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THE PLANCHEREL FORMULA FOR LINE BUNDLES ON COMPLEX HYPERBOLIC SPACES

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ABSTRACT. – In this paper we obtain the Plancherel formula for the spaces of L^2-sections of line bundles over the complex projective hyperboloids G/H with $G = \mathrm{U}(p,q;\mathbb{C})$ and $H = \mathrm{U}(1;\mathbb{C}) \times \mathrm{U}(p-1,q;\mathbb{C})$. The Plancherel formula is given in an explicit form by means of spherical distributions associated with a character χ of the subgroup H. We obtain the Plancherel formula by a special method which is also suitable for other problems, for example, for quantization in the spirit of Berezin. © 2000 Éditions scientifiques et médicales Elsevier SAS

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0. Introduction

In this paper we obtain the Plancherel formula for the spaces of L^2-sections of line bundles over the complex projective hyperboloids G/H with $G = \mathrm{U}(p,q;\mathbb{C})$ and $H = \mathrm{U}(1;\mathbb{C}) \times \mathrm{U}(p-1,q;\mathbb{C})$, i.e. we present the decomposition of L^2 into irreducible representations of the group G of class χ. In order to leave aside the well-known case of a hyperboloid with compact stabilizer subgroup, see [14], we assume $p > 1$, $q > 0$.

The Plancherel formula is given in an explicit form by means of spherical distributions associated with a character χ of the subgroup H. We obtain the Plancherel formula by Molchanov's method, see [9]. Namely, we follow the detailed scheme in [1], Sections 4, 7. This method deals with the spectral resolution of the radial part of the Laplace operator. The essential step is setting the boundary conditions at certain special points. Those conditions are prescribed by the behaviour of spherical distributions. Finally, it is necessary to perform various analytic continuations. This method is also suitable for other problems, for example, for quantization in the spirit of Berezin, namely, for the decomposition of the Berezin form. It is therefore why this method has to be preferred to the existing methods, described in [3].

We use our results from [13]. There we define χ-spherical distributions, study their asymptotic behaviour and express them by means of hypergeometric functions. We describe the irreducible unitary representations of the group G, of class χ associated with an isotropic cone. We give constructions for the Fourier and Poisson transform, define intertwining operators and diagonalize them. Some of those results are presented in Section 1.

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This problem was studied by several people. The case of the trivial bundle was studied in [3]. But the Plancherel formula was proved by a different method. Some important results about representations of the group G associated with a cone in the complex case were obtained in [8]. For non-trivial bundles the Plancherel formula is stated in [11]. The authors of [11] follow a method from [3] but do not determine the spherical distributions. They also do not present an explicit form of the discrete part of the Plancherel formula. As to the method used by them, it is not suitable for other important rank one spaces like $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$. Our method however is suitable not only for the determination of the Plancherel formula but also for the other problems, mentioned above.

We would like to emphasize an interesting, but maybe known, phenomenon which occurs due to the noncompactness of H. The spherical distributions have, as in the real case, see [7], nonintegrable singularities on the singular orbits of the group H and hence do not yield locally integrable functions. In some cases, namely in the first discrete series (Theorem 4.4), the spherical distributions turn out to be supported only on the singular orbits.

1. χ-spherical distributions on $\mathrm{U}(p,q)$ with respect to the subgroup $\mathrm{U}(1)\times\mathrm{U}(p-1,q)$

In this section we introduce some results from [13].

1.1. χ-spherical distributions

Let G denote the group $\mathrm{U}(p,q)$ of complex matrices of order $n=p+q$, preserving in $\mathbb{C}^n$ the Hermitian form:

$$[x,y]=x_1\bar{y}_1+\cdots+x_p\bar{y}_p-x_{p+1}\bar{y}_{p+1}-\cdots-x_n\bar{y}_n. \tag{1.1}$$

Consider the subgroup $H=H_0\times H_1$, where $H_0=\mathrm{U}(1)$, $H_1=\mathrm{U}(p-1,q)$. Define for $h\in H$, $h=h_0\cdot h_1$: $\chi(h)=h_0^l$, where $h_0\in H_0$, $h_1\in H_1$ and $l=0,1,2,\ldots$. We can define this for any integer l but it is sufficient and more convenient to take non-negative l.

Consider the space $X_1=G/H_1$. Let $\mathcal{D}(X_1)$ be the space of complex-valued $\mathbb{C}^\infty$-functions on X_1 with compact support. X_1 can be identified with the set $[x,x]=1$. It is a pseudo-Riemannian manifold (non-symmetric) and as such it has a Laplacian which we denote by $\square_1$. We define the χ-spherical distributions on $G=\mathrm{U}(p,q)$ with respect to the subgroup $H=\mathrm{U}(1)\times\mathrm{U}(p-1,q)$ as the distributions T on G which satisfy the following conditions:

(1) $T(hgh')=\chi(hh')T(g)$. This means that such distributions are H_1-invariant and hence can be seen as distributions on X_1 with an extra property with respect to the subgroup H_0: $T(x\,h_0)=\chi(h_0)T(x)$, where $x\in X_1$.

(2) They are eigendistributions of $\square_1$.

So χ-spherical distributions on G with respect to the subgroup H are distributions on X_1 with the following properties:

- T is invariant with respect to H_1,
- $\square_1 T=\lambda T$,
- $T(\mathrm{e}^{\mathrm{i}\theta}x)=\mathrm{e}^{\mathrm{i}l\theta}T(x)$, $\theta\in\mathbb{R}$.

