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On the realization of Riemannian symmetric spaces in Lie groups II ☆

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Abstract

In this paper we generalize a result in [J. An, Z. Wang, On the realization of Riemannian symmetric spaces in Lie groups, Topology Appl. 153 (7) (2005) 1008–1015, showing that an arbitrary Riemannian symmetric space can be realized as a closed submanifold of a covering group of the Lie group defining the symmetric space. Some properties of the subgroups of fixed points of involutions are also proved.

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1. Introduction

Suppose G is a connected Lie group with an involution σ. Then the Lie algebra $\mathfrak{g}$ of G has a canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of $d\sigma$ in $\mathfrak{g}$ with eigenvalues 1 and -1, respectively. Let $G^\sigma = \{g \in G \mid \sigma(g) = g\}$, and suppose K is an open subgroup of G^σ. Suppose moreover that $\mathrm{Ad}_G(K)|_{\mathfrak{p}}$ is compact, that is, G/K has a structure of Riemannian symmetric space. In the particular case that $K = G^\sigma$, it was proved in [1] that $P = \exp(\mathfrak{p})$ is a closed submanifold of G, and there is a natural isomorphism $G/G^\sigma \cong P$. This gave a realization of the symmetric space G/G^σ in G. In this paper we generalize this result to the case of arbitrary symmetric space G/K, that is, the case that K is an arbitrary open subgroup of G^σ such that $\mathrm{Ad}_G(K)|_{\mathfrak{p}}$ is compact.

In Section 2 we will make some preparation for this generalized realization. Some properties of the subgroup G^σ will be examined. We will prove the following results.

- There exists a covering group G' of G with covering homomorphism π such that there is an involution σ' on G' with $\pi \circ \sigma' = \sigma \circ \pi$, and such that $\pi^{-1}(K) = (G')^{\sigma'}$.

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- The quotient group G^σ/G_0^σ is isomorphic to $(\mathbb{Z}_2)^r$ for some non-negative integer r, where G_0^σ is the identity component of G^σ.
- G^σ is connected if $\pi_1(G)$ is finite with odd order.

In Section 3 we will give the precise statement of the realization of arbitrary symmetric spaces in Lie groups. Briefly speaking, a symmetric space G/K is diffeomorphic to a closed submanifold P' of a covering group G' of G, where G' is chosen such that $\pi^{-1}(K) = (G')^{\sigma'}$. The idea of the proof is that $G/K \cong G'/\pi^{-1}(K) = G'/(G')^{\sigma'} \cong \exp_{G'}(\mathfrak{p}) = P'$.

2. The subgroups of fixed points of involutions

For a Lie group H with an automorphism θ, we always denote $H^\theta = \{h \in H \mid \theta(h) = h\}$, and denote the identity component of H^θ by H_0^θ. In this section we prove the following two theorems. The realization of arbitrary symmetric spaces in Lie groups will be based on Theorem 2.1.

Theorem 2.1. *Let G be a connected Lie group with an involution σ, K an open subgroup of G^σ. Then there exists a covering group G' of G with covering homomorphism π such that there is an involution σ' on G' with $\pi \circ \sigma' = \sigma \circ \pi$, and such that $\pi^{-1}(K) = (G')^{\sigma'}$.*

Theorem 2.2. *Let G be a connected Lie group with an involution σ. Then the quotient group G^σ/G_0^σ is isomorphic to $(\mathbb{Z}_2)^r$ for some non-negative integer r, and $r = 0$ if $\pi_1(G)$ is finite with odd order.*

First we introduce some notations. Let G be a connected Lie group with an involution σ. For $g, h \in G$, denote the set of all continuous paths $\gamma : [0,1] \to G$ with $\gamma(0) = g$ and $\gamma(1) = h$ by $\Omega(G, g, h)$, and denote $\pi_1(G, g, h) = \{[\gamma] \mid \gamma \in \Omega(G, g, h)\}$, where $[\gamma]$ is the homotopy class relative to endpoints determined by γ. Note that $\pi_1(G, g, g)$ is the fundamental group $\pi_1(G, g)$ of G with basepoint g. For $g \in G^\sigma$ and $\gamma_g \in \Omega(G, e, g)$, let $\gamma_g^\sigma \in \Omega(G, e, e)$ be defined as

$$\gamma_g^\sigma(t) = \begin{cases} \gamma_g(2t), & t \in [0, \frac{1}{2}]; \\ \sigma(\gamma_g(2-2t)), & t \in [\frac{1}{2}, 1]. \end{cases}$$

The set $\pi_g = \{[\gamma_g^\sigma] \mid \gamma_g \in \Omega(G, e, g)\}$ is a subset of $\pi_1(G, e)$. For $g_1, g_2 \in G^\sigma$, if they belong to the same coset space of G_0^σ, it is obvious that $\pi_{g_1} = \pi_{g_2}$. So we can define $\pi_{[g]} = \pi_g$ for $g \in G^\sigma$, where $[g]$ denotes the coset space gG_0^σ. We denote the identity element of $\pi_1(G, e)$ by $\mathbf{e}$.

