and (5.15), and considering that $\sigma \leqslant \sigma_0$, we obtain

$$U_0'(z) - \sigma f'\big(U_0(z)\big) \geqslant \sigma m \quad \text{for } -\infty < z < \infty. \tag{5.18}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $\mathcal{L}u_\varepsilon^+ \geqslant 0$ provided that the constants ε_0 and L are appropriately chosen. We recall that $\alpha_- < U_0 < \alpha_+$. We go on under the following assumption

$$\varepsilon_0 M \leqslant 1, \qquad \varepsilon_0^2 L e^{LT} \leqslant 1. \tag{5.19}$$

Then, given any $\varepsilon \in (0, \varepsilon_0)$, we have $\varepsilon|U_1^\varepsilon(x,t,z)| \leqslant 1$ and, since $\sigma \leqslant \sigma_1$, $0 \leqslant q(t) \leqslant 1$, so that

$$\alpha_- - 2 \leqslant u_\varepsilon^\pm(x,t) \leqslant \alpha_+ + 2. \tag{5.20}$$

5.3.2. The term E_1

Direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + L e^{Lt}\big(I + \varepsilon^2 \sigma L\big),$$

where

$$I = U_0' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)\big(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\big).$$

In virtue of (5.18) and (5.20), we have

$$I \geqslant \sigma m - \frac{\sigma^2}{2} F_2\big(\beta + \varepsilon^2 L e^{LT}\big).$$

Combining this, (5.19) and the inequality $\sigma \leqslant \sigma_2$, we obtain $I \geqslant 2\sigma\beta$. Consequently, we have

$$E_1 \geqslant \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt}.$$

5.3.3. The term E_2

First, in the region where $|d| \leqslant d_0$, we have $|\nabla d| = 1$, hence $E_2 = 0$. Next we consider the region where $|d| \geqslant d_0$. We deduce from Lemma 2.1 and from (5.10) that

$$|E_2| \leqslant C\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) e^{-\lambda|d+\varepsilon p|/\varepsilon} \leqslant \frac{2C}{\varepsilon^2} e^{-\lambda(d_0/\varepsilon - |p|)}.$$

We remark that $0 < K - 1 \leqslant p \leqslant e^{LT} + K$. Consequently, if we assume

$$e^{LT} + K \leqslant \frac{d_0}{2\varepsilon_0}, \tag{5.21}$$

then $\frac{d_0}{\varepsilon} - |p| \geqslant \frac{d_0}{2\varepsilon}$, so that

$$|E_2| \leqslant \frac{2C}{\varepsilon^2} e^{-\lambda d_0/(2\varepsilon)} \leqslant C_2 := \frac{32C}{(e\lambda d_0)^2}.$$

5.3.4. The term E_3

By (5.4) and (2.16), we have

$$(d_t - \Delta d + \gamma)(x,t) = 0 \quad \text{on } \Gamma_t = \{x \in \Omega,\ d(x,t) = 0\}.$$

Since γ is of class $C^{1+\vartheta, \frac{1+\vartheta}{2}}$ by virtue of (1.5), we see that the interface Γ_t is of class $C^{3+\vartheta, \frac{3+\vartheta}{2}}$. Therefore both Δd and d_t are Lipschitz continuous near Γ_t. It follows that there exists a constant $N > 0$ such that

$$\left|(d_t - \Delta d + \gamma)(x,t)\right| \leqslant N\left|d(x,t)\right| \quad \text{for all } (x,t) \in Q_T.$$

Applying Lemma 2.1 and the estimate (5.10) we deduce that

$$\begin{aligned} |E_3| &\leqslant 2NC \frac{|d|}{\varepsilon} e^{-\lambda|d/\varepsilon + p|} \\ &\leqslant 2NC \max_{\xi \in \mathbb{R}} |\xi| e^{-\lambda|\xi + p|} \\ &\leqslant 2NC \max\left(|p|, \frac{1}{\lambda}\right). \end{aligned}$$

Thus, recalling that $|p| \leqslant e^{Lt} + K$, we obtain

$$|E_3| \leqslant C_3\left(e^{Lt} + K\right) + C_3',$$

where $C_3 := 2NC$ and $C_3' := 2NC/\lambda$.

5.3.5. The term E_4

In view of (1.4) and (5.10), both g_u and $|U^\varepsilon_{1z}|$ are bounded by some constant C. Hence, substituting the expression for p_t and q, we obtain

$$|E_4| \leqslant C_4\left(\frac{1}{\varepsilon}\beta e^{-\beta t/\varepsilon^2} + \varepsilon L e^{Lt}\right),$$

where $C_4 := C + \sigma(C + MF_2)$.

5.3.6. The term E_5

In view of (2.16), the term $|\gamma|$ is bounded by $c_0(\alpha_+ - \alpha_-)C$ on $\overline{\Omega} \times [0, T]$. Using (1.4) and (5.11), we easily obtain $|E_5| \leqslant C_5$, where C_5 depends only on C, M, F_2.

5.3.7. The term E_6

We use (5.9) to deduce that $|E_6| \leqslant 2C =: C_6$.

5.3.8. Finally the term E_7

We recall that $|g^\varepsilon - g| \leqslant C\varepsilon$ so that $|\gamma^\varepsilon - \gamma| \leqslant c_0(\alpha_+ - \alpha_-)C\varepsilon$. It then follows that

$$|E_7| \leqslant 2C + Cc_0(\alpha_+ - \alpha_-) =: C_7.$$

5.3.9. Completion of the proof

Collecting all these estimates gives

$$\mathcal{L}u^+_\varepsilon \geqslant \left(\frac{\sigma\beta^2}{\varepsilon^2} - \frac{C_4\beta}{\varepsilon}\right)e^{-\beta t/\varepsilon^2} + (2\sigma\beta L - C_3 - \varepsilon C_4 L)e^{Lt} - C_8, \tag{5.22}$$

where $C_8 := C_2 + KC_3 + C_3' + C_5 + C_6 + C_7$. Now we set

$$L := \frac{1}{T}\ln\frac{d_0}{4\varepsilon_0},$$

which, for ε_0 small enough, validates assumptions (5.19) and (5.21). For ε_0 small enough, the first term of the right-hand side of (5.22) is positive, hence

$$\mathcal{L}u^+_\varepsilon \geqslant [\sigma\beta L - C_3]e^{Lt} - C_8 \geqslant \frac{1}{2}\sigma\beta L - C_8 \geqslant 0.$$

The proof of Lemma 5.1 is now complete, with the choice of the constants β, σ as in (5.16), (5.17).

6. Proof of the main results

6.1. Proof of Theorem 1.3

Let $\eta \in (0, \eta_0)$ be arbitrary. Choose β and σ that satisfy (5.16), (5.17) and

$$\sigma\beta \leqslant \frac{\eta}{3}. \tag{6.1}$$

By Theorem 3.1, there exist positive constants ε_0 and M_0 such that (3.2)–(3.4) hold with the constant η replaced by $\sigma\beta/2$. Since $\nabla u_0 \cdot n \neq 0$ everywhere on $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ and since Γ_0 is a compact hypersurface, we can find a positive constant M_1 such that

$$\begin{aligned} &\text{if } d_0(x) \geqslant M_1\varepsilon \quad \text{then } u_0(x) \geqslant a + M_0\varepsilon, \\ &\text{if } d_0(x) \leqslant -M_1\varepsilon \quad \text{then } u_0(x) \leqslant a - M_0\varepsilon. \end{aligned} \tag{6.2}$$

Here $d_0(x) := \tilde{d}(x,0)$ denotes the signed distance function associated with the hypersurface Γ_0. Now we define functions $H^+(x)$, $H^-(x)$ by

$$H^+(x) = \begin{cases} \alpha_+ + \sigma\beta/2 & \text{if } d_0(x) \geqslant -M_1\varepsilon, \\ \alpha_- + \sigma\beta/2 & \text{if } d_0(x) < -M_1\varepsilon, \end{cases}$$

$$H^-(x) = \begin{cases} \alpha_+ - \sigma\beta/2 & \text{if } d_0(x) \geqslant M_1\varepsilon, \\ \alpha_- - \sigma\beta/2 & \text{if } d_0(x) < M_1\varepsilon. \end{cases}$$

Then from the above observation we see that

$$H^-(x) \leqslant u^\varepsilon\big(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\big) \leqslant H^+(x) \quad \text{for } x \in \Omega. \tag{6.3}$$

Next we fix a sufficiently large constant $K > 0$ such that

$$U_0(-M_1 + K) \geqslant \alpha_+ - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(M_1 - K) \leqslant \alpha_- + \frac{\sigma\beta}{3}. \tag{6.4}$$

For this K, we choose ε_0 and L as in Lemma 5.1. We claim that

$$u_\varepsilon^-(x,0) \leqslant H^-(x), \qquad H^+(x) \leqslant u_\varepsilon^+(x,0) \quad \text{for } x \in \Omega. \tag{6.5}$$

We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x,0) = U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) + \varepsilon U_1^\varepsilon\left(x, 0, \frac{d_0(x)}{\varepsilon} - K\right) - \sigma\big(\beta + \varepsilon^2 L\big) \leqslant H^-(x). \tag{6.6}$$

By (5.8) we have $|U_1^\varepsilon| \leqslant M$. Therefore, by choosing ε_0 small enough so that $\varepsilon_0 M \leqslant \sigma\beta/6$, we see that

$$\begin{aligned} u_\varepsilon^-(x,0) &\leqslant U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) + \varepsilon M - \sigma\big(\beta + \varepsilon^2 L\big) \\ &\leqslant U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta. \end{aligned}$$

In the range where $d_0(x) < M_1\varepsilon$, the second inequality in (6.4) and the fact that U_0 is an increasing function imply

$$U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta \leqslant \alpha_- - \frac{\sigma\beta}{2} = H^-(x).$$

On the other hand, in the range where $d_0(x) \geqslant M_1\varepsilon$, we have

$$U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta \leqslant \alpha_+ - \frac{5}{6}\sigma\beta \leqslant H^-(x).$$

This proves (6.6), hence (6.5) is established.

