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Local derivations of nest subalgebras of von Neumann algebras ☆

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Abstract

In this paper, it is proved that every norm continuous linear local derivation of a nest subalgebra of a factor von Neumann algebra is a derivation, and that every linear 2-local derivation of a nest subalgebra of a factor von Neumann algebra is a derivation.

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1. Introduction

Let $\mathscr{A}$ be an (associative) algebra and $\mathscr{E}$ be a $\mathscr{A}$-bimodule, and $\delta : \mathscr{A} \to \mathscr{E}$ be a linear mapping. Recall that δ is a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ holds for all a in $\mathscr{A}$, and that δ is a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ holds for all a and b in $\mathscr{A}$, and that δ is an inner derivation if there exists an element m in $\mathscr{E}$ such that $\delta(a) = ma - am$ holds for all a in $\mathscr{A}$. Also, δ is a local derivation if for each a in $\mathscr{A}$ there is a derivation δ_a from $\mathscr{A}$ into $\mathscr{E}$, depending on a, such that $\delta(a) = \delta_a(a)$.

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A mapping $\theta : \mathscr{A} \to \mathscr{E}$ is called a 2-local derivation if for each a and b in $\mathscr{A}$ there is a derivation $\theta_{a,b}$ from $\mathscr{A}$ into $\mathscr{E}$, depending on a and b, such that $\theta(a) = \theta_{a,b}(a)$ and $\theta(b) = \theta_{a,b}(b)$. These two notions were introduced by Kadison [2] and Larson and Sourour [6], and Šemrl [7].

The study of local mappings represents one of the most active research areas in operator theory. In the past decade, similar questions have received a fair amount of attention [3–6]. The interest in these types of mappings has been sparked by two lines of research. One is the study of the reflexivity of the space of linear maps from an algebra to itself. Let $\mathscr{B}$ be a subspace of the space of linear mappings $B(\mathscr{X})$ on a vector space $\mathscr{X}$. $\mathscr{B}$ is said to be algebraically reflexive if, given $T \in B(\mathscr{X})$ with the property that $Tx \in \mathscr{B}x$ for all $x \in \mathscr{X}$, then $T \in \mathscr{B}$. Larson [1] asked which algebras have a reflexive derivation space and when is the space of inner derivations relatively reflexive in the space of derivations. Another is the study of Hochschild cohomology for various operator algebras. These maps arise naturally when looking for sufficient conditions to ensure that a map is a derivation. Kadison set out a program of study for local maps in [2], suggesting that local derivations could prove useful in building derivations with particular properties. In his study of the local derivations of von Neumann algebras, Kadison proved that every norm continuous linear local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. Crist [3] proved the same result for the direct limit of finite dimensional CSL algebras via *-extendable embedding (e.g., a triangular AF algebras). Without the linearity assumption, Šemrl [7] proved that every 2-local derivation on $B(H)$, the algebra of all bounded linear operators on a Hilbert space H, is a derivation. However, Šemrl's method was based on the fact that a 2-local derivation of $B(H)$, which vanishes at two special operators, must be the zero mapping. It is clear that any attempt to extend Šemrl's results to general operator algebras must use different techniques. In this paper we consider linear local derivations and linear 2-local derivations of nest subalgebras of factor von Neumann algebras. More precisely, we prove that every norm continuous linear local derivation of a nest subalgebra of a factor von Neumann algebra is a derivation, and that every linear 2-local derivation of a nest subalgebra of a factor von Neumann algebra (no continuity is assumed) is a derivation.

Let M be a von Neumann algebra acting on a separable Hilbert space H. A nest β in M is a totally ordered family of (selfadjoint) projections in M which is closed in the strong operator topology, and which includes 0 and I. Let P be a projection in β, we define

$$P_+ = \inf\{Q \in \beta : Q > P\} \quad \text{and} \quad P_- = \sup\{Q \in \beta : Q < P\}.$$