Lemma 2.3. *For $g \in G^\sigma$, $\pi_{[g]}^{-1} = \pi_{[g]}$, that is, $x \in \pi_{[g]} \Longleftrightarrow x^{-1} \in \pi_{[g]}$.*

Proof. It is obvious from the equation $[\gamma_g^\sigma]^{-1} = [(\sigma \circ \gamma_g)^\sigma]$. □

Lemma 2.4. *For $g \in G^\sigma$, $\pi_{[g]} = \pi_{[g^{-1}]}$.*

Proof. For $[\gamma_g^\sigma] \in \pi_{[g]}$, $[(\gamma_g^{-1})^\sigma] \in \pi_{[g^{-1}]}$, where $\gamma_g^{-1}(t) = \gamma_g(t)^{-1}$. But by Lemma 16.7 in [4], $[\gamma_g^\sigma] \cdot [(\gamma_g^{-1})^\sigma] = [\gamma_g^\sigma] \cdot [(\gamma_g^\sigma)^{-1}] = [\gamma_g^\sigma(\gamma_g^\sigma)^{-1}] = \mathbf{e}$, where $(\gamma_g^\sigma(\gamma_g^\sigma)^{-1})(t) = \gamma_g^\sigma(t)(\gamma_g^\sigma)^{-1}(t)$. So by Lemma 2.3, $[\gamma_g^\sigma] = [(\gamma_g^{-1})^\sigma]^{-1} \in \pi_{[g^{-1}]}^{-1} = \pi_{[g^{-1}]}$. This proves $\pi_{[g]} \subset \pi_{[g^{-1}]}$. By symmetry, we have the equality. □

Lemma 2.5. *For $g_1, g_2 \in G^\sigma$, $[\gamma_{g_1}^\sigma] \cdot \pi_{[g_2]} = \pi_{[g_1 g_2]}$ for each $[\gamma_{g_1}^\sigma] \in \pi_{[g_1]}$. Hence $\pi_{[g_1]} \cdot \pi_{[g_2]} = \pi_{[g_1 g_2]}$.*

Proof. Let $[\gamma_{g_2}^\sigma] \in \pi_{[g_2]}$. By Lemma 16.7 in [4], $[\gamma_{g_1}^\sigma] \cdot [\gamma_{g_2}^\sigma] = [\gamma_{g_1}^\sigma \gamma_{g_2}^\sigma] = [(\gamma_{g_1}\gamma_{g_2})^\sigma] \in \pi_{[g_1 g_2]}$. So we have $[\gamma_{g_1}^\sigma] \cdot \pi_{[g_2]} \subset \pi_{[g_1 g_2]}$. To prove the equality, denote $x = [\gamma_{g_1}^\sigma]$, and consider the map $l_x : \pi_{[g_2]} \to \pi_{[g_1 g_2]}$ defined by $l_x(y) = xy$. Since $x^{-1} \in \pi_{[g_1]}^{-1} = \pi_{[g_1]} = \pi_{[g_1^{-1}]}$, $x^{-1} \cdot \pi_{[g_1 g_2]} \subset \pi_{[g_2]}$. So we can also define the map $l_{x^{-1}} : \pi_{[g_1 g_2]} \to \pi_{[g_2]}$ by $l_{x^{-1}}(y) = x^{-1}y$. Since $l_x \circ l_{x^{-1}} = \mathrm{id}$, l_x is surjective. This means $x \cdot \pi_{[g_2]} = \pi_{[g_1 g_2]}$, hence $\pi_{[g_1]} \cdot \pi_{[g_2]} = \pi_{[g_1 g_2]}$. □

Lemma 2.6. *Let K be an open subgroup of G^σ. Then $\pi_K := \bigcup_{k \in K} \pi_{[k]}$ is a subgroup of $\pi_1(G, e)$.*

Proof. By Lemma 2.3, each $\pi_{[k]}$ is closed under the inverse operation, hence so is π_K. By Lemma 2.5, π_K is also closed under multiplication. □