Denote this space by $\mathcal{D}'_{\lambda,l}(X_1)$. Suppose $T=x_1^l\cdot S$, where S is an H-invariant distribution and x_1 is the first coordinate of $x\in X_1$. The equation $\square_1 T=\lambda T$ can be reduced to a differential equation for S and then to a singular second order differential equation on $[0;\infty[$:

$$Ly=\lambda y, \tag{1.2}$$

where:

$$L = 4t(t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 4\big[(\mu+l+2)t-(l+1)\big]\frac{\mathrm{d}}{\mathrm{d}t} + l\big(2(\mu+1)+l\big),$$
$$\mu = p+q-2,\ \lambda = s^2-(\mu+1)^2. \tag{1.3}$$

The solutions of this equation at points $t=0$ and $t=1$ respectively are the following hypergeometric functions:

$$\begin{aligned} F(t,s,\mu,l) &= {}_2F_1\left(\frac{\mu+s+l+1}{2}, \frac{\mu-s+l+1}{2}; l+1; t\right),\\ \Phi(t,s,\mu,l) &= {}_2F_1\left(\frac{\mu+s+l+1}{2}, \frac{\mu-s+l+1}{2}; \mu+1; 1-t\right). \end{aligned} \tag{1.4}$$

1.2. The averaging map

It turns out that any χ-spherical distribution is associated with an H-invariant distribution (multiplied by x_1^l), see [13].

First, we find the basis of the space of these H-invariant distributions. We use the averaging map $\mathcal{M}$ from [3]. Consider the surface $\{|x_1|^2 = t,\ t\in\mathbb{R},\ t\geqslant 0\}$. The average of a function $f\in\mathcal{D}(G/H)$ over this surface belongs to the space $\mathcal{H}$ of functions on $[0;\infty[$ of the following form:

$$\varphi(t) = \varphi_0(t) + \eta(t)\varphi_1(t), \quad \varphi_0(t),\ \varphi_1(t)\in\mathcal{D}\big([0;\infty[\big), \tag{1.5}$$

where η is the “singularity” function $\eta(t) = Y(1-t)(1-t)^\mu$ and Y is the Heaviside function: $Y(t)=1$ if $t\geqslant 0$, $Y(t)=0$ if $t<0$. Using the results from [6] we solve the equation $LS=\lambda S$ in $\mathcal{H}'$ and find a basis of the space of distribution solutions. Then we take the pull back and get a basis of the space of H-invariant distributions. Finally, we return to χ-spherical distributions by multiplying with x_1^l. Let us denote this procedure by $\mathcal{A}'$. In this way we find a basis of $\mathcal{D}'_{\lambda,l}(X_1)$.

The basis in $\mathcal{H}'$ is built by means of functions (1.4). Introduce the following notations:

$$\begin{aligned} F^0(t,\lambda,\mu,l) &= \lim_{\varepsilon\to 0}\frac{F(t+\mathrm{i}\varepsilon,\lambda,\mu,l)+F(t-\mathrm{i}\varepsilon,\lambda,\mu,l)}{2},\\ G(t,\lambda,\mu,l) &= \lim_{\varepsilon\to 0}\frac{F(t+\mathrm{i}\varepsilon,\lambda,\mu,l)-F(t-\mathrm{i}\varepsilon,\lambda,\mu,l)}{2\mathrm{i}},\\ A(\lambda,\mu,l) &= \frac{\Gamma(l+1)\Gamma(\mu)}{\Gamma(\frac{l+\mu+s+1}{2})\Gamma(\frac{l+\mu-s+1}{2})}, \end{aligned} \tag{1.6}$$

where $\lambda,\ \mu$ are as in (1.3). The functions F^0 and G are defined for $t\neq 1$. Moreover G is zero on $(-\infty,1)$ and for $t>1$ and integer $\mu\geqslant 0$ it can be rewritten as follows:

$$G(t,\lambda,\mu,l) = \frac{(-1)^\mu\pi}{\mu!\Gamma(-\mu)}\cdot A(\lambda,-\mu,l)\cdot\Phi(t,\lambda,\mu,l) \tag{1.7}$$

with $\Phi(t,\lambda,\mu,l)$ as in (1.4). Notice that for $\lambda=\lambda_r=(l+2r)(l+2r+2\mu+2),\ r=0,1,2,\ldots,$ i.e. for $s=\pm(l+2r+\mu+1)$ the function $F^0(t,\lambda,\mu,l)$ is a polynomial of degree r and the function $G(t,\lambda,\mu,l)$ is zero.

Define as in [3]:

$$S_{\lambda,l}(\varphi) = \lim_{\varepsilon\to 0}\int_0^\infty \frac{1}{2}\big[F(t+\mathrm{i}\varepsilon,\lambda,\mu,l)+F(t-\mathrm{i}\varepsilon,\lambda,\mu,l)\big]\varphi(t)\,\mathrm{d}t,$$

$$T_{\lambda,l}(\varphi)=\lim_{\varepsilon\to 0}\int_0^\infty \frac{1}{2\mathrm{i}}\bigl[F(t+\mathrm{i}\varepsilon,\lambda,\mu,l)-F(t-\mathrm{i}\varepsilon,\lambda,\mu,l)\bigr]\varphi(t)\,\mathrm{d}t;$$

$T_{\lambda,l}$ can be expressed for integer $\mu\geqslant 0$ as:

$$\begin{aligned} T_{\lambda,l}(\varphi) \;=\; & A(\lambda,\mu,l)\sum_{k=0}^{\mu-1}(-1)^{\mu-k-1}\pi\cdot a_k(\lambda,-\mu,l)\frac{\varphi^{(\mu-k-1)}(t)|_{t=1}}{(\mu-k-1)!}\\ & +\int_1^\infty G(t,\lambda,\mu,l)\varphi(t)\,\mathrm{d}t \end{aligned}\tag{1.8}$$

with

$$\alpha_k(\lambda,\mu,l)=\frac{((\mu+s+l+1)/2)_k((\mu-s+l+1)/2)_k}{(\mu+1)_k k!},$$

where $(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}=\gamma(\gamma+1)\cdots(\gamma+k-1)$, see [3, p. 385].