If $P_+ - P \neq 0$ or $P - P_- \neq 0$, then $P_+ - P$ or $P - P_-$ is called an atom of β. A nest is said to be continuous if it has no atoms. The nest subalgebra of M associated to a nest β is the set $\mathrm{alg}_M\beta = \{T \in M : PTP = TP \text{ for all } P \in \beta\}$. The diagonal $\mathscr{D}_M(\beta)$ of a nest subalgebra $\mathrm{alg}_M\beta$ is the von Neumann subalgebra $(\mathrm{alg}_M\beta) \cap (\mathrm{alg}_M\beta)^*$. The core $\mathscr{C}(\beta)$ of the nest subalgebra $\mathrm{alg}_M\beta$ is the von Neumann algebra generated by the projections $\{P : P \in \beta\}$. Let $\mathscr{R}_M(\beta)$ denote the norm closed algebra generated by $\{PTP^{\perp} : T \in M, P \in \beta\}$. It is clear that $\mathscr{R}_M(\beta)$ is a norm closed ideal of the nest subalgebra $\mathrm{alg}_M\beta$. As a notational convenience, if E is an idempotent, we let $E^{\perp}$ denote $I - E$ throughout this paper.

We refer the reader to [13,14] for background information about von Neumann algebras, and to [15] for the theory of nest algebras.

2. Local derivations

In this section, our main result is the following theorem.

Theorem 1. *Let β be a nest in an arbitrary factor von Neumann algebra M, and δ be a norm continuous linear local derivation from the nest subalgebra $\mathrm{alg}_M\beta$ into M. Then $\delta(AB) = \delta(A)B + A\delta(B)$ for all A and B in $\mathrm{alg}_M\beta$.*

To prove Theorem 1, we need some lemmas. We assume that δ is a norm continuous linear local derivation from $\mathrm{alg}_M\beta$ into the factor von Neumann algebra M. The proof of the following lemma is similar to that of [3]. It is shown that, without loss of generality, we may assume that δ annihilates the diagonal $\mathscr{D}_M(\beta)$ of the nest subalgebra $\mathrm{alg}_M\beta$.

Lemma 2.1. $\delta = \eta + \delta'$, *where η is an inner derivation and δ' is a local derivation that annihilates $\mathscr{D}_M(\beta)$ the diagonal of* $\mathrm{alg}_M\beta$.

Applying the same proof as in [2], we have the following lemma.

Lemma 2.2. (a) $\delta(E) = \delta(E)E + E\delta(E)$ *for every idempotent E in* $\mathrm{alg}_M\beta$; (b) *If $A, B, D \in \mathrm{alg}_M\beta$, are such that $AB = BD = 0$, then $A\delta(B)D = 0$.*

Lemma 2.3. $\delta(QAQ^{\perp}) = Q\delta(QAQ^{\perp})Q^{\perp}$ *for all A in $\mathrm{alg}_M\beta$ and all projections Q in $\mathscr{D}_M(\beta)$.*

Proof. Let $E = Q + QAQ^{\perp}$, it is clear that E is an idempotent in $\mathrm{alg}_M\beta$. By Lemma 2.2(a), we have

$$\begin{aligned}\delta(QAQ^{\perp}) &= \delta(QAQ^{\perp})(Q + QAQ^{\perp}) + (Q + QAQ^{\perp})\delta(QAQ^{\perp})\\ &= \delta(QAQ^{\perp})Q + Q\delta(QAQ^{\perp}),\end{aligned}$$

which implies that

$$Q\delta(QAQ^{\perp})Q = Q^{\perp}\delta(QAQ^{\perp})Q^{\perp} = 0. \tag{1}$$

By Lemma 2.2(b), we have $Q^{\perp}\delta(QAQ^{\perp})Q = 0$. This together with Eq. (1) gives us that $\delta(QAQ^{\perp}) = Q\delta(QAQ^{\perp})Q^{\perp}$. The proof is complete. $\square$

Lemma 2.4. $\delta(XTY) = \delta(XT)Y + X\delta(TY) - X\delta(T)Y$ *for all* T *in* $\text{alg}_M\beta$ *and* X, Y *in* $\mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$.