By Lemma 2.6, $\pi_{[e]} = \pi_{(G_0^\sigma)}$ and $\pi_{(G^\sigma)}$ are subgroups of $\pi_1(G, e)$. By Lemma 2.5, for each $g \in G^\sigma$ and each $x \in \pi_{[g]}$, $\pi_{[g]} = x \cdot \pi_{[e]}$. So $\pi_{[g]}$ is a coset space of $\pi_{[e]}$ in $\pi_{(G^\sigma)}$, that is, an element in the quotient group $\Pi^\sigma = \pi_{(G^\sigma)}/\pi_{[e]}$ (note that the fundamental group of a Lie group is always Abelian). Define the map $f : G^\sigma/G_0^\sigma \to \Pi^\sigma$ by $f([g]) = \pi_{[g]}$. By Lemma 2.5, f is a homomorphism.

Lemma 2.7. *Let G be a connected Lie group with an involution σ. Then G^σ has finite many connected components. If moreover G is simply connected, then G^σ is connected.*

Proof. The subgroup $\Sigma = \{\mathrm{id}, \sigma\}$ of the automorphism group $\mathrm{Aut}(G)$ of G has a natural action on G, which we also denote by σ. We form the semidirect product $\mathbf{G} = G \times_\sigma \Sigma$. Denote $x = (e, \sigma) \in \mathbf{G}$. Since the identity component $\mathbf{G}_0$ of $\mathbf{G}$ is naturally isomorphic to G by $i : (g, \mathrm{id}) \mapsto g$, and $\sigma(i(y)) = i(xyx^{-1})$, $\forall y \in \mathbf{G}_0$, to prove the lemma, it is sufficient to show that $\mathbf{G}_0^x = \{y \in \mathbf{G}_0 \mid xyx^{-1} = y\}$ has finite many connected components, and is connected when $\mathbf{G}_0$ is simply connected.

Since $x = (e, \sigma)$ is an element of order two in $\mathbf{G}$, it lies in some maximal compact subgroup $\mathbf{L}$ of $\mathbf{G}$. By Theorem 3.1 of Chapter XV in [3], there exist some linear subspaces $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ (which is isomorphic to the Lie algebra of G) such that

(1) $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_k$, where $\mathfrak{l}$ is the Lie algebra of $\mathbf{L}$;
(2) $\mathrm{Ad}(l)(\mathfrak{m}_i) = \mathfrak{m}_i$, $\forall l \in \mathbf{L}$, $i \in \{1, \dots, k\}$ (in particular, $\mathrm{Ad}(x)(\mathfrak{m}_i) = \mathfrak{m}_i$);
(3) the map $\psi : \mathbf{L}_0 \times \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_k \to \mathbf{G}_0$ defined by $\psi(l, X_1, \dots, X_k) = le^{X_1} \cdots e^{X_k}$ is a diffeomorphism, where $\mathbf{L}_0$ is the identity component of $\mathbf{L}$.

Denote $\mathbf{L}_0^x = \{l \in \mathbf{L}_0 \mid xlx^{-1} = l\}$, $\mathfrak{m}_i^x = \{X \in \mathfrak{m}_i \mid \mathrm{Ad}(x)(X) = X\}, i = 1, \dots, k$. We claim that $\mathbf{G}_0^x = \psi(\mathbf{L}_0^x \times \mathfrak{m}_1^x \times \cdots \times \mathfrak{m}_k^x)$. In fact, for $y \in \mathbf{G}_0^x$, write $y = le^{X_1} \cdots e^{X_k}$, where $l \in \mathbf{L}_0$, $X_i \in \mathfrak{m}_i$. Then $y = xyx^{-1} = (xlx^{-1})e^{Ad(x)(X_1)} \cdots e^{Ad(x)(X_k)}$. By (3), we have $xlx^{-1} = l$, $\mathrm{Ad}(x)(X_i) = X_i, i = 1, \dots, k$. That is, $l \in \mathbf{L}_0^x$, $X_i \in \mathfrak{m}_i^x$. Hence we have $\mathbf{G}_0^x \subset \psi(\mathbf{L}_0^x \times \mathfrak{m}_1^x \times \cdots \times \mathfrak{m}_k^x)$. The other inclusion is obvious.