Combining (6.3) and (6.5), we obtain

$$u_\varepsilon^-(x,0) \leqslant u^\varepsilon\bigl(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\bigr) \leqslant u_\varepsilon^+(x,0).$$

Since u_ε^- and u_ε^+ are sub- and super-solutions of (P^ε) thanks to Lemma 5.1, the comparison principle yields

$$u_\varepsilon^-(x,t) \leqslant u^\varepsilon\bigl(x, t + t^\varepsilon\bigr) \leqslant u_\varepsilon^+(x,t) \quad \text{for } 0 \leqslant t \leqslant T - t^\varepsilon, \tag{6.7}$$

where $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$. Note that, in view of (5.13), this is enough to prove Corollary 1.4. Now let C be a positive constant such that

$$U_0\bigl(C - e^{LT} - K\bigr) \geqslant \alpha_+ - \frac{\eta}{2} \quad \text{and} \quad U_0\bigl(-C + e^{LT} + K\bigr) \leqslant \alpha_- + \frac{\eta}{2}. \tag{6.8}$$

One then easily checks, using (6.7) and (6.1), that, for ε_0 small enough, for $0 \leqslant t \leqslant T - t^\varepsilon$, we have

$$\begin{aligned} &\text{if } d(x,t) \geqslant C\varepsilon \quad \text{then } u^\varepsilon\bigl(x, t + t^\varepsilon\bigr) \geqslant \alpha_+ - \eta, \\ &\text{if } d(x,t) \leqslant -C\varepsilon \quad \text{then } u^\varepsilon\bigl(x, t + t^\varepsilon\bigr) \leqslant \alpha_- + \eta, \end{aligned} \tag{6.9}$$

and

$$u^\varepsilon\bigl(x, t + t^\varepsilon\bigr) \in [\alpha_- - \eta, \alpha_+ + \eta],$$

which completes the proof of Theorem 1.3.

6.2. *Proof of Theorem 1.5*

In the case where $\mu^{-1}\varepsilon^2|\ln\varepsilon| \leqslant t \leqslant T$, the assertion of the theorem is a direct consequence of Theorem 1.3. Thus, all we have to consider is the case where $0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|$. We first need the following lemma concerning Y, the solution of the perturbed ordinary differential equation (4.4).

Lemma 6.1. *There exists a constant $C_8 > 0$ such that*

$$\begin{aligned} &\textit{if } \xi \geqslant a + C_8\varepsilon \quad \textit{then } Y(\tau, \xi; \pm\varepsilon\mathcal{G}) > a \quad \textit{for } 0 \leqslant \tau \leqslant \mu^{-1}|\ln\varepsilon|, \\ &\textit{if } \xi \leqslant a - C_8\varepsilon \quad \textit{then } Y(\tau, \xi; \pm\varepsilon\mathcal{G}) < a \quad \textit{for } 0 \leqslant \tau \leqslant \mu^{-1}|\ln\varepsilon|. \end{aligned} \tag{6.10}$$

Proof. We only prove the first inequality. In view of estimates (4.7) and (4.1), we obtain, for $\xi \geqslant a + C_8\varepsilon$,

$$\begin{aligned} Y(\tau,\xi;\pm\varepsilon\mathcal{G}) &\geqslant a(\pm\varepsilon\mathcal{G}) + C_1 e^{\mu(\pm\varepsilon\mathcal{G})\tau}\big(a + C_8\varepsilon - a(\pm\varepsilon\mathcal{G})\big) \\ &\geqslant a - C\mathcal{G}\varepsilon + C_1(-C\mathcal{G}\varepsilon + C_8\varepsilon) \\ &\geqslant a + \varepsilon\big(C_1C_8 - C\mathcal{G}(C_1+1)\big) \\ &> a, \end{aligned}$$

if we choose C_8 large enough. □

Now we turn to the proof of Theorem 1.5. We first claim that there exists a positive constant M_2 such that for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \tag{6.11}$$

To see this, we choose M_0 large enough, so that $M_0 \geqslant C_8 + 2C_6$ holds in addition to (3.2), (3.3) and (3.4). We then choose $M_2 > M_1$, where M_1 is as defined in (6.2). In view of this last condition, we see that if ε_0 is small enough and if $d_0(x) \geqslant M_2\varepsilon$, then for $0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{aligned} u_0(x) - \varepsilon^2 r\left(-\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}\right) &\geqslant a + M_0\varepsilon - \varepsilon^2 C_6\big[e^{\mu(-\varepsilon\mathcal{G})|\ln\varepsilon|/\mu} - 1\big] \\ &\geqslant a + \varepsilon\big[M_0 - C_6\varepsilon^{(\mu-\mu(\pm\varepsilon\mathcal{G}))/\mu} + \varepsilon C_6\big] \\ &\geqslant a + \varepsilon(M_0 - 2C_6) \quad \big(\leftarrow \text{thanks to (4.16)}\big) \\ &\geqslant a + C_8\varepsilon. \end{aligned}$$

This inequality and Lemma 6.1 imply $w_\varepsilon^-(x,t) > a$, where w_ε^- is the sub-solution defined in (3.19). Consequently, by (3.20),

$$u^\varepsilon(x,t) > a \quad \text{if } d_0(x) \geqslant M_2\varepsilon.$$

In the case where $d_0(x) \leqslant -M_2\varepsilon$, similar arguments lead to $u^\varepsilon(x,t) < a$. This completes the proof of (6.11). Note that we have proved that, for all $0 \leqslant t \leqslant \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{aligned} &u^\varepsilon(x,t) > a \quad \text{if } x \in \Omega_0^+ \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \\ &u^\varepsilon(x,t) < a \quad \text{if } x \in \Omega_0^- \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \end{aligned} \tag{6.12}$$

Since Γ_t depends on t smoothly, there is a constant $\tilde{C} > 0$ such that, for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_0 \subset \mathcal{N}_{\tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t), \tag{6.13}$$

and

$$\Omega_t^+ \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^+ \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0),$$
$$\Omega_t^- \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^- \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \tag{6.14}$$

As a consequence of (6.11) and (6.13) we get

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon+\tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t) \subset \mathcal{N}_{C\varepsilon}(\Gamma_t),$$

which completes the proof of Theorem 1.5.

Proof of Corollary 1.6. In view of Theorem 1.5 and the definition of the Hausdorff distance, to prove this corollary we only need to show that

$$\Gamma_t \subset \mathcal{N}_{C'\varepsilon}\big(\Gamma_t^\varepsilon\big) \quad \text{for } 0 \leqslant t \leqslant T, \tag{6.15}$$

for some constant $C' > 0$. To that purpose let C' be a constant satisfying $C' > \max(\tilde{C}, C)$, where C is as in Theorem 1.3 and $\tilde{C}$ as in (6.14). Choose $t \in [0,T]$ and $x_0 \in \Gamma_t$ arbitrarily and, n being the Euclidean normal vector exterior to Γ_t at point x_0, define a pair of points:

$$x^+ := x_0 + C'\varepsilon n \quad \text{and} \quad x^- := x_0 - C'\varepsilon n.$$

Since $C' > C$ and since the curvature of Γ_t is uniformly bounded as t varies over $[0,T]$, we see that

$$x^+ \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{and} \quad x^- \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t),$$

if ε is sufficiently small. Therefore, if $t \in [\mu^{-1}\varepsilon^2|\ln\varepsilon|, T]$, then, by Theorem 1.3, we have

$$u^\varepsilon(x^-,t) < a < u^\varepsilon(x^+,t). \tag{6.16}$$

On the other hand, if $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, then from (6.12), (6.14) and the fact that $C' > \tilde{C}$, we again obtain (6.16). Thus (6.16) holds for all $t \in [0,T]$. Now, by the mean value theorem, we see that for each $t \in [0,T]$ there exists a point x_1 on the line segment $[x^-, x^+]$ such that $u^\varepsilon(x_1,t) = a$. This implies $x_1 \in \Gamma_t^\varepsilon$. Furthermore we have $d(x_0,x_1) \leqslant C'\varepsilon$, since x_1 lies on the line segment $[x^-, x^+]$. This proves (6.15). □

7. Application to reaction–diffusion systems

In this section we discuss the singular limit of the reaction–diffusion system (RD^ε) and prove Theorems 1.11, 1.13 and their corollaries. Our strategy is to regard the first equation of (RD^ε) as a perturbed Allen–Cahn equation and apply what we have already proved for this equation.

7.1. Preliminaries: Global existence

Before studying the singular limit of (RD^ε), we first show that the solution of this system exists globally for $t \geqslant 0$, provided that ε is sufficiently small. Recall that the system (RD^ε) is written in the form

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big(f(u) + \varepsilon f_1(u,v) + O(\varepsilon^2)\big), \\ v_t = D\Delta v + h(u,v), \end{cases}$$

where $h(u,v)$ satisfies the hypothesis (H). The standard parabolic theory guarantees the existence of local solutions for (RD^ε). In order to prove that the solution exists globally for $t \geqslant 0$, it suffices to show that the solution remains uniformly bounded. This will be done by using the well-known method of invariant rectangles.