1.3. A basis of the space of χ-spherical distributions

Set for $\lambda_r=(l+2r)(l+2r+2\mu+2)$

$$T'_{\lambda_r,l}(f)=\lim_{\lambda\to\lambda_r}\frac{1}{A(\lambda,\mu,l)}T_{\lambda,l}.\tag{1.9}$$

We have the important result:

THEOREM 1.1. – *For any* $\lambda\neq\lambda_r$ $\dim\mathcal{D}'_{\lambda,l}(X_1)=1$ *and* $\mathcal{D}'_{\lambda,l}(X_1)$ *is generated by* $\mathcal{A}'T_{\lambda,l}$. *For* $\lambda=\lambda_r$ $\dim\mathcal{D}'_{\lambda,l}(X_1)=2$ *and* $\mathcal{D}'_{\lambda,l}(X_1)$ *is generated by* $\mathcal{A}'S_{\lambda_r,l}$ *and* $\mathcal{A}'T'_{\lambda_r,l}$.

1.4. Representations of the group G

We consider representations of the group G induced from maximal parabolic subgroup of G. These representations are associated with an isotropic cone and were studied in [8,3] for the case $\chi=1$. A description of representations of class χ was given in [13], see also [12,10].

Some subgroups of our group G can be introduced as usual, see [3,8,10]. Namely, let K be the maximal compact subgroup $\mathrm{U}(p)\times\mathrm{U}(q)$ of G.

Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ denote the Lie algebras of the groups G, K, H respectively. Let $\mathfrak{p}$, $\mathfrak{q}$ be the orthogonal complements to $\mathfrak{k}$, $\mathfrak{h}$ with respect to the Killing form. All maximal Abelian subalgebras in $\mathfrak{p}\cap\mathfrak{q}$ are one-dimensional. Consider one of them with basis:

$$L=\begin{pmatrix}0&0&1\\0&O_{n-2}&0\\1&0&0\end{pmatrix},$$

where O_{n-2} is the zero matrix of size $n-2$. Let A denote the group of matrices $a_t=\exp\{tL\}$, $t\in\mathbb{R}$, and let M be its centralizer in H. The Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\sum_{k=-2}^{2}\mathfrak{g}_k$, where the $\mathfrak{g}_k$ are eigensubspaces of the operator $\operatorname{ad}L$: $\operatorname{ad}L|_{\mathfrak{g}_k}=k\cdot\mathrm{id}$. Denote by $N=\exp\mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent subalgebra $\mathfrak{n}=\mathfrak{g}_1+\mathfrak{g}_2$, generated by the positive roots.

Define for $s\in\mathbb{C}$ and $l=0,1,2,\ldots$ the character $\chi_{s,l}$ of the parabolic subgroup MAN as

$$\chi_{s,l}(ma_tn)=\mathrm{e}^{-\mathrm{i}l\theta}\cdot\mathrm{e}^{st},\quad t\in\mathbb{R},$$

where $e^{-il\theta}$ is one-dimensional representation of the group M given by:

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \longmapsto e^{-il\theta} \tag{1.10}$$

with $v \in \mathrm{U}(p-1, q-1)$, $\theta \in \mathbb{R}$. The representation $\pi_{s,l}$ of the group G induced by the character $\chi_{s,l}$ can be realized in the space $\mathcal{E}_{s,l}(\Xi)$ defined by:

$$\mathcal{E}_{s,l}(\Xi) = \left\{ f \in \mathbb{C}^{\infty}(\Xi) \mid f\left(gma_t\xi^0\right) = e^{(s-\rho)t} \cdot e^{-il\theta} \cdot f\left(g\xi^0\right) \right\},$$
$$\left(\xi^0 = eM_0N\right),$$
$$\left[\pi_{s,l}(g)f\right](\xi) = f\left(g^{-1}\xi\right).$$

Here $\rho = n-1$ and M_0 is the subgroup of M, consisting of the matrices as in (1.10) but with $\theta = 0$. As $\Xi = G/M_0N$ can be identified with the cone $\Gamma = \{y \in \mathbb{C}^n \mid [y, y] = 0,\ y \neq 0\}$, a function of $\mathcal{E}_{s,l}(\Xi)$ can be seen as a function on the cone Γ satisfying:

$$\forall \lambda \in \mathbb{C},\ \lambda \neq 0 \quad f(\gamma\lambda) = \lambda^{s-\rho,-l} \cdot f(\gamma). \tag{1.11}$$

Then, according to the method of Molchanov, see [9,7] we define the χ-spherical distributions by means of these representations:

$$\zeta_{s,l}(\varphi) = \left\langle \pi'_{-s,-l}(\varphi)u_{s,l},\ u_{-s,-l} \right\rangle \quad \left(\varphi \in \mathcal{D}(G)\right).$$

Here $\langle f, h\rangle = \int_B f(b)h(b)\,db$ is the G-invariant linear form on $\mathcal{E}_{s,l}(\Xi) \times \mathcal{E}_{-s,-l}(\Xi)$, where $B = \{k\xi^0 \mid k \in K\}$ is the K-orbit which can be identified with the direct product of two unit spheres $\Sigma = S^{2p-1} \times S^{2q-1}$:

$$b \leftrightarrow \gamma, \quad b \in B,\ \gamma \in \Sigma,$$

and db is the K-invariant measure on B.

$u_{s,l}(f)$ is the following entire function of s:

$$u_{s,l}(f) = \frac{1}{\Gamma(\frac{s-\rho+|l|+2}{2})} \int_{\Sigma} \overline{\gamma_1}^{\,s-\rho,l} f(\gamma)\,d\gamma,$$

where $f \in \mathcal{E}_{-s,-l}(\Xi)$ seen as a function on the cone, see (1.11), γ_1 is the first coordinate of $\gamma \in \Sigma$.

In [13], Section 5, we showed that $\zeta_{s,l}$ is a well-defined χ-spherical distribution on G.