Since $\mathbf{L}_0$ is a connected compact Lie group, $\mathbf{L}_0^x$ has finitely many connected components, hence so does $\mathbf{G}_0^x = \psi(\mathbf{L}_0^x \times \mathfrak{m}_1^x \times \cdots \times \mathfrak{m}_k^x)$. If $\mathbf{G}_0$ is simply connected, by (3), $\mathbf{L}_0$ is also simply connected. By Theorem 8.2 of Chapter VII in [2], $\mathbf{L}_0^x$ is connected, hence so is $\mathbf{G}_0^x$. This proves the lemma. □

Lemma 2.8. *For $g \in G^\sigma$, if $\mathbf{e} \in \pi_{[g]}$, then $[g] = G_0^\sigma$.*

Proof. We endow $\widetilde{G} = \bigcup_{g \in G} \pi_1(G, e, g)$ with the canonical smooth manifold structure and the canonical group structure such that $\widetilde{G}$ is the universal covering group of G with covering homomorphism $\pi([\gamma]) = \gamma(1)$. The induced involution of σ on $\widetilde{G}$ is $\tilde{\sigma}([\gamma]) = [\sigma \circ \gamma]$. Let $g \in G^\sigma$. If $\mathbf{e} \in \pi_{[g]}$, by definition of $\pi_{[g]}$, there is a $\gamma_g \in \Omega(G, e, g)$ such that $[\gamma_g] = [\sigma \circ \gamma_g]$ in $\pi_1(G, e, g)$. That is, $[\gamma_g] \in (\widetilde{G})^{\tilde{\sigma}}$. By Lemma 2.7, $(\widetilde{G})^{\tilde{\sigma}}$ is connected. So $g = \pi([\gamma_g]) \in \pi((\widetilde{G})^{\tilde{\sigma}}) = G_0^\sigma$, that is, $[g] = G_0^\sigma$. □

Proposition 2.9. *Each non-trivial element of Π^σ has order 2, and the homomorphism $f : G^\sigma/G_0^\sigma \to \Pi^\sigma$ is an isomorphism.*

Proof. The first assertion follows from Lemma 2.3, the surjectivity of f follows from the definition of Π^σ, and the injectivity of f follows from Lemma 2.8. □

In particular, we have

Lemma 2.10. *For $g_1, g_2 \in G^\sigma$, if $[g_1] \neq [g_2]$, then $\pi_{[g_1]} \cap \pi_{[g_2]} = \emptyset$.*

Now we are prepared to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.2. By Lemma 2.7, G^σ/G_0^σ is a finite group. By Proposition 2.9, G^σ/G_0^σ is Abelian, and each non-trivial element of G^σ/G_0^σ has order 2. By the structure theorem of finitely generated Abelian groups, G^σ/G_0^σ is isomorphic to $(\mathbb{Z}_2)^r$ for some non-negative integer r. If $\pi_1(G)$ is finite with odd order, so are $\pi_{(G^\sigma)}$, Π^σ, and G^σ/G_0^σ. This forces $r=0$. □

Proof of Theorem 2.1. We define an equivalence relation on the set $\bigcup_{g\in G}\Omega(G,e,g)$ as follows. For $\gamma_1,\gamma_2\in\bigcup_{g\in G}\Omega(G,e,g)$, $\gamma_1\sim\gamma_2$ if and only if $\gamma_1(1)=\gamma_2(1)$ and $[\gamma_{12}]\in\pi_K$, where $\gamma_{12}\in\Omega(G,e,e)$ is defined by

$$\gamma_{12}(t)=\begin{cases}\gamma_1(2t), & t\in[0,\frac{1}{2}];\\ \gamma_2(2-2t), & t\in[\frac{1}{2},1].\end{cases}$$

Denote $G'=(\bigcup_{g\in G}\Omega(G,e,g))/{\sim}$, and denote the equivalence class of a $\gamma\in\bigcup_{g\in G}\Omega(G,e,g)$ by $\langle\gamma\rangle$. Endow G' with a smooth manifold structure in the canonical way, and define the group operation on G' as follows. For $\langle\gamma_1\rangle,\langle\gamma_2\rangle\in G'$, let $\alpha(t)=\gamma_1(t)\gamma_2(t)$ and $\beta(t)=\gamma_1(t)^{-1}$, and define $\langle\gamma_1\rangle\langle\gamma_2\rangle=\langle\alpha\rangle$, $\langle\gamma_1\rangle^{-1}=\langle\beta\rangle$. It is easy to check that these are well defined, making G' a Lie group. Let $\pi:G'\to G$ be the map $\pi(\langle\gamma\rangle)=\gamma(1)$, then π is a covering homomorphism. Since $\sigma_*(\pi_K)=\pi_K$, the map $\sigma':G'\to G'$, $\sigma'(\langle\gamma\rangle)=\langle\sigma\circ\gamma\rangle$ is a well defined involution of G'. It is obvious that $\pi\circ\sigma'=\sigma\circ\pi$, that is, σ' is just the induced involution of σ on G'. Now we have

$$\begin{aligned}\langle\gamma\rangle\in(G')^{\sigma'} &\Longleftrightarrow \langle\sigma\circ\gamma\rangle=\langle\gamma\rangle\\ &\Longleftrightarrow \sigma\circ\gamma\sim\gamma\\ &\Longleftrightarrow \gamma(1)\in G^\sigma,\ [\gamma^\sigma_{\gamma(1)}]\in\pi_K\\ &\Longleftrightarrow \gamma(1)\in K\quad\text{(by Lemma 2.10)}\\ &\Longleftrightarrow \langle\gamma\rangle\in\pi^{-1}(K).\end{aligned}$$