Given arbitrary $u_0, v_0 \in C(\overline{\Omega})$, we choose a constant $L > 0$ such that

$$f(-L) > 0 > f(L), \qquad -L \leqslant u_0(x) \leqslant L \quad \text{for } x \in \overline{\Omega}. \tag{7.1}$$

Such a constant L exists since $f(u) > 0$ for $u < \alpha_-$ and $f(u) < 0$ for $u > \alpha_+$. By hypothesis (H), we can choose a constant M_1 satisfying

$$M_1 \geqslant \|v_0\|_{L^\infty(\Omega)},$$

along with the condition (1.18), namely

$$h(u, -M_1) \geqslant 0 \geqslant h(u, M_1) \quad \text{for } |u| \leqslant L. \tag{7.2}$$

Now we consider the rectangle

$$\mathcal{R} := \big\{(u,v) \in \mathbb{R}^2 \;\big|\; |u| \leqslant L,\ |v| \leqslant M_1\big\}.$$

It follows from (7.1) that, for all sufficiently small $\varepsilon > 0$,

$$f^\varepsilon(-L, v) > 0 > f^\varepsilon(L, v) \quad \text{for } |v| \leqslant M_1. \tag{7.3}$$

The inequalities (7.2) and (7.3) imply that the rectangle $\mathcal{R}$ is a positively invariant region for the system of ordinary differential equations

$$\begin{cases} u_t = \dfrac{1}{\varepsilon^2} f^\varepsilon(u,v), \\ v_t = h(u,v), \end{cases}$$

since the vector field $(\varepsilon^{-2} f^\varepsilon(u,v), h(u,v))$ points inwards everywhere on the boundary of $\mathcal{R}$. The maximum principle then implies that $\mathcal{R}$ is also positively invariant for the system (RD^ε). Consequently, since $(u_0(x), v_0(x)) \in \mathcal{R}$ for $x \in \overline{\Omega}$, we have

$$\big(u(x,t), v(x,t)\big) \in \mathcal{R} \quad \text{for } x \in \overline{\Omega},\ t \geqslant 0,$$

so long as the solution is defined. This uniform bound then implies that the solution exists globally for $t \geqslant 0$.

In the case of equations for which only nonnegative solutions are to be considered (see Remark 1.9), we can argue just similarly, by replacing $\mathcal{R}$ by the rectangle $\mathcal{R}_+ := \{(u,v) \mid 0 \leqslant u \leqslant L,\ 0 \leqslant v \leqslant M_1\}$. Summarizing, we have proved the following proposition:

Proposition 7.1. *Let* $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. *In the case where the conditions of Remark* 1.9 *apply, assume further that* $u_0, v_0 \geqslant 0$. *Then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, *the solution of* (RD^ε) *exists globally for* $t \geqslant 0$ *and is uniformly bounded.*

Remark 7.2. For the details of the method of invariant rectangles, we refer the reader to the book [25, Chapter 14, Corollary 14.8]. See also [11] and [10]. It should be noted that [19] makes a much earlier study of invariant rectangles for a finite-difference scheme for reaction–diffusion systems.

7.2. *Proof of the main results*

Now we turn to the reaction–diffusion system (RD^ε) and explain our strategy for proving Theorems 1.11, 1.13 and their corollaries.

In what follows, we fix the initial data (u_0, v_0) and denote by $(u^\varepsilon, v^\varepsilon)$ the solution of the system (RD^ε). The solution of the associated moving boundary problem (RD^0) will be denoted by $(\Gamma, \tilde{v})$.

Given a function $v(x,t)$ on $\overline{\Omega} \times [0, +\infty)$, we set

$$g^\varepsilon[v](x,t,u) := -f_1\big(u, v(x,t)\big) - \varepsilon f_2^\varepsilon\big(u, v(x,t)\big),$$
$$g[v](x,t,u) := -f_1\big(u, v(x,t)\big), \tag{7.4}$$

where f_1, f_2^ε are as in (1.17). The first equation of (RD^ε) is then written in the form

$$u_t = \Delta u + \frac{1}{\varepsilon^2}\big(f(u) - \varepsilon g^\varepsilon[v](x,t,u)\big), \tag{7.5}$$

so that $u^\varepsilon(x,t)$ is the solution of (P^ε) with the choice of the perturbation term $g^\varepsilon(x,t,u) = g^\varepsilon[v^\varepsilon](x,t,u)$. On the other hand, the equation of surface motion in the limit problem (RD^0) is written in the form

$$V_n = -(N-1)\kappa + c_0 \int_{\alpha_-}^{\alpha_+} g[\tilde{v}](x,t,r)\,dr \quad \text{on } \Gamma_t, \tag{7.6}$$

so that Γ is the solution of (P^0) with the choice $g(x,t,u) = g[\tilde{v}](x,t,u)$.

Thus Theorems 1.11, 1.13 and their corollaries will follow from what we have shown for the single equation (P^ε). In order for Theorems 1.3 and 1.5 for (P^ε) to be applicable to the present reaction–diffusion system (RD^ε), all we have to do is to verify the conditions (1.3) to (1.6). More precisely, we have to show that, for all small $\varepsilon > 0$,

$$\big|\Delta_x g^\varepsilon\big[v^\varepsilon\big](x,t,u)\big| \leqslant C\varepsilon^{-1} \quad \text{and} \quad \big|\partial_t g^\varepsilon\big[v^\varepsilon\big](x,t,u)\big| \leqslant C\varepsilon^{-1},$$
$$\big|\partial_u g^\varepsilon\big[v^\varepsilon\big](x,t,u)\big| \leqslant C,$$

$$\left\| g^\varepsilon\left[v^\varepsilon\right](\cdot,\cdot,u)\right\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leqslant C,$$
$$\left| g^\varepsilon\left[v^\varepsilon\right](x,t,u) - g[\tilde{v}](x,t,u)\right| \leqslant C\varepsilon.$$

Since $g[v]$, $g^\varepsilon[v]$ are defined by (7.4) and since f_1, f_2^ε are smooth, it suffices to prove the following estimates for some $C > 0$ and for all small $\varepsilon > 0$:

$$\left|\Delta_x v^\varepsilon(x,t)\right| \leqslant C\varepsilon^{-1} \quad \text{and} \quad \left|\partial_t v^\varepsilon(x,t)\right| \leqslant C\varepsilon^{-1}, \tag{7.7}$$

$$\left\| v^\varepsilon\right\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leqslant C, \tag{7.8}$$

$$\left| v^\varepsilon(x,t) - \tilde{v}(x,t)\right| \leqslant C\varepsilon. \tag{7.9}$$

The estimates (7.7) and (7.8) are elementary, but (7.9) requires far more elaborate analysis. In this subsection we prove (7.7), (7.8) and give an outline of the proof of (7.9). A full proof of (7.9) will be given later.

Proof of (7.8) and (7.7). Since v^ε satisfies

$$v_t^\varepsilon = D\Delta v^\varepsilon + h\left(u^\varepsilon, v^\varepsilon\right) \quad \text{in } \Omega\times(0,T], \tag{7.10}$$

along with the Neumann boundary conditions, it can be expressed as

$$v^\varepsilon(x,t) = I_1 + I_2, \tag{7.11}$$

where

$$I_1 := \int_\Omega G(x,y,t)v_0(y)\,dy,$$

$$I_2 := \int_0^t \int_\Omega G(x,y,t-s)h\left(u^\varepsilon(y,s), v^\varepsilon(y,s)\right)dy\,ds,$$

with $G(x,y,t)$ being the fundamental solution for equation $v_t = D\Delta v$ under the Neumann boundary conditions. Since $h(u^\varepsilon, v^\varepsilon)$ is uniformly bounded, standard estimates of $G(x,y,t)$ imply (7.8) for any $\vartheta\in(0,1)$.

In the mean while, the same rescaling argument as in Remark 1.7 yields

$$\left\| u^\varepsilon\right\|_{C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leqslant C\varepsilon^{-\vartheta}. \tag{7.12}$$

Indeed, since $\nabla_y u$, u_τ are bounded, where $y = x/\varepsilon$, $\tau = t/\varepsilon^2$, we have $\nabla_x u = O(1/\varepsilon)$, $u_t = O(1/\varepsilon^2)$. Consequently we have

$$\begin{aligned}\frac{|u(x,t)-u(x',t')|}{|x-x'|^{\vartheta}+|t-t'|^{\vartheta/2}} &\leqslant \frac{|u(x,t)-u(x',t)|}{|x-x'|^{\vartheta}}+\frac{|u(x',t)-u(x',t')|}{|t-t'|^{\vartheta/2}} \\ &\leqslant \big|u(x,t)-u(x',t)\big|^{1-\vartheta}\frac{|u(x,t)-u(x',t)|^{\vartheta}}{|x-x'|^{\vartheta}} \\ &\quad + \big|u(x',t)-u(x',t')\big|^{1-\vartheta/2}\frac{|u(x',t)-u(x',t')|^{\vartheta/2}}{|t-t'|^{\vartheta/2}} \\ &\leqslant \big(2\|u\|_{L^\infty}\big)^{1-\vartheta}\|\nabla_x u\|_{L^\infty}^{\vartheta}+\big(2\|u\|_{L^\infty}\big)^{1-\vartheta/2}\|u_t\|_{L^\infty}^{\vartheta/2} \\ &\leqslant C\varepsilon^{-\vartheta}.\end{aligned}$$

Combining (7.12) and (7.8), we see that $\|h(u^\varepsilon,v^\varepsilon)\|_{C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])}\leqslant C\varepsilon^{-\vartheta}$, hence, by the Schauder estimate,

$$\|I_2\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])}\leqslant C\varepsilon^{-\vartheta}.$$

Here the constant C may depend on the choice of $\vartheta\in(0,1)$. On the other hand, I_1 is bounded in $C^{2,1}(\overline{\Omega}\times[0,T])$ since $v_0\in C^2(\overline{\Omega})$. Combining these, we obtain $|\Delta_x v^\varepsilon(x,t)|=O(\varepsilon^{-\vartheta})$, hence $O(\varepsilon^{-1})$. Substituting this into (7.10) yields the second inequality in (7.7). □