1.5. Poisson kernel, intertwining operators, Fourier transform

We define Poisson kernel, intertwining operators and Fourier transform in the same way as in [3]. So we have "χ-analogues" of the main theorems and propositions from [3]. For example, the χ-spherical distributions can be given by means of the Fourier transform $\widehat{\varphi}$ of a function φ from $\mathcal{D}(X_1)$

$$\zeta_{s,l}(\varphi) = \left\langle \widehat{\varphi}(\cdot, s, l),\ u_{-s,-l} \right\rangle.$$

To diagonalize intertwining operators we use [12,10].

But we remark the following. In his paper Faraut gave detailed proofs for the real case. On the other hand, according to [8], the complex case differs quite seriously from the real case.

Namely, the K-types have a more complicated form. Because of it we can not use the methods of computation used by Faraut. We have to prove some important theorems in an alternative way.

1.6. The behaviour of spherical distributions at infinity

First we consider the asymptotic behaviour of spherical distributions.

PROPOSITION 1.2. – *Let φ be a function in $\mathcal{D}(X_1)$. Put $\varphi_t(x) = \varphi(a_t x)$. We have for* $\operatorname{Re} s > \rho - 1$ *and* $l = 0, 1, 2, \ldots$

$$\lim_{t\to\infty} \mathrm{e}^{-(s-\rho)t} \cdot \zeta_{s,l}(\varphi_t) = c(s,l) \cdot \gamma(s,l) \cdot \widehat{\varphi}\big(\xi^0, s, l\big)$$

with

$$c(s,l) = \frac{2^{\rho-s} \cdot 4 \cdot \pi^{p+q} \cdot \Gamma(s)}{\Gamma(\frac{s+l-q+p+1}{2}) \cdot \Gamma(\frac{s+l+\rho}{2}) \cdot \Gamma(\frac{s-l+q-p+1}{2})}. \tag{1.12}$$

The factor $\gamma(s,l) = \Gamma(\frac{s-\rho+l+2p}{2})/\Gamma(\frac{-s-\rho+l+2p}{2})$ appears in the relation $A_{s,-l} \circ \mathcal{F}_{-s,l} = \gamma(s,l) \cdot \mathcal{F}_{s,l}$, where A and $\mathcal{F}$ denote an intertwining operator and the Fourier transform respectively.

Later on we would like to express the coefficients of the Plancherel formula by means of c-function, see (1.12). We shall need the following formulae:

$$\begin{aligned}
&\frac{1}{c(s,l)c(-s,l)} = -\frac{s \sin \pi s\, \Gamma(\frac{s+l+\rho}{2}) \Gamma(\frac{-s+l+\rho}{2})}{4^{\rho+2} \pi^{2\rho+1} \sin \pi (\frac{-s+l+\rho}{2}) \sin \pi (\frac{s+l+\rho}{2})},\\
&\operatorname{Res}\left(\frac{1}{c(s,l)c(-s,l)};\ \rho+l+2r\right) = \frac{(\rho+l+2r)\Gamma(\rho+l+r)\Gamma(-r)}{4^{\rho+1}\pi^{2\rho+2}},\\
&c_{-2}\left(\frac{1}{c(s,l)c(-s,l)};\ \rho+l+2r\right) = \frac{2(-1)^{r+1}(\rho+l+2r)\Gamma(\rho+l+r)}{4^{\rho+1}\pi^{2\rho+2} r!}.
\end{aligned} \tag{1.13}$$

Here $c_{-2}(f(s);\ s_0)$ denotes the coefficient of the term $(s-s_0)^{-2}$ in the Laurent expansion of f in a neighbourhood of $s = s_0$.

Proposition 1.2 helps us to express spherical distributions by means of basis distributions, see Theorem 1.1.

PROPOSITION 1.3. – *Put $\lambda = s^2 - \rho^2$. Then for all $s \neq \pm(l+n-1+2r)$, $r = 0, 1, 2, \ldots,$ $l = 0, 1, 2, \ldots,$*

$$\zeta_{s,l} = \frac{4 \cdot \pi^{\rho}}{l!} \cdot (-1)^{q+1} \cdot \mathcal{A}' T_{\lambda,l}. \tag{1.14}$$

Moreover, for $\lambda = \lambda_r = (l+2r)(l+2r+2\mu+2)$, we have:

$$\zeta_{s,l} = 0. \tag{1.15}$$

Because of (1.15) for $\lambda = (l+2r)(l+2r+2\mu+2)$ we define spherical distributions $\eta_{r,l}$ and $\theta_{r,l}$, cf. [3], which correspond to the exceptional series of representations of G. Considering their behaviour at infinity we have:

PROPOSITION 1.4. – *For $\lambda = \lambda_r = (l+2r)(l+2r+2\rho)$*

$$\begin{aligned}\eta_{r,l} &= \frac{(-1)^q}{\Gamma(l+1)} 2\pi^{\rho+1} \cdot \mathcal{A}' S_{\lambda_r,l};\\ \theta_{r,l} &= 2\pi^\rho (-1)^q \cdot \left[(-1)^r \frac{r!\Gamma(\rho-1)}{\Gamma(\rho+l+r)} \cdot \mathcal{A}' T'_{\lambda_r,l} - \frac{\pi}{l!} \cdot \mathcal{A}' S_{\lambda_r,l}\right].\end{aligned} \tag{1.16}$$

2. Eigenfunctions of the radial part of the Laplace–Beltrami operator

This section follows §28, [9], for a detailed version see Section 3, [1].