Proof. Let E, F be idempotents in $\text{alg}_M\beta$, it follows from Lemma 2.2(b) that $E^\perp\delta(ETF)F^\perp = E^\perp\delta(ETF^\perp)F = E\delta(E^\perp TF)F^\perp = E\delta(E^\perp TF^\perp)F = 0$ for all T in $\text{alg}_M\beta$. Hence

$$\begin{aligned}\delta(ETF) &= \delta(ETF)F + E\delta(ETF) - E\delta(ETF)F\\ &= \delta(ET)F - \delta(ETF^\perp)F + E\delta(TF) - E\delta(E^\perp TF) - E\delta(ETF)F\\ &= \delta(ET)F + E\delta(TF) - E\delta(T)F.\end{aligned}$$

Let P, Q be projections in β, it is clear that $F = P + PSP^\perp$ and $G = Q + QRQ^\perp$ are idempotents in $\text{alg}_M\beta$ for all S and R in M. Then

$$\delta(PSP^\perp TE) = \delta(PSP^\perp T)E + PSP^\perp\delta(TE) - PSP^\perp\delta(T)E$$

and

$$\delta(ETQRQ^\perp) = \delta(ET)QRQ^\perp + E\delta(TQRQ^\perp) - E\delta(T)QRQ^\perp$$

for all projections E in $\mathscr{D}_M(\beta)$, and hence

$$\begin{aligned}\delta(PSP^\perp TQRQ^\perp) = {}&\delta(PSP^\perp T)QRQ^\perp + PSP^\perp\delta(TQRQ^\perp)\\ &- PSP^\perp\delta(T)QRQ^\perp.\end{aligned}$$

Since δ is norm continuous and the set of finite linear combinations of projections in $\mathscr{D}_M(\beta)$ is norm dense in $\mathscr{D}_M(\beta)$ and the linear space $\{PSP^\perp : P \in \beta, S \in M\}$ is norm dense in $\mathscr{R}_M(\beta)$, we have

$$\delta(CTD) = \delta(CT)D + C\delta(TD) - C\delta(T)D, \tag{2}$$

$$\delta(CTB) = \delta(CT)B + C\delta(TB) - C\delta(T)B, \tag{3}$$

$$\delta(ATD) = \delta(AT)D + A\delta(TD) - A\delta(T)D, \tag{4}$$

and

$$\delta(ATB) = \delta(AT)B + A\delta(TB) - A\delta(T)B \tag{5}$$

for all C, D in $\mathscr{D}_M(\beta)$ and all A, B in $\mathscr{R}_M(\beta)$. We then have by Eqs. (2)–(5) that

$$\delta(XTY) = \delta(XT)Y + X\delta(TY) - X\delta(T)Y$$

for all T in $\text{alg}_M\beta$ and all X and Y in $\mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$. The proof is complete. $\square$

Taking $T = I$ in Lemma 2.4, we have the following corollary.

Corollary 2.1. $\delta(XY) = \delta(X)Y + X\delta(Y)$ *for all* X, Y *in* $\mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$.

Let β be a finite nest in a factor von Neumann algebra M, it is clear that $\mathscr{D}_M(\beta) + \mathscr{R}_M(\beta) = \mathrm{alg}_M\beta$. Hence by Corollary 2.1, we have the following corollary.

Corollary 2.2. *Let β be a finite nest in M. Then every norm continuous linear local derivation from* $\mathrm{alg}_M\beta$ *into M is a derivation.*

Lemma 2.5. *Let $A \in \mathrm{alg}_M\beta$ and $B \in \mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$. If $AB \in \mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$, then $\delta(AB) = \delta(A)B + A\delta(B)$.*

Proof. If $P = 0_+ \neq 0$, then $PT \in \mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$ for all T in M. Hence by Lemma 2.4, we have

$$\delta(PTAB) = \delta(PTA)B + PT\delta(AB) - PT\delta(A)B.$$

On the other hand, it follows from Corollary 2.1 that

$$\delta(PTAB) = \delta(PTA)B + PTA\delta(B).$$

This implies that $PT[\delta(AB) - \delta(A)B - A\delta(B)] = 0$ for all T in M. Noticing that M is a factor von Neumann algebra, we have $\delta(AB) = \delta(A)B + A\delta(B)$.