Outline of the proof of (7.9). We decouple the system (RD^ε) as follows. As mentioned earlier, $u^\varepsilon(x,t)$ is the solution of (P^ε) with the choice of the perturbation term $g^\varepsilon(x,t,u)=g^\varepsilon[v^\varepsilon](x,t,u)$, that is,

$$(\star)\quad \begin{cases} u_t=\Delta u+\dfrac{1}{\varepsilon^2}\big(f(u)-\varepsilon g^\varepsilon(x,t,u)\big),\\ \dfrac{\partial u}{\partial\nu}=0,\\ u(x,0)=u_0(x).\end{cases}$$

Once the solution u^ε is determined, v^ε is the solution of the problem

$$(\star\star)\quad \begin{cases} v_t=D\Delta v+h(u,v),\\ \dfrac{\partial v}{\partial\nu}=0,\\ v(x,0)=v_0(x),\end{cases}$$

with the choice $u=u^\varepsilon$. This means that $v^\varepsilon(x,t)$ is a fixed point of the following map $\Phi^\varepsilon:=\Phi_2\circ\Phi_1^\varepsilon$:

$$\Phi^\varepsilon: v\xrightarrow{\Phi_1^\varepsilon\text{ via }(\star)}\bar{u}^\varepsilon\xrightarrow{\Phi_2\text{ via }(\star\star)}\bar{v}^\varepsilon,$$

where Φ_1^ε maps a function $v(x,t)$ to the solution $\bar{u}^\varepsilon(x,t)$ of $(\star)$ for the choice $g^\varepsilon(x,t,u)=g^\varepsilon[v](x,t,u)$, and Φ_2 maps a function $\bar{u}(x,t)$ to the solution $\bar{v}(x,t)$ of $(\star\star)$ for the choice $u=\bar{u}$.

On the other hand, as for the limit problem (RD^0), the solution $\tilde{v}(x,t)$ can be regarded as a fixed point of the map $\Phi^0:=\Phi_2\circ\Phi_1^0$:

$$\Phi^0: v\xrightarrow{\Phi_1^0\text{ via }(\star\star\star)}\hat{u}\xrightarrow{\Phi_2\text{ via }(\star\star)}\hat{v},$$

where Φ_1^0 maps a function $v(x,t)$ to the step function

$$\hat{u}(x,t) = \begin{cases} \alpha_+ & \text{in } \Omega^+(\Gamma_t[v]), \\ \alpha_- & \text{in } \Omega^-(\Gamma_t[v]), \end{cases}$$

where $\Gamma_t[v]$ is the solution of the equation of surface motion

$$(\star\star\star) \quad \begin{cases} V_n = -(N-1)\kappa - c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} f_1\big(r, v(x,t)\big)\,dr \quad \text{on } \Gamma_t, \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

and $\Omega^-(\Gamma)$ denotes the region enclosed by the hypersurface Γ and $\Omega^+(\Gamma)$ the region between $\partial\Omega$ and Γ.

In what follows we set

$$Q_t := \Omega \times (0,t) \quad \text{for } 0 < t \leqslant T.$$

Given $\delta_0 > 0$, we define $t_{\max} = t_{\max}(\delta_0) > 0$ by

$$t_{\max} = \max\big\{t \in [0,T],\ \big\|v^\varepsilon - \tilde{v}\big\|_{L^\infty(Q_t)} \leqslant \delta_0\big\}. \tag{7.13}$$

The key estimates for proving (7.9) are the following:

Claim 7.3. *There exist constants $\delta_0 > 0$ and $C > 0$ such that, for any $t \in (0, t_{\max}]$, we have*

$$\big\|\Phi^0\big(v^\varepsilon\big) - \Phi^0(\tilde{v})\big\|_{L^\infty(Q_t)} \leqslant C \int_0^t \frac{1}{\sqrt{t-s}} \big\|v^\varepsilon - \tilde{v}\big\|_{L^\infty(Q_s)}\,ds. \tag{7.14}$$

Claim 7.4. *There exists a constant $A > 0$ such that, for any v satisfying the estimates* (7.7), (7.8) *and the Neumann boundary conditions, and for any $t \in (0,T]$, we have*

$$\big\|\Phi^\varepsilon(v) - \Phi^0(v)\big\|_{L^\infty(Q_t)} \leqslant A\varepsilon. \tag{7.15}$$

The proof of these claims will be given later. For the moment, let us simply mention that Claim 7.4 can be shown by the following two-step argument: first, our results on the single equation (P^ε) yields

$$\Phi_1^\varepsilon(v) - \Phi_1^0(v) = O(\varepsilon),$$

in the sense that the transition layer of $\Phi_1^\varepsilon(v)$ and that of $\Phi_1^0(v)$ are within an $O(\varepsilon)$ distance; this observation and an estimate of the heat kernel yield (7.15). To prove Claim 7.3, we also use a similar estimate of the heat kernel, see Lemma 7.6 and Section 7.7 for details.

Combining these estimates, we obtain, for any $t \in (0, t_{\max}]$,

$$\begin{aligned}\|v^\varepsilon - \tilde{v}\|_{L^\infty(Q_t)} &= \|\Phi^\varepsilon(v^\varepsilon) - \Phi^0(\tilde{v})\|_{L^\infty(Q_t)} \\ &\leqslant \|\Phi^\varepsilon(v^\varepsilon) - \Phi^0(v^\varepsilon)\|_{L^\infty(Q_t)} + \|\Phi^0(v^\varepsilon) - \Phi^0(\tilde{v})\|_{L^\infty(Q_t)} \\ &\leqslant A\varepsilon + C\int_0^t \frac{1}{\sqrt{t-s}}\|v^\varepsilon - \tilde{v}\|_{L^\infty(Q_s)}\,ds.\end{aligned}$$

As we will see later in Lemma 7.7, this implies

$$\|v^\varepsilon - \tilde{v}\|_{L^\infty(Q_t)} \leqslant A\varepsilon\bar{k}(t) \quad (0 \leqslant t \leqslant t_{\max}), \tag{7.16}$$

where $\bar{k}$ is the function determined by the integral equality:

$$\bar{k}(t) = 1 + C\int_0^t \frac{\bar{k}(s)}{\sqrt{t-s}}\,ds. \tag{7.17}$$

Since $\bar{k}(t)$ is bounded on any finite interval $[0, T]$, we obtain $\|v^\varepsilon - \tilde{v}\|_{L^\infty(Q_t)} = O(\varepsilon)$, for $0 \leqslant t \leqslant t_{\max}$. This implies, first of all, that $t_{\max} = T$ if ε is small enough, hence it proves (7.9), for $0 \leqslant t \leqslant T$. □

The rest of this section gives a detailed account of the proof of (7.9). We begin with some notations to clarify the statements of the above claims.

7.3. Some notations

Given any function $\bar{g}(x,t,u)$ satisfying the conditions (1.3) and (1.5), we can define a classical solution of the interface equation (P^0) on some time interval $0 \leqslant t < T^{\max}(\bar{g})$. We denote this solution by $\Gamma_t[\bar{g}]$ in order to clarify its dependence on $\bar{g}$. More precisely, $\Gamma_t[\bar{g}]$ is a solution of the problem

$$(P^0_{\bar{g}}) \quad \begin{cases} V_n = -(N-1)\kappa + c_0\displaystyle\int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr & \text{on } \Gamma_t, \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases}$$

Also, we denote by $u^\varepsilon[\bar{g}](x,t)$ the solution of the problem

$$(P^\varepsilon_{\bar{g}}) \quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big(f(u) - \varepsilon\bar{g}(x,t,u)\big) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Once the interface $\Gamma_t[\bar{g}]$ is given, we denote by $\Omega_t^-[\bar{g}]$, $\Omega_t^+[\bar{g}]$ the region enclosed by $\Gamma_t[\bar{g}]$ and the one enclosed between $\partial\Omega$ and $\Gamma_t[\bar{g}]$, respectively. As in (1.13), we define the step function $\tilde{u}[\bar{g}](x,t)$ by

$$\tilde{u}[\bar{g}](x,t) = \begin{cases} \alpha_+ & \text{in } \Omega_t^+[\bar{g}], \\ \alpha_- & \text{in } \Omega_t^-[\bar{g}], \end{cases} \quad \text{for } t \in \big[0, T^{\max}(\bar{g})\big). \tag{7.18}$$

Next, given any function $u(x,t)$ on $\overline{\Omega} \times [0,T]$, we denote by $V[u](x,t)$ the solution of the problem

$$\begin{cases} V_t = D\Delta V + h\big(u(x,t), V\big) & \text{in } \Omega \times (0,T], \\ \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \\ V(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{7.19}$$

In view of (7.4) and the above notations, the solution $(u^\varepsilon, v^\varepsilon)$ of (RD^ε) is expressed as

$$u^\varepsilon = u^\varepsilon\big[g^\varepsilon\big[v^\varepsilon\big]\big], \qquad v^\varepsilon = V\big[u^\varepsilon\big].$$

On the other hand the solution $(\Gamma, \tilde{v})$ of (RD^0) is expressed as

$$\Gamma_t = \Gamma_t\big[g[\tilde{v}]\big], \qquad \tilde{v} = V[\tilde{u}],$$

the step function $\tilde{u}$ in (RD^0) being given by

$$\tilde{u} = \tilde{u}\big[g[\tilde{v}]\big].$$

Finally, the maps Φ_1^ε, Φ_1^0 and Φ_2 are now written as

$$\Phi_1^\varepsilon : v \to u^\varepsilon\big[g^\varepsilon[v]\big], \qquad \Phi_1^0 : v \to \tilde{u}\big[g[v]\big], \qquad \Phi_2 : u \to V[u].$$