Changing the variable in equation (1.2) as $t = (c+1)/2$ and introducing new notations $\mu = 2\alpha,\ l = 2\beta$ we get the following operator (cf. [9], (28.1)):

$$L_1 = 4\left(c^2-1\right)\frac{\mathrm{d}^2}{\mathrm{d}c^2} + 8\left[(\alpha+\beta+1)c + (\alpha-\beta)\right]\frac{\mathrm{d}}{\mathrm{d}c} + 4\beta(2\alpha+\beta+1). \tag{2.1}$$

We consider the following solutions of equation (1.2) $P(c,\tau,\alpha,\beta),\ Q(c,\tau,\alpha,\beta),\ T(c,\tau,\alpha,\beta)$ functions of the complex variable c (cf. [9], p. 105):

$$\begin{aligned}P(c) &= \frac{2^{-\alpha-\beta}}{\Gamma(1+2\alpha)} {}_2F_1\left(\alpha+\beta+\tau+1, \alpha+\beta-\tau; 1+2\alpha; \frac{1-c}{2}\right),\\ Q(c) &= 2^\tau \frac{\Gamma(\alpha+\beta+\tau+1)\Gamma(\alpha-\beta+\tau+1)}{\Gamma(2\tau+2)} (c+1)^{-\alpha-\beta-\tau-1}\\ &\quad \times {}_2F_1\left(\alpha+\beta+\tau+1, \alpha-\beta+\tau+1; 2\tau+2; \frac{2}{1+c}\right),\\ T(c) &= \frac{2^{\alpha-\beta}}{\Gamma(1-2\alpha)} (c-1)^{-2\alpha} {}_2F_1\left(-\alpha+\beta+\tau+1, -\alpha+\beta-\tau; 1-2\alpha; \frac{1-c}{2}\right),\end{aligned} \tag{2.2}$$

where $\alpha,\ \beta,\ \tau$ are complex parameters, ${}_2F_1$ is the hypergeometric function, Γ is the Euler gamma-function and the power has to be interpreted as the principal value.

They are defined and analytic in the c-plane with the cut $(-\infty, -1]$ for P and $(-\infty, 1]$ for Q and T. On the cuts we define $P(c)$ and $Q(c)$ as half sum of the limit values from above and below, and for $T(c)$ we set (cf. [9], p. 106)

$$T(c) = \lim_{\varepsilon\to 0} \frac{\mathrm{e}^{\mathrm{i}2\pi\alpha} \cdot T(c+i\varepsilon) + \mathrm{e}^{-\mathrm{i}2\pi\alpha} \cdot T(c-i\varepsilon)}{2}.$$

If it is necessary to indicate the parameters, then we write $P(c,\tau,\alpha,\beta), \ldots$.

In addition to these functions we define functions $\widehat{P},\ \widehat{Q},\ \widehat{T},\ P^*,\ Q^*$ and T^*, where, for example:

$$\widehat{P}(c,\tau,\alpha,\beta) = P(-c,\tau,\beta,\alpha), \qquad P^*(c,\tau,\alpha,\beta) = P(c,-\tau-1,\alpha,\beta). \tag{2.3}$$

All these functions satisfy certain relations (see [1], Section 3) on each interval I_j:

$$I_j = \begin{cases}(1,\ \infty), & j=1,\\ (-1,\ 1), & j=2,\\ (-\infty,\ -1), & j=3.\end{cases} \tag{2.4}$$

We shall use some of those relations. We also use the same regularization process for defining functionals associated with these functions. For example:

$$
\begin{aligned}
\left(P^{(j)}, h\right) &= \int_{I_j} P(c)h(c)\,\mathrm{d}c,\\
\left(T^{(j)}, h\right) &= (-1)^{jn}\cdot \int_{I_j} T(c,\tau,0,\beta-\alpha_0)h^{(n-2)}(c)\,\mathrm{d}c.
\end{aligned}
\tag{2.5}
$$

Notice that in our case $\beta \neq 0$. The regularization process for P differs from the one for T. That's why the relations for functionals differ from the ones for functions by terms concentrated at $c = 1$. For example:

$$
\begin{aligned}
&T^{(1)} - W\cdot P^{(1)} = Z;\\
&(-1)^n T^{(2)} - W\cdot P^{(2)} = -Z,
\end{aligned}
\tag{2.6}
$$

where

$$
W = \frac{\Gamma(\alpha+\beta+\tau+1)\cdot\Gamma(\alpha-\beta+\tau+1)}{\Gamma(-\alpha-\beta+\tau+1)\cdot\Gamma(-\alpha+\beta+\tau+1)},
$$

$$
(Z,\,h) = \sum_{m=0}^{n-3} \frac{(-1)^m 2^{1-\alpha_0-\beta+m}\Gamma(\alpha_0+\beta+\tau-m)\Gamma(\alpha_0-\beta+\tau+1)}{\Gamma(-\alpha_0-\beta+\tau+m+2)\Gamma(-\alpha_0+\beta+\tau+1)\Gamma(n-2-m)} h^{(m)}(c)|_{c=1}.
\tag{2.7}
$$

To prove this we use the formulae similar to (3.55), (3.56), [1] but with $\beta \neq 0$ and some formulae for functions $T,\ P$, for example, (3.23), (3.25), [1]. Notice that with $\beta = 0,\ \tau = -\sigma - \alpha_0 - 1,\ \sigma^* = 1 - n - \sigma$ we have the same Z as in (3.50), [1].

3. A spectral decomposition

In this section we follow §30, [9], i.e. we obtain the spectral decomposition of the operator L_1, see (2.1). The main result is the Parseval equality in Theorem 3.2.

The operator L_1 is not symmetric. So we make the change

$$
y = G^{-1}z, \quad \text{where } G(c) = |c-1|^\alpha \cdot |c+1|^\beta.
\tag{3.1}
$$

We obtain the following differential operator (cf. [9], p. 113):

$$
L_2 = \left(c^2-1\right)\frac{\mathrm{d}^2}{\mathrm{d}c^2} + 2c\frac{\mathrm{d}}{\mathrm{d}c} + \frac{2\cdot\alpha^2}{1-c} + \frac{2\cdot\beta^2}{1+c}.
$$

The operator L_2 is symmetric, provided α^2 and β^2 are real. With the notation $\tau = (s-1)/2$ the equation $Ly = \lambda y$ gives us the following differential equation:

$$
L_2 z = \tau(\tau+1)z.
$$

The eigenfunctions f and F of the operators L_2 and L respectively are connected by the equality $f = GF$, where G is as in (3.1).