If $0_+ = 0$, then there exists a decreasing sequence P_n of projections in $\beta\backslash\{0\}$ which converges strongly to 0. Hence by Lemma 2.4, we have

$$\delta(P_nTP_n^{\perp}AB) = \delta(P_nTP_n^{\perp}A)B + P_nTP_n^{\perp}\delta(AB) - P_nTP_n^{\perp}\delta(A)B$$

for all T in M. On the other hand, it follows from Corollary 2.1 that

$$\delta(P_nTP_n^{\perp}AB) = \delta(P_nTP_n^{\perp}A)B + P_nTP_n^{\perp}A\delta(B).$$

Thus $P_n^{\perp}[\delta(AB) - \delta(A)B - A\delta(B)] = 0$. Letting $n \to \infty$, we have $\delta(AB) = \delta(A)B + A\delta(B)$. The proof is complete. □

Proof of Theorem 1. Let A, B be operators in $\mathrm{alg}_M\beta$. If $P = I_- \neq I$, then $TP^{\perp} \in \mathscr{D}_M(\beta) + \mathscr{R}_M(\beta)$ for all T in M. Hence by Lemma 2.5, we have

$$\delta(ABTP^{\perp}) = \delta(AB)TP^{\perp} + AB\delta(TP^{\perp}).$$

On the other hand, we have from Lemma 2.5 again

$$\begin{aligned}\delta(ABTP^{\perp}) &= \delta(A)BTP^{\perp} + A\delta(BTP^{\perp})\\ &= \delta(A)BTP^{\perp} + A[\delta(B)TP^{\perp} + B\delta(TP^{\perp})]\\ &= \delta(A)BTP^{\perp} + A\delta(B)TP^{\perp} + AB\delta(TP^{\perp}).\end{aligned}$$

This implies that $[\delta(AB) - \delta(A)B - A\delta(B)]TP^{\perp} = 0$ for all T in the factor von Neumann algebra M. Thus $\delta(AB) = \delta(A)B + A\delta(B)$.

If $I_- = I$, then there exists an increasing sequence P_n of projections in $\beta\backslash\{I\}$ which converges strongly to I. Hence by Lemma 2.5, we have

$$\delta(ABP_nTP_n^{\perp}) = \delta(AB)P_nTP_n^{\perp} + AB\delta(P_nTP_n^{\perp})$$

for all T in M. On the other hand, it follows from Lemma 2.5 that

$$\begin{aligned}\delta(ABP_nTP_n^{\perp}) &= \delta(A)BP_nTP_n^{\perp} + A\delta(BP_nTP_n^{\perp})\\ &= \delta(A)BP_nTP_n^{\perp} + A\delta(B)P_nTP_n^{\perp} + AB\delta(P_nTP_n^{\perp}).\end{aligned}$$

Thus $[\delta(AB) - \delta(A)B - A\delta(B)]P_n = 0$. Letting $n \to \infty$, we have $\delta(AB) = \delta(A)B + A\delta(B)$ for all A and B in $\mathrm{alg}_M\beta$. The proof is complete. □

From the proofs of Lemma 2.5 and Theorem 1, we have the following corollary.

Corollary 2.3. *Let β be a nest in a factor von Neumann algebra M with $0_+ = 0$ and $I_- = I$. Then every linear local derivation δ from $\mathrm{alg}_M\beta$ into M (no continuity of δ is assumed) is a derivation.*

Christensen [9] proved that every derivation of nest algebras is an inner derivation. Hence by Theorem 1 and Corollary 3.1, we have the following corollaries.

Corollary 2.4. *Every norm continuous linear local derivation of nest algebras is an inner derivation.*

Corollary 2.5. *Every linear local derivation of neat algebras with nests that satisfy $0_+ = 0$ and $I_- = I$ is an inner derivation.*

By Theorem 1, we can obtain the following corollary, which partially answers the question posed by Larson [1, Remark 6.4].

Corollary 2.6. *Let β be any nest in an arbitrary factor von Neumann algebra M. Then the nest subalgebra $\mathrm{alg}_M\beta$ has an algebraic reflexive derivation space.*

3. 2-local derivations

In this section, our main result is the following theorem.