7.4. Interface motion under perturbation

In this subsection we show that the interface $\Gamma_t[\bar{g}]$ depends continuously on the pressure term induced by $\bar{g}$. To this end, we first fix constants $C_*, T_* > 0$, $\vartheta \in (0,1)$, and denote by $\mathcal{Y}$ the set of functions $\bar{g}(x,t,u)$ on $\overline{\Omega} \times [0,T_*] \times \mathbb{R}$ satisfying

$$\sup_{u\in\mathbb{R}} \big\| \bar{g}(\cdot,\cdot,u) \big\|_{C^{1+\vartheta, \frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T_*])} \leqslant C_*. \tag{7.20}$$

Proposition 7.5. *Let $\bar{g} \in \mathcal{Y}$. Let $T \in (0, T^{\max}(\bar{g}))$. Then there exist positive constants δ, K, M such that, for any $\tilde{g} \in \mathcal{Y}$ satisfying*

$$\|\tilde{g} - \bar{g}\|_{L^\infty(\Omega\times(0,T)\times\mathbb{R})} \leqslant \delta, \tag{7.21}$$

it holds that $T^{\max}(\tilde{g}) > T$, where we recall that $T^{\max}(\tilde{g})$ is the maximum time of existence of a classical solution of problem $(P^0_{\tilde{g}})$. Furthermore, for each $t \in [0,T]$,

$$d_{\mathcal{H}}\big(\Gamma_t[\tilde{g}], \Gamma_t[\bar{g}]\big) \leqslant K\big(e^{Mt} - 1\big)\|\tilde{g} - \bar{g}\|_{L^\infty(\Omega\times(0,t)\times\mathbb{R})}. \tag{7.22}$$

Proof. First, the assertion that $T^{\max}(\tilde g) > T$, for $\tilde g$ sufficiently close to $\bar g$, follows from the standard local existence theory for quasi-linear parabolic equations. In fact, by using appropriate parametrization, one can express $\Gamma_t[\bar g]$ and $\Gamma_t[\tilde g]$, as graphs over $\mathcal{M}$, where $\mathcal{M}$ is an $(N-1)$-dimensional manifold without boundary, and transfer the motion equations $(P^0_{\bar g})$ and $(P^0_{\tilde g})$, into quasi-linear parabolic equations on the manifold $\mathcal{M}$, at least locally in time. For more details we refer to [8]. Since $\tilde g$ and $\bar g$ satisfy (7.20), and since the embedding

$$C^{1+\vartheta,\frac{1+\vartheta}{2}} \hookrightarrow C^{1+\vartheta',\frac{1+\vartheta'}{2}}$$

is compact if $0 < \vartheta' < \vartheta$, the assumption $\|\tilde g - \bar g\|_{L^\infty} \leqslant \delta$ implies

$$\big\|\tilde g(\cdot,\cdot,u) - \bar g(\cdot,\cdot,u)\big\|_{C^{1+\vartheta',\frac{1+\vartheta'}{2}}} \leqslant c(\delta),$$

where $c(\delta)$ is a constant satisfying $c(\delta) \to 0$, as $\delta \to 0$. Consequently, the coefficients appearing in $(P^0_{\bar g})$ and $(P^0_{\tilde g})$ satisfy

$$\left\| \int_{\alpha_-}^{\alpha_+} \tilde g(\cdot,\cdot,r)\,dr - \int_{\alpha_-}^{\alpha_+} \bar g(\cdot,\cdot,r)\,dr \right\|_{C^{1+\vartheta',\frac{1+\vartheta'}{2}}} \leqslant (\alpha_+ - \alpha_-)c(\delta).$$

Hence, the two solutions $\Gamma_t[\bar g]$ and $\Gamma_t[\tilde g]$ stay close to each other, at least locally in time, and, by repeating this argument, one can prove that $T^{\max}(\tilde g) > T$, for δ sufficiently small.

Next we prove the estimate (7.22). This will be done by using the maximum principle. Let us introduce some notation. For each $\bar g \in \mathcal{Y}$, we denote by $d(x,t;\bar g)$ the signed distance function associated with the interface $\Gamma_t[\bar g]$. By $\bar\Gamma_t \preccurlyeq \tilde\Gamma_t$ we mean that $\bar\Gamma_t$ lies inside of $\tilde\Gamma_t$. Clearly we have

$$\Gamma_t[\bar g] \preccurlyeq \Gamma_t[\tilde g] \quad\Longleftrightarrow\quad d(x,t;\bar g) \geqslant d(x,t;\tilde g) \quad \text{for } x \in \overline{\Omega}. \tag{7.23}$$

Now we choose $t_0 \in [0,T]$ arbitrarily and put

$$\eta_0 := \|\tilde g - \bar g\|_{L^\infty(\Omega\times(0,t_0)\times\mathbb{R})}.$$

Then

$$\bar g(x,t,u) - \eta_0 \leqslant \tilde g(x,t,u) \leqslant \bar g(x,t,u) + \eta_0 \quad \text{for } 0 \leqslant t \leqslant t_0.$$

The comparison principle then yields

$$\Gamma_t[\bar g - \eta_0] \preccurlyeq \Gamma_t[\tilde g] \preccurlyeq \Gamma_t[\bar g + \eta_0] \quad \text{for } 0 \leqslant t \leqslant t_0.$$

Thus, in order to prove (7.22), it suffices to show that there exist constants $K, M > 0$ such that, for all small $\eta_0 > 0$,

$$\begin{cases} d_{\mathcal{H}}\big(\Gamma_t[\bar g - \eta_0], \Gamma_t[\bar g]\big) \leqslant K\eta_0\big(e^{Mt} - 1\big), \\ d_{\mathcal{H}}\big(\Gamma_t[\bar g + \eta_0], \Gamma_t[\bar g]\big) \leqslant K\eta_0\big(e^{Mt} - 1\big), \end{cases} \tag{7.24}$$

for $0 \leqslant t \leqslant t_0$. We will only show the latter inequality for $\Gamma_t[\bar{g}+\eta_0]$ since the former can be shown in the same manner.

Recall that $d(x,t;\bar{g})$ satisfies Eq. (5.4), namely

$$d_t = \Delta d - c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr \quad \text{on } \Gamma_t[\bar{g}]. \tag{7.25}$$

Choose a constant $d_0 > 0$ such that $d(x,t;\bar{g})$ is smooth—say, C^3 in x and $C^{3/2}$ in t—in the neighborhood $\mathcal{N}_{d_0}(\Gamma_t[\bar{g}])$, $0 \leqslant t \leqslant T$. By the smoothness of $d(x,t;\bar{g})$ and equality (7.25), there exists a constant $N > 0$ such that

$$\left| d_t - \Delta d + c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr \right| \leqslant N|d| \quad \text{in } \mathcal{N}_{d_0/2}\big(\Gamma_t[\bar{g}]\big).$$

Now we put

$$d^{\mathrm{new}}(x,t) := d(x,t;\bar{g}) - K\eta_0\big(e^{2Nt}-1\big),$$
$$\tilde{\Gamma}_t := \big\{x \in \Omega \;\big|\; d^{\mathrm{new}}(x,t) = 0\big\},$$

where the constant K is to be determined later. If

$$\eta_0 \leqslant \eta_0^* := \frac{e^{-2NT} d_0}{4} K^{-1},$$

then $\tilde{\Gamma}_t$ lies within the neighborhood $\mathcal{N}_{d_0/2}(\Gamma_t[\bar{g}])$. Observe that

$$\begin{aligned}\big(d^{\mathrm{new}}\big)_t - \Delta d^{\mathrm{new}} &= d_t - 2NK\eta_0 e^{2Nt} - \Delta d \\ &\leqslant -c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr + N|d| - 2NK\eta_0 e^{2Nt}.\end{aligned}$$

Since $d = K\eta_0(e^{2Nt}-1)$ on $\tilde{\Gamma}_t$, we obtain

$$\big(d^{\mathrm{new}}\big)_t - \Delta d^{\mathrm{new}} \leqslant -c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr - NK\eta_0 \quad \text{on } \tilde{\Gamma}_t.$$

Now we set

$$K = (\alpha_+ - \alpha_-)c_0 N^{-1}.$$

Then it follows from the above inequality that

$$\left(d^{\text{new}}\right)_t \leqslant \Delta d^{\text{new}} - c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)\,dr - (\alpha_+ - \alpha_-)c_0\eta_0 \quad \text{on } \tilde{\Gamma}_t.$$

This inequality and the fact that $d^{\text{new}}(x,0) = d(x,0;\bar{g})$ imply that $\tilde{\Gamma}_t$ satisfies

$$\begin{cases} V_n \geqslant -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} \left(\bar{g}(x,t,r) + \eta_0\right) dr & \text{on } \tilde{\Gamma}_t, \\ \tilde{\Gamma}_t|_{t=0} = \Gamma_0. \end{cases}$$

On the other hand, $\Gamma_t[\bar{g} + \eta_0]$ satisfies

$$\begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} \left(\bar{g}(x,t,r) + \eta_0\right) dr & \text{on } \Gamma_t[\bar{g} + \eta_0], \\ \Gamma_t[\bar{g} + \eta_0]|_{t=0} = \Gamma_0. \end{cases}$$

By the comparison principle, we obtain

$$\Gamma_t[\bar{g}] \preccurlyeq \Gamma_t[\bar{g} + \eta_0] \preccurlyeq \tilde{\Gamma}_t \quad \text{for } 0 \leqslant t \leqslant t_0.$$

Consequently,

$$d_{\mathcal{H}}\left(\Gamma_t[\bar{g} + \eta_0], \Gamma_t[\bar{g}]\right) \leqslant d_{\mathcal{H}}\left(\tilde{\Gamma}_t, \Gamma_t[\bar{g}]\right) \leqslant K\eta_0\left(e^{2Nt} - 1\right),$$

for $0 \leqslant t \leqslant t_0$. The proposition is proved. □

7.5. *Proof of Claim 7.4*

For a function $v(x,t)$ satisfying the estimates (7.7), (7.8) and the Neumann boundary conditions, we compare below $\Phi^\varepsilon(v) = \Phi_2 \circ \Phi_1^\varepsilon(v)$ and $\Phi^0(v) = \Phi_2 \circ \Phi_1^0(v)$.