We are going to define an extension of the operator L_2 and obtain its spectral decomposition. Suppose:

$$
|\mathrm{Re}\,\alpha| < 1/2, \quad \alpha \neq 0, \qquad |\mathrm{Re}\,\beta| < 1/2.
$$

Then the eigenfunctions of the operator L_2 are square-integrable on each bounded interval. For every function $\varphi \in L^2(-1,\infty)$ which is absolutely continuous on each compact set not containing ± 1 and for which $L_2\varphi \in L^2(-1,\infty)$ we can define the following boundary values at points $c=\pm 1$:

$$
\begin{aligned}
A_j^+(\varphi) &= \lim_{c\to 1,\, c\in I_j,\ j=1,2} |c-1|^{-\alpha}\bigl\{\alpha\varphi(c)+(c-1)\varphi'(c)\bigr\},\\
B_j^+(\varphi) &= \lim_{c\to 1,\, c\in I_j,\ j=1,2} |c-1|^{\alpha}\bigl\{\alpha\varphi(c)-(c-1)\varphi'(c)\bigr\},\\
A_j^-(\varphi) &= \lim_{c\to -1,\, c\in I_j,\ j=2,3} |c+1|^{-\beta}\bigl\{\beta\varphi(c)+(c+1)\varphi'(c)\bigr\},\\
B_j^-(\varphi) &= \lim_{c\to -1,\, c\in I_j,\ j=2,3} |c+1|^{\beta}\bigl\{\beta\varphi(c)-(c+1)\varphi'(c)\bigr\},
\end{aligned}
\tag{3.2}
$$

where I_j as in (2.4).

Then we set the following boundary conditions at points $c=\pm 1$ (cf. [9], p. 114):

$$
A_1^+ = A_2^+ = B_2^- = 0. \tag{3.3}
$$

And at infinity $\varphi \in L^2$, we denote this as follows:

$$
\infty(\varphi) = 0. \tag{3.4}
$$

These boundary conditions define an operator on $L^2(-1,\infty)$ which we denote by $\widetilde{L}$. Now we are going to write explicitly the resolvent $R_\omega = (\omega I - \widetilde{L})^{-1}$ of this operator by means of hypergeometric functions, see (2.2).

LEMMA 3.1. – *Let* $\operatorname{Re}\tau > -1/2,\ \operatorname{Im}\tau \neq 0$. *The resolvent of the operator* $\widetilde{L}$ *is an integral operator with a symmetric kernel given for* $c>y$ *by the following formula*:

$$
R_{\tau(\tau+1)}(c,y) = G(c)\cdot G(y)\cdot \begin{cases} W^{-1}\cdot Q(c)\cdot T(y), & \textit{if } 1<y,\ c<\infty,\\ W_1\cdot T(c)\cdot \widehat{P}(y), & \textit{if } -1<y,\ c<1 \end{cases} \tag{3.5}
$$

with $G(c)$ *as in* (3.1), W *as in* (2.7) *and*

$$
W_1 = -\frac{1}{2}\cdot\Gamma(-\alpha+\beta-\tau)\cdot\Gamma(-\alpha+\beta+\tau+1). \tag{3.6}
$$

Proof. – We take $\operatorname{Re}\tau > -1/2,\ \operatorname{Im}\tau\neq 0$, this is the same as $\operatorname{Im}\tau(\tau+1)\neq 0$. The method of variation of constants gives us the symmetric kernel $R_{\tau(\tau+1)}(c,y) = R_{\tau(\tau+1)}(y,c)$ and for $c>y$:

$$
R_{\tau(\tau+1)}(c,y) = \begin{cases} \frac{v_1(c)u_1(y)}{[u_1,v_1]}, & \text{if } 1<y,\ c<\infty,\\ \frac{v(c)u(y)}{[u,v]}, & \text{if } -1<y,\ c<1. \end{cases} \tag{3.7}
$$

Here u,v and u_1,v_1 are linearly independent solutions to $\widetilde{L}g = \tau(\tau+1)\,g$ on the intervals $(-1,1)$ and $(1,\infty)$ respectively. Moreover, they are chosen to satisfy our boundary conditions

(3.3), (3.4) in the following way:

$$
\begin{aligned}
&B_2^-(u)=0, \quad A_2^+(v)=0, \quad A_1^+(u_1)=0, \quad \infty(v_1)=0,\\
&A_2^+(u)\neq 0, \quad B_2^-(v)\neq 0, \quad \infty(u_1)\neq 0, \quad A_1^+(v_1)\neq 0.
\end{aligned}
$$

And $[u,v]$ denotes $[u,v]=(c^2-1)(u'(c)v(c)-u(c)v'(c))$.

Now we are going to choose suitable candidates for $u,\ v,\ u_1,\ v_1$. Using formulae for the analytic continuation of the hypergeometric series from [2], 2.10, pp. 108–109, one can write down the explicit formulae for $GP,\ GT,\ G\widehat{P}$ and $G\widehat{T}$ near points $c=\pm 1$ (here G is as in (3.1)) by means of hypergeometric series. Then one can evaluate the boundary values $A_{1,2}^+$ and B_2^- of the functions $GP,\ GT,\ G\widehat{P}$ and $G\widehat{T}$ according to formulae (3.2):

	B_2^-	A_2^+	A_1^+
GP	$\beta 2^{\alpha+1} d_{\alpha,-\beta}$	$\frac{2^{-\alpha}}{\Gamma(2\alpha)}$	$\frac{2^{-\alpha}}{\Gamma(2\alpha)}$
$G\widehat{P}$	0	$\alpha\, 2^{\beta+1} d_{\beta,\alpha}$	$\alpha 2^{\beta+1} d_{\beta,\alpha}$
GT	$\beta 2^{-\alpha+1} d_{-\alpha,-\beta}$	0	0
$G\widehat{T}$	$-\frac{2^{\beta}}{\Gamma(-2\beta)}$	$\alpha\, 2^{-\beta+1} d_{-\beta,\alpha}$	$\alpha 2^{-\beta+1} d_{-\beta,\alpha}$

Here $d_{x,y}=\frac{2^{-(x+y)/2}\cdot\Gamma(-y)}{\Gamma(\frac{x-y}{2}-\tau)\cdot\Gamma(\frac{x-y}{2}+\tau+1)}$. Notice that for evaluating A^+ one has to use that $\alpha<1/2$.