That is, $(G')^{\sigma'}=\pi^{-1}(K)$. This completes the proof of Theorem 2.1. □

Remark 2.1. The covering group G' of G satisfying the conclusion of Theorem 2.1 is not necessarily unique. For example, let G be a semisimple Lie group with trivial center, and let σ be a global Cartan involution of G. Then $K=G^\sigma$ is a maximal compact subgroup of G. Let G' be any finite cover of G with an involution σ' such that $\pi\circ\sigma'=\sigma\circ\pi$, then $(G')^{\sigma'}$ is a maximal compact subgroup of G'. Since $\pi^{-1}(G^\sigma)$ is a compact subgroup of G' containing $(G')^{\sigma'}$, it must equal $(G')^{\sigma'}$. In fact, using the same method as in the proof of Theorem 2.1, it is easy to see that a covering group G' of G with covering homomorphism π satisfying $\pi^{-1}(K)=(G')^{\sigma'}$ if and only if $\pi_*(\pi_1(G',e))\cap\pi_{(G^\sigma)}=\pi_K$, and the group G' that we constructed in the proof of Theorem 2.1 is just the one satisfying $\pi_*(\pi_1(G',e))=\pi_K$.

Remark 2.2. Professor Jiu-Kang Yu pointed out to the first author proofs of Theorems 2.1 and 2.2 using nonabelian cohomology after he read the first draft of the paper.

3. Covering groups and the realization of symmetric spaces

Let G be a connected Lie group with an involution σ, and let K be an open subgroup of G^σ. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the differential of the involution σ. Suppose $\mathrm{Ad}_G(K)|_{\mathfrak{p}}$ is compact, that is, G/K has a structure of Riemannian symmetric space. Suppose $\pi:G'\to G$ is a covering homomorphism of Lie groups. Then the twisted conjugate action τ of G on G, which is defined by $\tau_g(h)=gh\sigma(g)^{-1}$, can be lifted to a well defined action τ' of G on G', that is, $\tau'_g(h')=g'h'\sigma'(g')^{-1}$, $g\in G$, $h'\in G'$, where $g'\in\pi^{-1}(g)$, σ' is the induced involution of σ on G'.

Theorem 3.1. *Under the above assumptions, there is a covering group G' of G with covering homomorphism π such that $P'=\exp'_G(\mathfrak{p})$ is a closed submanifold of G', and such that the map $\varphi:G/K\to P'$ defined by $\varphi(gK)=g'\sigma'(g')^{-1}$ is a diffeomorphism, where $g'\in\pi^{-1}(g)$, σ' is the induced involution of σ on G'. With respect to the actions of G by left multiplication on G/K and by the action τ' on P', φ is equivariant.*

Proof. By Theorem 2.1, there is a covering group G' of G with covering homomorphism π such that $\pi^{-1}(K) = (G')^{\sigma'}$. So π induces a G-equivariant diffeomorphism $\bar{\pi}: G'/(G')^{\sigma'} \to G/K$ with respect to left multiplications (note that the left multiplication of G on $G'/(G')^{\sigma'}$ is well defined). Since $\mathrm{Ad}_{G'}((G')^{\sigma'})|_{\mathfrak{p}} = \mathrm{Ad}_G(K)|_{\mathfrak{p}}$ is compact, by Corollary 2.6 in [1], $P' = \exp_{G'}(\mathfrak{p})$ is a closed submanifold of G', and the map $\varphi': G'/(G')^{\sigma'} \to P'$, $\varphi'(g'(G')^{\sigma'}) = g'\sigma'(g')^{-1}$ is a G'-equivariant diffeomorphism with respect to left multiplication and twisted conjugate action of G'. Let $\varphi = \varphi' \circ (\bar{\pi})^{-1}$. Then $\varphi(gK) = \varphi'(g'(G')^{\sigma'}) = g'\sigma'(g')^{-1}$. It is obviously a G-equivariant diffeomorphism with respect to left multiplication on G/K and the action τ' on P'. $\square$

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