Action of $\boldsymbol{\Phi_1^\varepsilon}$ and $\boldsymbol{\Phi_1^0}$. Let us compare $\Phi_1^\varepsilon(v) = u^\varepsilon[g^\varepsilon[v]]$ with the step function $\Phi_1^0(v) = \tilde{u}[g[v]]$. By the definitions in (7.4) we have $g^\varepsilon[v] = g[v] + O(\varepsilon)$, and all the conditions in (1.3)–(1.7) are satisfied. It follows that our results for the single equation apply and, in particular,

$$u_\varepsilon^-(x,t) \leqslant u^\varepsilon\left[g^\varepsilon[v]\right]\left(x, t + t^\varepsilon\right) \leqslant u_\varepsilon^+(x,t) \quad \text{for } 0 \leqslant t \leqslant T - t^\varepsilon,$$

where $u_\varepsilon^\pm$ are as in (5.12), d being the signed distance function associated with the interface $\Gamma_t[g[v]]$. Since the term $e^{-\beta t/\varepsilon^2}$ in $q(t)$—that appears in (5.12)—quickly becomes small,

$$\left|u^\varepsilon\left[g^\varepsilon[v]\right] - \tilde{u}\left[g[v]\right]\right|(x,t) \leqslant \alpha_+ - U_0\left(\frac{d(x,t) - \varepsilon p(t)}{\varepsilon}\right) + O(\varepsilon) \quad \text{in } \Omega_t^+\left[g[v]\right],$$

$$\left|u^\varepsilon\left[g^\varepsilon[v]\right] - \tilde{u}\left[g[v]\right]\right|(x,t) \leqslant U_0\left(\frac{d(x,t) + \varepsilon p(t)}{\varepsilon}\right) - \alpha_- + O(\varepsilon) \quad \text{in } \Omega_t^-\left[g[v]\right],$$

for $\mu_1\varepsilon^2|\ln\varepsilon| \leqslant t \leqslant T$, provided that we choose the constant μ_1 large enough. Consequently, by Lemma 2.1, there exist constants $B, C > 0$ such that

$$\left|\Phi_1^\varepsilon(v) - \Phi_1^0(v)\right|(x,t) \leqslant B\exp\left(-\lambda\frac{|d(x,t)|}{\varepsilon}\right) + C\varepsilon, \tag{7.26}$$

for $(x,t) \in \overline{\Omega} \times [\mu_1\varepsilon^2|\ln\varepsilon|, T]$.

Action of $\boldsymbol{\Phi_2}$. Next we compare $\Phi^\varepsilon(v) = V[\Phi_1^\varepsilon(v)]$ and $\Phi^0(v) = V[\Phi_1^0(v)]$. Set $w := \Phi^\varepsilon(v) - \Phi^0(v)$. By subtracting the equations for $V[\Phi_1^\varepsilon(v)]$ and $V[\Phi_1^0(v)]$, we obtain

$$w_t = D\Delta w + \left(h\left(\Phi_1^\varepsilon(v), \Phi^\varepsilon(v)\right) - h\left(\Phi_1^0(v), \Phi^0(v)\right)\right).$$

Since $|h(\Phi_1^\varepsilon(v), \Phi^\varepsilon(v)) - h(\Phi_1^0(v), \Phi^0(v))| \leqslant C|w| + C|\Phi_1^\varepsilon(v) - \Phi_1^0(v)|$ for some constant $C > 0$, the function $\tilde{w} := e^{-Ct}w$ satisfies

$$\tilde{w}_t \leqslant D\Delta\tilde{w} + Ce^{-Ct}\left|\Phi_1^\varepsilon(v)(x,t) - \Phi_1^0(v)(x,t)\right| + C\left(|\tilde{w}| - \tilde{w}\right),$$

hence

$$\tilde{w}_t \leqslant D\Delta\tilde{w} + C\left|\Phi_1^\varepsilon(v)(x,t) - \Phi_1^0(v)(x,t)\right| + C\left(|\tilde{w}| - \tilde{w}\right). \tag{7.27}$$

Now let $W(x,t)$ be the solution of the equation

$$W_t = D\Delta W + C\left|\Phi_1^\varepsilon(v)(x,t) - \Phi_1^0(v)(x,t)\right| + C\left(|W| - W\right),$$

with initial data $W(x,0) = 0$. Then since (7.27) implies that $\tilde{w}$ is a sub-solution of the above equation, and since $\tilde{w}(x,0) = 0$, we have

$$\tilde{w}(x,t) \leqslant W(x,t) \quad \text{for } x \in \overline{\Omega},\ t \geqslant 0. \tag{7.28}$$

Moreover, since $W \geqslant 0$, the above equation for W can be reduced to

$$W_t = D\Delta W + C\left|\Phi_1^\varepsilon(v)(x,t) - \Phi_1^0(v)(x,t)\right|.$$

In view of this, we see that

$$W(x,t) = C\int_0^t\int_\Omega G(x,y,t-s)\left|\Phi_1^\varepsilon(v)(y,s) - \Phi_1^0(v)(y,s)\right|dy\,ds,$$

$G(x,y,t)$ being the fundamental solution that appears in (7.11). This and (7.28) yield

$$\left|w(x,t)\right| \leqslant Ce^{Ct}\int_0^t\int_\Omega G(x,y,t-s)\left|\Phi_1^\varepsilon(v)(y,s) - \Phi_1^0(v)(y,s)\right|dy\,ds. \tag{7.29}$$

Combining this and (7.26), we obtain

$$\left|w(x,t)\right| \leqslant BCe^{Ct} \int_0^t \int_\Omega G(x,y,t-s) \exp\left(-\lambda \frac{|d(y,s)|}{\varepsilon}\right) dy\, ds + O(\varepsilon). \tag{7.30}$$

In order to estimate the above integral, we need the following lemma:

Lemma 7.6. *Let Γ be a smooth closed hypersurface in Ω and denote by $d(x)$ the signed distance function associated with Γ. Then there exist constants $C, r_0 > 0$ such that for any function $\eta(r) \geqslant 0$ on $\mathbb{R}$, it holds that*

$$\int_{|d| \leqslant r_0} G(x,y,t)\eta\big(d(y)\big)\, dy \leqslant \frac{C}{\sqrt{t}} \int_{-r_0}^{r_0} \eta(r)\, dr \quad \textit{for } 0 < t \leqslant T. \tag{7.31}$$

The proof of this lemma will be given in the next subsection. As is easily seen from its proof, the above estimate remains to hold if Γ depends on t smoothly; in other words, the constant C can be chosen uniformly as Γ varies. Applying the above estimate to $\Gamma_t[g[v]]$, $0 < t \leqslant T$, we obtain

$$\begin{aligned}
&\int_\Omega G(x,y,t-s) \exp\left(-\lambda \frac{|d(y,s)|}{\varepsilon}\right) dy \\
&\quad = \int_{|d|<r_0} + \int_{|d| \geqslant r_0} G(x,y,t-s) \exp\left(-\lambda \frac{|d(y,s)|}{\varepsilon}\right) dy \\
&\quad = O\left(\frac{\varepsilon}{\sqrt{t-s}}\right) + O\left(e^{-\lambda r_0/\varepsilon}\right) \\
&\quad = O\left(\frac{\varepsilon}{\sqrt{t-s}}\right).
\end{aligned}$$

It follows from this and (7.30) that

$$\left|w(x,t)\right| = O\left(\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\, ds\right) + O(\varepsilon) = O(\varepsilon), \tag{7.32}$$

which completes the proof of Claim 7.4. □

7.6. Proof of Lemma 7.6

We first show that

$$\int_\Gamma G(x,y,t)\, dS_y \leqslant \frac{C}{\sqrt{t}} \quad \text{for } x \in \Omega,\ 0 < t \leqslant T. \tag{7.33}$$

It suffices to prove this estimate on a small interval $[0, t_0]$, since the estimate for the remaining interval $[t_0, T]$ will follow by simply choosing a large constant C (since G is bounded for t large). Hereafter, we choose t_0 sufficiently small. Then, for $0 < t \leqslant t_0$, $G(x, y, t)$ is well approximated by the fundamental solution on the entire space $\mathbb{R}^N$:

$$G_0(x, y, t) := \frac{1}{(4\pi Dt)^{N/2}} \exp\left(-\frac{|x-y|^2}{4Dt}\right).$$

In particular, there exists a constant $C > 0$ such that

$$0 < G(x, y, t) \leqslant C G_0(x, y, t) \quad \text{for } x, y \in \overline{\Omega},\ 0 < t \leqslant t_0$$

(see, for example, [14, Section IV.2]). Thus it suffices to prove (7.33) for G_0 instead of G.