According to the table we should take $u=G\widehat{P}$, $v=GT$. Then $[u,v]=1/W_1$, see (3.6). And we can choose $u_1=GT,\ v_1=GQ$ because at least $A_1^+(GT)=0$ and one can check, using formula 1.5(2), p. 9 from [2], that $\infty(GQ)=0$, see (3.4). To show that $A_1^+(GQ)\neq 0$ we use the following formula (cf. [9], p. 105), which is true for $c>1$:

$$\frac{2}{\pi}\sin 2\alpha\pi\cdot Q(c)=T(c)-W\cdot P(c), \tag{3.8}$$

where W is as in (2.7). To show that $\infty(GT)\neq 0$ we use the analytic continuation of T to $c=+\infty$, see formula 2.10(3) from [2], p. 109.

Calculating $[u_1,v_1]$ we have $[u_1,v_1]=W$. Substituting u,v,u_1,v_1 into (3.7) we get (3.5). □

Now we are going to determine the spectral decomposition of the operator $\widetilde{L}$, i.e. we write the Parseval equality for $\varphi,\ \psi\in L^2(-1,\infty)$.

THEOREM 3.2. – *For* $|\mathrm{Re}\,\alpha|<1/2,\ \alpha\neq 0,\ |\mathrm{Re}\,\beta|<1/2,\ \mathrm{Re}(\alpha\pm\beta)<1/2$ *we have the following Parseval equality*:

$$
\begin{aligned}
(\varphi\mid\psi) = {}&\int_{-\infty}^{+\infty}\Omega\cdot\bigl(GT^{(1)},\varphi\bigr)\cdot\bigl(GT^{(1)},\bar{\psi}\bigr)\big|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu\\
&+\sum_{k=0}^{\infty}\Omega_1\cdot\bigl(GT^{(2)},\varphi\bigr)\cdot\bigl(GT^{(2)},\bar{\psi}\bigr)\big|_{\tau=-\alpha+\beta+k}
\end{aligned}
\tag{3.9}
$$

with

$$\begin{aligned}\Omega &= \frac{2\tau+1}{8\pi^2}\sin 2\tau\pi \\ &\quad\times \Gamma(-\alpha-\beta+\tau+1)\Gamma(-\alpha+\beta+\tau+1)\Gamma(-\alpha-\beta-\tau)\Gamma(-\alpha+\beta-\tau), \\ \Omega_1 &= \frac{2\tau+1}{2}\,\frac{\Gamma(-\alpha-\beta+\tau+1)\cdot\Gamma(-\alpha+\beta+\tau+1)}{\Gamma(\alpha+\beta+\tau+1)\cdot\Gamma(\alpha-\beta+\tau+1)}.\end{aligned} \tag{3.10}$$

Proof. – We use the following formula:

$$\int_{\mathbb{R}} h(\omega)\big(E(\mathrm{d}\omega)\varphi \mid \psi\big) = -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} h(\omega)\big((R_{\omega+\mathrm{i}\varepsilon}-R_{\omega-\mathrm{i}\varepsilon})\varphi \mid \psi\big)\,\mathrm{d}\omega \tag{3.11}$$

for any continuous h with compact support.

Passing from ω to τ by $\omega=\tau^2+\tau$ we change the notation $R_\omega=\widetilde{R}_\tau$. Split the τ-contour into two parts: the “imaginary” part and the “real” part, see Fig. 1.

The “Imaginary” and the “real” parts will give us the continuous and the discrete spectrum of the operator $\widetilde{L}$ respectively.

Taking $h\equiv 1$ in (3.11) we write the Parseval equality:

$$(\varphi \mid \psi) = \int_{-\infty}^{+\infty} (K_\tau\varphi \mid \psi)|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu + \sum\big(\mathrm{Res}\big[K_\tau^1\big]\varphi \mid \psi\big), \tag{3.12}$$

where

$$K_\tau = \frac{1}{4\pi}(2\tau+1)(\widetilde{R}_\tau-\widetilde{R}_{-\tau-1}), \qquad K_\tau^1=(2\tau+1)\widetilde{R}_\tau \tag{3.13}$$

and the sum is taken over all points inside the P-contour at which the function K_τ^1 has a pole.

Now rewrite equality (3.12) in a more explicit form. First consider the continuous spectrum. Using formula (3.5) for $R_\omega=\widetilde{R}_\tau$ we have for $c>y$:

$$K_\tau(c,y) = G(c)\cdot G(y)\cdot\begin{cases}\Omega T(c)T(y), & \text{if } 1<y,\ c<\infty,\\ 0, & \text{if } -1<y,\ c<1\end{cases}$$

with Ω as in (3.10). Here we used formula (3.8) for Q and the fact that u, T, P, $\widehat{P}$ are invariant under the transformation $\tau\to-\tau-1$, see (2.3), i.e., $u^*=u$, $T^*=T$, $P^*=P$, $\widehat{P}^*=\widehat{P}$.

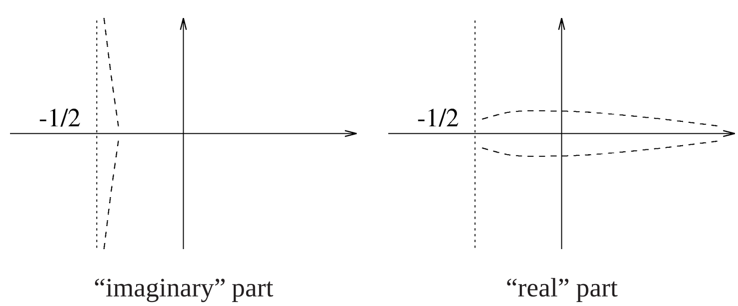

Fig. 1.