Theorem 2. *Let β be a nest in an arbitrary factor von Neumann algebra M, and θ be a linear 2-local derivation from the nest subalgebra $\mathrm{alg}_M\beta$ into M (no continuity of θ is assumed). Then $\theta(AB) = \theta(A)B + A\theta(B)$ for all A and B in $\mathrm{alg}_M\beta$.*

Clearly, every linear 2-local derivation is a linear local derivation. Therefore, some results of the above section can apply to linear 2-local derivations. To prove Theorem 2, we also need some lemmas. We assume that θ is a linear 2-local derivation from $\mathrm{alg}_M\beta$ into the factor von Neumann algebra M. It follows from the following lemma that, without loss of generality, we may assume that θ is a Jordan derivation that annihilates $\mathscr{C}(\beta)$ the core of $\text{alg}_M\beta$.

Lemma 3.1. *θ is a Jordan derivation and $\theta = \eta + \theta'$, where η is an inner derivation and θ' is a Jordan derivation that annihilates $\mathscr{C}(\beta)$ the core of* $\text{alg}_M\beta$.

Proof. Let A be any operator in $\text{alg}_M\beta$, then

$$\theta(A^2) = \theta_{A,A^2}(A^2) = \theta_{A,A^2}(A)A + A\theta_{A,A^2}(A) = \theta(A)A + A\theta(A).$$

This shows that θ is a Jordan derivation. Hence by Herstein's results [10], we have

$$\begin{aligned}\theta(ABC + CBA) = {} & \theta(A)BC + A\theta(B)C + AB\theta(C)\\ & + \theta(C)BA + C\theta(B)A + CB\theta(A)\end{aligned}$$

for all A, B and C in $\text{alg}_M\beta$. Next we prove that $\theta : \mathscr{C}(\beta) \to M$ is an inner derivation.

Let A, B and X be operators in $\mathscr{C}(\beta)$, we define

$$\omega(X) = \theta(AX) - (\theta(A)X + A\theta(X)).$$

Noticing that $\mathscr{C}(\beta)$ is an abelian von Neumann algebra, we have

$$\begin{aligned}2\theta(AXB) &= \theta(X)AB + X\theta(A)B + XA\theta(B) + \theta(B)AX + B\theta(A)X + BA\theta(X)\\ &= \theta(A)XB + A\theta(X)B + AX\theta(B) + \theta(B)XA + B\theta(X)A + BX\theta(A).\end{aligned}$$

This implies that $\omega(X)B - B\omega(X) = 0$ for all B in $\mathscr{C}(\beta)$, and hence $\omega(X) \in \mathscr{D}_M(\beta)$. Now let $X, Y \in \mathscr{C}(\beta)$, then

$$2\omega(XY) = (\theta(Y)X + Y\theta(X))A - A(\theta(Y)X + Y\theta(X)) + XY\theta(A) - \theta(A)XY.$$

Since $\theta(Y)X + Y\theta(X) - (\theta(X)Y + X\theta(Y)) \in \mathscr{D}_M(\beta)$, we have

$$\begin{aligned}2(\omega(X)Y + X\omega(Y)) &= (\theta(X)Y + X\theta(Y))A - \theta(A)XY\\ &\quad - A(\theta(X)Y + X\theta(Y)) + XY\theta(A)\\ &= (\theta(Y)X + Y\theta(X))A - \theta(A)XY\\ &\quad - A(\theta(Y)X + Y\theta(X)) + XY\theta(A)\\ &= 2\omega(XY).\end{aligned}$$

Then $\omega : \mathscr{C}(\beta) \to \mathscr{D}_M(\beta)$ is a derivation, and hence by [15, Theorem 10.8] we have $\omega = 0$. Thus $\theta : \mathscr{C}(\beta) \to M$ is a derivation, and so there exists an operator T in M such that $\theta(X) = TX - XT$ for all X in $\mathscr{C}(\beta)$. For every A in $\text{alg}_M\beta$, we set $\eta(A) = TA - AT$ and $\theta'(A) = \theta(A) - \eta(A)$, then $\theta = \eta + \theta'$ and θ' is a Jordan derivation that annihilates $\mathscr{C}(\beta)$ the core of $\text{alg}_M\beta$. The proof is complete. $\square$

From Lemma 3.1 and the results of Brešar [11] and Sakai [12], we have the following corollary.