Given $x \in \Omega$, let x_0 be the point on Γ that is closest to x, and let $n(x_0)$ be the outward normal to Γ at x_0. Then $x - x_0 = d(x)n(x_0)$. Define

$$\tilde{Y} := \left\{y \in \mathbb{R}^N,\ y \cdot n(x_0) = 0\right\}, \qquad Y_0 = \operatorname{span}\left\langle n(x_0)\right\rangle,$$

where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^N$ and $\operatorname{span}\langle w\rangle$ the line spanned by the vector w. This gives an orthogonal decomposition $\mathbb{R}^N = \tilde{Y} \oplus Y_0$, and $x_0 + \tilde{Y}$ is the tangent hyperplane of Γ at x_0. Since Γ is smooth, it is expressed locally as the graph of a map defined on a subset of $\tilde{Y}$. More precisely, there exist a smooth map $h : \tilde{Y} \to Y_0$ and a constant $\delta > 0$ such that $h(0) = 0$, $\nabla h(0) = 0$, and that

$$\begin{gathered} S := \left\{x_0 + \tilde{y} + h(\tilde{y}),\ \tilde{y} \in \tilde{Y},\ |\tilde{y}| < \delta\right\} \subset \Gamma, \\ \operatorname{dist}(x_0, \Gamma \setminus S) \geqslant \delta. \end{gathered} \tag{7.34}$$

Now we decompose the integral (7.33) for G_0 as

$$\int_\Gamma G_0(x, y, t)\, dS_y = \frac{1}{(4\pi Dt)^{N/2}} \left(\int_S + \int_{\Gamma\setminus S} \exp\left(-\frac{|x-y|^2}{4Dt}\right) dS_y\right).$$

Since $|x - y| \geqslant |d(x)|$ for every $y \in \Gamma$ and since

$$|x - y| \geqslant \bigl||x - x_0| - |y - x_0|\bigr| = \bigl|\bigl|d(x)\bigr| - |y - x_0|\bigr|,$$

we have

$$|x - y| \geqslant \frac{|d(x)| + ||d(x)| - |y - x_0||}{2} \geqslant \frac{|y - x_0|}{2}.$$

This and (7.34) yield

$$|x - y| \geqslant \frac{\delta}{2} \quad \text{for } y \in \Gamma \setminus S.$$

Consequently,

$$\int_{\Gamma\setminus S} \exp\left(-\frac{|x-y|^2}{4Dt}\right) dS_y \leqslant e^{-\delta^2/16Dt}|\Gamma|, \tag{7.35}$$

where $|\Gamma|$ denotes the total area of Γ.

On the other hand, for each $y \in S$, we can express $y - x_0$ as

$$y - x_0 = \tilde{y} + h(\tilde{y}) \quad \big(\tilde{y} \in \tilde{Y},\ h(\tilde{y}) \in Y_0\big),$$

and $\tilde{Y}$ can be identified with $\mathbb{R}^{N-1}$. Thus

$$\int_S \exp\left(-\frac{|x-y|^2}{4Dt}\right) dS_y$$
$$= \int_{|\tilde{y}|<\delta} \exp\left(-\frac{|x-x_0-\tilde{y}-h(\tilde{y})|^2}{4Dt}\right)\sqrt{1+\left|\nabla h(\tilde{y})\right|^2}\, d\tilde{y}.$$

Since $\nabla h(0) = 0$, there exists a constant $C_1 > 0$ such that

$$\left|\nabla h(\tilde{y})\right| \leqslant C_1|\tilde{y}| \quad \text{for } |\tilde{y}| < \delta. \tag{7.36}$$

Note also that the orthogonality $(x - x_0 - h(\tilde{y})) \perp \tilde{y}$ implies

$$\left|x - x_0 - \tilde{y} - h(\tilde{y})\right|^2 = \left|x - x_0 - h(\tilde{y})\right|^2 + |\tilde{y}|^2 \geqslant |\tilde{y}|^2.$$

Combining these, we obtain

$$\int_S \exp\left(-\frac{|x-y|^2}{4Dt}\right) dS_y \leqslant \int_{|\tilde{y}|<\delta} \exp\left(-\frac{|\tilde{y}|^2}{4Dt}\right)\sqrt{1+C_1^2|\tilde{y}|^2}\, d\tilde{y}$$
$$= t^{(N-1)/2} \int_{|z|<\sqrt{t}^{-1}\delta} e^{-|z|^2/4D}\sqrt{1+tC_1^2|z|^2}\, dz,$$

where $z := \tilde{y}/\sqrt{t}$. Observe that, as $t \to 0$,

$$\int_{|z|<\sqrt{t}^{-1}\delta} e^{-|z|^2/4D}\sqrt{1+tC_1^2|z|^2}\, dz \to \int_{\mathbb{R}^{N-1}} e^{-|z|^2/4D}\, dz = (4D\pi)^{(N-1)/2}.$$

Consequently,

$$\frac{1}{(4\pi Dt)^{N/2}} \int_S \exp\left(-\frac{|x-y|^2}{4Dt}\right) dS_y \leqslant \frac{1}{\sqrt{4\pi Dt}} + o\left(\frac{1}{\sqrt{t}}\right).$$

Combining the estimate above and (7.35), we obtain

$$\int_{\Gamma} G_0(x,y,t)\,dS_y = O\left(\frac{1}{\sqrt{t}}\right) + O\left(\frac{1}{(\sqrt{t}\,)^N} e^{-\delta^2/16Dt}\right) = O\left(\frac{1}{\sqrt{t}}\right).$$

Since Γ is a smooth compact hypersurface, its curvature is bounded. Therefore, the constants δ and C_1 that appear in (7.35), (7.36) can be chosen independent of the choice of $x_0 \in \Gamma$. Hence the above $O(1/\sqrt{t}\,)$ estimate is uniform with respect to the choice of $x \in \Omega$. This proves the estimate (7.33).

Now, choose a sufficiently small constant $r_0 > 0$ such that the signed distance function $d(x)$ is smooth in the region $\{d(x) < 2r_0\}$. For each $r \in [-r_0, r_0]$, we define a hypersurface $\Gamma(r)$ by

$$\Gamma(r) := \{x \in \Omega,\ d(x) = r\}.$$

Then the curvatures of $\Gamma(r)$ are uniformly bounded as r varies, which implies that there exists some constant $C > 0$ such that

$$\int_{\Gamma(r)} G(x,y,t)\,dS_y \leqslant \frac{C}{\sqrt{t}} \quad \text{for } 0 < t \leqslant T,\ r \in [-r_0, r_0].$$

The estimate (7.31) now follows by integrating in r.

7.7. Proof of Claim 7.3

We compare below $\Phi^0(v^\varepsilon) = \Phi_2 \circ \Phi_1^0(v^\varepsilon)$ and $\Phi^0(\tilde{v}) = \Phi_2 \circ \Phi_1^0(\tilde{v})$.

Action of Φ_1^0. Let us compare the two step functions $\Phi_1^0(v^\varepsilon) = \tilde{u}[g[v^\varepsilon]]$ and $\Phi_1^0(\tilde{v}) = \tilde{u}[g[\tilde{v}]]$. We want to apply Proposition 7.5, with $g[\tilde{v}]$ and $g[v^\varepsilon]$, playing the role of $\bar{g}$ and $\tilde{g}$, respectively. (Hence, the role of $T^{\max}(\bar{g})$ is played by $T^{\max}(g[\tilde{v}])$, which corresponds to $T^{\max}$ in Lemma 1.10.) First, we choose $C_* > 0$ large enough so that both $g[v^\varepsilon](x,t,u)$ and $g[\tilde{v}](x,t,u)$ satisfy (7.20). For $T \in (0, T^{\max})$, we choose δ, K and M as in Proposition 7.5. Next, we define $K_1 = \max_{(u,v)\in\mathcal{R}} |\partial_v f_1(u,v)|$, with $\mathcal{R}$ being the rectangle defined in Section 7.1, and $\delta_0 = \delta/K_1$. We observe that, using the definition of $t_{\max}$ in (7.13),

$$\big\|g[v^\varepsilon] - g[\tilde{v}]\big\|_{L^\infty(\Omega\times(0,t_{\max})\times\mathbb{R})} \leqslant K_1 \big\|v^\varepsilon - \tilde{v}\big\|_{L^\infty(Q_{t_{\max}})} \leqslant K_1\delta_0 = \delta.$$

By (7.22), it follows that, for any $t \in [0, t_{\max}]$,

$$d_{\mathcal{H}}\big(\Gamma_t[g[v^\varepsilon]], \Gamma_t[g[\tilde{v}]]\big) \leqslant K\big(e^{Mt} - 1\big)\big\|g[v^\varepsilon] - g[\tilde{v}]\big\|_{L^\infty}.$$

Combining these, we obtain

$$d_{\mathcal{H}}\big(\Gamma_t[g[v^\varepsilon]], \Gamma_t[g[\tilde{v}]]\big) \leqslant KK_1\big(e^{Mt} - 1\big)\big\|v^\varepsilon - \tilde{v}\big\|_{L^\infty(Q_t)}. \tag{7.37}$$

Action of Φ_2. Next we compare the two functions $\Phi^0(v^\varepsilon) = V[\Phi_1^0(v^\varepsilon)] = V[\tilde{u}[g[v^\varepsilon]]]$ and $\Phi^0(\tilde{v}) = V[\Phi_1^0(\tilde{v})] = V[\tilde{u}[g[\tilde{v}]]]$. Since

$$\left|\tilde{u}\left[g\left[v^\varepsilon\right]\right] - \tilde{u}\left[g[\tilde{v}]\right]\right| \leqslant \alpha_+ - \alpha_-,$$

and since the two step functions differ only in the region enclosed between the two surfaces $\Gamma_t[g[v^\varepsilon]]$ and $\Gamma_t[g[\tilde{v}]]$, the estimates (7.29) and (7.31) imply that there exists a constant $B_1 > 0$ such that

$$\left\| V\left[\tilde{u}\left[g\left[v^\varepsilon\right]\right]\right] - V\left[\tilde{u}\left[g[\tilde{v}]\right]\right]\right\|_{L^\infty(Q_t)} \leqslant B_1 \int_0^t \frac{d_{\mathcal{H}}(\Gamma_s[g[v^\varepsilon]], \Gamma_s[g[\tilde{v}]])}{\sqrt{t-s}}\, ds.$$

Combining this and (7.37), we obtain, for any $t \in [0, t_{\max}]$,

$$\left\|\Phi^0\left(v^\varepsilon\right) - \Phi^0(\tilde{v})\right\|_{L^\infty(Q_t)} \leqslant C \int_0^t \frac{1}{\sqrt{t-s}} \left\|v^\varepsilon - \tilde{v}\right\|_{L^\infty(Q_s)}\, ds, \tag{7.38}$$

with $C = B_1 K K_1(e^{MT} - 1)$. The proof of Claim 7.3 is complete.

7.8. Estimate of $\bar{k}(t)$

In this subsection we justify the estimate (7.16). Let $\bar{k}(t)$ be the function satisfying (7.17), namely,

$$\bar{k}(t) = 1 + C \int_0^t \frac{\bar{k}(s)}{\sqrt{t-s}}\, ds \quad \text{for } t \geqslant 0.$$

We will show below that $\bar{k}$ is given by

$$\bar{k}(t) = e^{C^2\pi t}\left(1 + C \int_0^t \frac{e^{-C^2\pi s}}{\sqrt{s}}\, ds\right). \tag{7.39}$$

The following lemma justifies (7.16):

Lemma 7.7. *Let $k(t)$ be a continuous function satisfying*

$$0 \leqslant k(t) \leqslant A + C \int_0^t \frac{k(s)}{\sqrt{t-s}}\, ds \quad (0 < t \leqslant T),$$

for some constant $A > 0$ and $T > 0$. Then

$$0 \leqslant k(t) \leqslant A\bar{k}(t) \quad \text{for } 0 < t \leqslant T. \tag{7.40}$$

Proof. Define

$$\bar{k}_\varepsilon(t) := (1+\varepsilon)A\bar{k}(t).$$

Then this function satisfies

$$\bar{k}_\varepsilon(t) = (1+\varepsilon)A + C\int_0^t \frac{\bar{k}_\varepsilon(s)}{\sqrt{t-s}}\,ds \quad (0 < t \leqslant T). \tag{7.41}$$

In particular, we have $\bar{k}_\varepsilon(0) = (1+\varepsilon)A > A \geqslant k(0)$. Let us show that

$$k(t) < \bar{k}_\varepsilon(t) \quad \text{for } 0 < t \leqslant T. \tag{7.42}$$

Suppose that (7.42) does not hold. Then there exists $t_0 \in (0, T]$ such that

$$k(t) < \bar{k}_\varepsilon(t) \quad \text{for } 0 \leqslant t < t_0, \qquad k(t_0) = \bar{k}_\varepsilon(t_0). \tag{7.43}$$

Combining the first part of (7.43) and (7.41), we get

$$\bar{k}_\varepsilon(t_0) = (1+\varepsilon)A + C\int_0^{t_0} \frac{\bar{k}_\varepsilon(s)}{\sqrt{t_0-s}} > A + C\int_0^{t_0} \frac{k(s)}{\sqrt{t_0-s}} \geqslant k(t_0),$$

but this contradicts the second part of (7.43), establishing (7.42). Letting $\varepsilon \to 0$, we obtain (7.40). □

Corollary 7.8. *Let $k(t)$ be a continuous function satisfying*

$$0 \leqslant k(t) = A + C\int_0^t \frac{k(s)}{\sqrt{t-s}}\,ds \quad (0 < t \leqslant T),$$

for some constant $A > 0$ and $T > 0$. Then $k(t) \equiv A\bar{k}(t)$. In particular, the function $\bar{k}(t)$ is uniquely determined by the integral identity (7.17)*.*

Proof. Define $\hat{k}(t) := A^{-1}k(t)$. Then $\hat{k}(t)$ satisfies

$$\hat{k}(t) = 1 + C\int_0^t \frac{\hat{k}(s)}{\sqrt{t-s}}\,ds \quad (0 < t \leqslant T).$$

By Lemma 7.7 we have $\hat{k}(t) \leqslant \bar{k}(t)$. Exchanging the role of $\hat{k}$ and $\bar{k}$, we obtain the opposite inequality, hence $\hat{k}(t) \equiv \bar{k}(t)$. Thus $k(t) = A\bar{k}(t)$. □

Now let us prove (7.39). Integration by parts gives

$$\bar{k}(t) = 1 + 2C\sqrt{t} + 2C\int_0^t \sqrt{t-s}\,\bar{k}'(s)\,ds.$$

Hence

$$\bar{k}'(t) = \frac{C}{\sqrt{t}} + C\int_0^t \frac{\bar{k}'(s)}{\sqrt{t-s}}\,ds.$$

Thus the function $m(t) := \bar{k}'(t) - C/\sqrt{t}$ satisfies

$$m(t) = C\int_0^t \frac{m(s) + C(\sqrt{s})^{-1}}{\sqrt{t-s}}\,ds = C^2\int_0^t \frac{1}{\sqrt{s(t-s)}}\,ds + C\int_0^t \frac{m(s)}{\sqrt{t-s}}\,ds.$$

Since the first integral on the right-hand side is equal to π, we obtain

$$m(t) = C^2\pi + C\int_0^t \frac{m(s)}{\sqrt{t-s}}\,ds.$$

It follows from Corollary 7.8 that $m(t) = C^2\pi\bar{k}(t)$, hence

$$\bar{k}'(t) = C^2\pi\bar{k}(t) + \frac{C}{\sqrt{t}}.$$

This and the equality $\bar{k}(0) = 1$ yield (7.39).

References

[1] N.D. Alikakos, P.W. Bates, X. Chen, Convergence of the Cahn–Hilliard equation to the Hele–Shaw model, Arch. Ration. Mech. Anal. 128 (1994) 165–205.
[2] S. Allen, J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metallica 27 (1979) 1084–1095.
[3] G. Barles, H.M. Soner, P.E. Souganidis, Front propagation and phase field theory, SIAM J. Control Optim. 31 (1993) 439–469.
[4] G. Barles, P.E. Souganidis, A new approach to front propagation problems: Theory and applications, Arch. Ration. Mech. Anal. 141 (1998) 237–296.
[5] L. Bronsard, R.V. Kohn, Motion by mean curvature as the singular limit of Ginzburg–Landau dynamics, J. Differential Equations 90 (1991) 211–237.
[6] X. Chen, Generation and propagation of interfaces for reaction–diffusion equations, J. Differential Equations 96 (1992) 116–141.
[7] X. Chen, Generation and propagation of interfaces for reaction–diffusion systems, Trans. Amer. Math. Soc. 334 (1992) 877–913.
[8] X. Chen, F. Reitich, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling, J. Math. Anal. Appl. 164 (1992) 350–362.
[9] X.-Y. Chen, Dynamics of interfaces in reaction–diffusion systems, Hiroshima Math. J. 21 (1991) 47–83.

[10] K.N. Chueh, A compact positively invariant set of solutions of the Nagumo equation, J. Differential Equations 28 (1978) 35–42.
[11] K.N. Chueh, C.C. Conley, J.A. Smoller, Positively invariant regions for systems of nonlinear diffusion equations, Indiana Univ. Math. J. 26 (1977) 373–392.
[12] S.I. Ei, M. Iida, E. Yanagida, Dynamics of interfaces in a scalar parabolic equation with variable diffusion coefficients, Japan. J. Indust. Appl. Math. 14 (1997) 1–25.
[13] L.C. Evans, H.M. Soner, P.E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math. 45 (1992) 1097–1123.
[14] S.D. Eydel'man, Parabolic Systems, North-Holland, Amsterdam, London, 1969.
[15] P.C. Fife, Dynamics of Internal Layers and Diffusive Interfaces, CBMS-NSF Reg. Conf. Ser. Appl. Math., SIAM, Philadelphia, 1988.
[16] D. Hilhorst, G. Karali, H. Matano, K. Nakashima, Singular limit of a spatially inhomogeneous Lotka–Volterra competition-diffusion system, Comm. Partial Differential Equations 32 (2007) 879–933.
[17] D. Hilhorst, H. Matano, R. Schätzle, Singular limit of an inhomogeneous reaction–diffusion equation, in preparation.
[18] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, Mem. Amer. Math. Soc. 108 (1994).
[19] Y. Kametaka, M. Mimura, M. Yamaguti, On a certain difference scheme for some semilinear diffusion system, Proc. Japan Acad. 47 (1971) 385–387.
[20] K. Kawasaki, T. Ohta, Kinetic drumhead model of interface I, Progr. Theoret. Phys. 67 (1982) 147–163.
[21] P. de Mottoni, M. Schatzman, Development of interfaces in $\mathbb{R}^n$, Proc. Roy. Soc. Edinburgh Sect. A 116 (1990) 207–220.
[22] P. de Mottoni, M. Schatzman, Geometrical evolution of developed interfaces, Trans. Amer. Math. Soc. 347 (1995) 1533–1589.
[23] K.-I. Nakamura, H. Matano, D. Hilhorst, R. Schätzle, Singular limit of a reaction–diffusion equation with a spatially inhomogeneous reaction term, J. Stat. Phys. 95 (1999) 1165–1185.
[24] J. Rubinstein, P. Sternberg, J.B. Keller, Fast reaction, slow diffusion and curve shortening, SIAM J. Appl. Math. 49 (1989) 116–133.
[25] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer-Verlag, New York, 1994.