Corollary 3.1. *Every linear* 2*-local derivation* θ *from a von Neumann algebra into itself* (*no continuity of* θ *is assumed*) *is an inner derivation.*

Lemma 3.2. *Let* P *be a projection in* β, *then* $\theta(PA) = P\theta(A)$ *and* $\theta(AP^{\perp}) = \theta(A)P^{\perp}$ *for all* A *in* $\operatorname{alg}_M \beta$.

Proof. It follows from the fact that θ is a Jordan derivation and $\theta(P) = 0$ that $\theta(PAP) = P\theta(A)P$, $\theta(P^{\perp}AP^{\perp}) = P^{\perp}\theta(A)P^{\perp}$ and

$$\theta(PAP^{\perp}) = \theta(PAP^{\perp} + P^{\perp}AP) = P\theta(A)P^{\perp} + P^{\perp}\theta(A)P. \tag{6}$$

By Lemma 2.3 and Eq. (6), we have $\theta(PAP^{\perp}) = P\theta(A)P^{\perp}$. Hence $\theta(PA) = \theta(PAP) + \theta(PAP^{\perp}) = P\theta(A)$ and similarly, $\theta(AP^{\perp}) = \theta(A)P^{\perp}$. The proof is complete. □

Lemma 3.3. *Let* P *be a projection in* β, *then for all* A *in* $\operatorname{alg}_M \beta$ *and all* X *in* M, *we have*

(a) $\theta(APXP^{\perp}) = \theta(A)PXP^{\perp} + A\theta(PXP^{\perp})$;
(b) $\theta(PXP^{\perp}A) = \theta(PXP^{\perp})A + PXP^{\perp}\theta(A)$.

Proof. (a) It is clear that if A and B belong to $\operatorname{alg}_M \beta$ with $AB = 0$, then $\theta(A)B + A\theta(B) = \theta_{A,B}(AB) = 0$. Hence by Lemmas 3.1, 3.2 and 2.3, we have

$$\begin{aligned}\theta(APXP^{\perp}) &= \theta(PAPXP^{\perp} + PXP^{\perp}PA)\\ &= \theta(PA)PXP^{\perp} + PA\theta(PXP^{\perp})\\ &= \theta(A)PXP^{\perp} + A\theta(PXP^{\perp}).\end{aligned}$$

Similarly, we can show that (b) holds. The proof is complete. □

Proof of Theorem 2. If β is a trivial nest, then $\operatorname{alg}_M \beta = M$, and hence θ is a derivation by Corollary 3.1. Now we suppose that β is a non-trivial nest. Let P be a fixed non-trivial projection in β, and A and B be operators in $\operatorname{alg}_M \beta$. Then by Lemma 3.3(a),

$$\theta(ABPXP^{\perp}) = \theta(AB)PXP^{\perp} + AB\theta(PXP^{\perp}) \tag{7}$$

for all X in M. On the other hand, we have

$$\begin{aligned}\theta(ABPXP^{\perp}) &= \theta(A)BPXP^{\perp} + A\theta(BPXP^{\perp})\\ &= \theta(A)BPXP^{\perp} + A\theta(B)PXP^{\perp} + AB\theta(PXP^{\perp}).\end{aligned}$$

Note that M is a factor von Neumann algebra, then

$$[\theta(AB) - \theta(A)B - A\theta(B)]P = 0. \tag{8}$$

Similarly, by Lemma 3.3(b), we can obtain that

$$P^{\perp}[\theta(AB) - \theta(A)B - A\theta(B)] = 0. \tag{9}$$

It follows from Lemma 3.2 that

$$\begin{aligned} P\theta(AB)P^{\perp} &= \theta(PABP^{\perp}) = \theta(PABP^{\perp} + BP^{\perp}PA) \\ &= \theta(PA)BP^{\perp} + PA\theta(BP^{\perp}) \\ &= P\theta(A)BP^{\perp} + PA\theta(B)P^{\perp}. \end{aligned}$$

This and Eqs. (8) and (9) give us that $\theta(AB) = \theta(A)B + A\theta(B)$ for all A and B in $\mathrm{alg}_M\beta$. The proof is complete. □

By Theorem 2 and Christensen's result [9], we have the following corollary.

Corollary 3.2. *Every linear* 2*-local derivation of nest algebras is an inner derivation*.

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