$K_\tau(c,y)$ is symmetric and one can easily write

$$(K_\tau\varphi \mid \psi) = \Omega \cdot \left(GT^{(1)}, \varphi\right) \cdot \left(GT^{(1)}, \overline{\psi}\right),$$

where $T^{(j)}$ as in (2.5). We also recall our notations:

$$(\varphi,\psi) = \int_{-1}^{\infty} \varphi(c)\psi(c)\,\mathrm{d}c, \qquad (\varphi \mid \psi) = \int_{-1}^{\infty} \varphi(c)\bar{\psi}(c)\,\mathrm{d}c.$$

So the continuous part of the spectral decomposition of our operator $\widetilde{L}$ can be written as follows:

$$\int_{-\infty}^{+\infty} (K_\tau\varphi \mid \psi)|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu = \int_{-\infty}^{+\infty} \Omega \cdot \left(GT^{(1)}, \varphi\right) \cdot \left(GT^{(1)}, \bar{\psi}\right)\Big|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu. \tag{3.14}$$

Now we consider the discrete part of the spectral decomposition. First of all using formula (3.5) we write for $c > y$:

$$K_\tau^1(c,y) = G(c)\cdot G(y)\cdot(2\tau+1)\cdot \begin{cases} W^{-1}Q(c)T(y), & \text{if } 1 < y,\ c < \infty,\\ W_1 T(c)\widehat{P}(y), & \text{if } -1 < y,\ c < 1. \end{cases}$$

We see that K_τ^1 has the series of poles because of W_1 and W. But if in addition to $|\mathrm{Re}\,\alpha| < 1/2$, $|\mathrm{Re}\,\beta| < 1/2$ we suppose that $\mathrm{Re}(\alpha \pm \beta) < 1/2$ then only the poles $\tau = -\alpha + \beta + k$, $k = 0, 1, 2, \ldots$ (given by W_1), lie in the half plane $\mathrm{Re}\,\tau \geqslant -1/2$.

If we denote $K'_\tau = \mathrm{Res}(K_\tau^1;\ \tau = -\alpha+\beta+k)$ then for $c > y$:

$$K'_\tau = \begin{cases} 0, & \text{if } 1 < y,\ c < \infty,\\ \frac{2\tau+1}{2W} G(c)T(c)G(y)T(y)|_{\tau=-\alpha+\beta+k}, & \text{if } -1 < y,\ c < 1. \end{cases}$$

Here we used formula (3.29), [1] for $\widehat{P}$. Now it is clear that K'_τ is symmetric and in a similar way as above one can write:

$$\left(K'_\tau\varphi \mid \psi\right) = \frac{2\tau+1}{2W} \cdot \left(GT^{(2)}, \varphi\right) \cdot \left(GT^{(2)}, \bar{\psi}\right)\Big|_{\tau=-\alpha+\beta+k}.$$

Thus the discrete part can be written as follows:

$$\sum \left(\mathrm{Res}\left[K_\tau^1\right]\varphi \mid \psi\right) = \sum_{k=0}^{\infty} \frac{2\tau+1}{2W} \cdot \left(GT^{(2)}, \varphi\right) \cdot \left(GT^{(2)}, \bar{\psi}\right)\Big|_{\tau=-\alpha+\beta+k}. \tag{3.15}$$

Finally, substituting (3.14) and (3.15) into (3.12) we have (3.9). □

4. The Plancherel formula

The main formula of this section is the decomposition of the function δ_l into spherical distributions (Theorem 4.4) which implies the Plancherel formula.

Let δ be the following distribution in $\mathcal{D}'(G/H)$ (the delta function concentrated at the origin x^0):

$$(\delta \mid f) = \overline{f(x^0)} \quad \big(f \in \mathcal{D}(G/H)\big).$$

Let us consider the following functions on the real line depending on the complex variable σ:

$$N_{\sigma,n}(c) = \begin{cases} |c-1|^\sigma, & \text{for } n \text{ odd},\\ |c-1|^\sigma \cdot \operatorname{sgn}(c-1), & \text{for } n \text{ even}. \end{cases}$$

By [4] these functionals are analytic functions of σ within the domain $\operatorname{Re}\sigma > -1$ and can be analytically extended to the whole σ-plane as meromorphic functions. At $\sigma = 1-n$ they have a pole of the first order. The same is true for the distributions:

$$\mathcal{M}' N_{\sigma,n} = \begin{cases} \big|c(x)-1\big|^\sigma, & \text{for } n \text{ odd},\\ \big|c(x)-1\big|^\sigma \cdot \operatorname{sgn}\big(c(x)-1\big), & \text{for } n \text{ even}, \end{cases}$$

where, according to our averaging map $c(x) = 2|x_1|^2 - 1$ (cf. Section 1 and the notation $t = (c+1)/2$). Here x_1 is the first coordinate of $x \in X_1$. Moreover, when $\sigma \to 1-n$ we have:

$$(\sigma + n - 1)\mathcal{M}' N_{\sigma,n} \to E \cdot \delta, \tag{4.1}$$

where

$$E = \begin{cases} (-1)^q \frac{2^{2-n}\pi^{n-1}}{\Gamma(n-1)}, & \text{for } n \text{ odd},\\ (-1)^{q+1} \frac{2^{2-n}\pi^{n-1}}{\Gamma(n-1)}, & \text{for } n \text{ even}. \end{cases} \tag{4.2}$$

To evaluate the coefficients before δ-function we wrote:

$$\mathcal{M}' N_{\sigma,n} = \begin{cases} 2^\sigma \cdot |S|^\sigma = 2^\sigma \cdot \big[S_+^\sigma + S_-^\sigma\big], & \text{for } n \text{ odd},\\ 2^\sigma \cdot |S|^\sigma \operatorname{sgn} S = 2^\sigma \cdot \big[S_+^\sigma - S_-^\sigma\big], & \text{for } n \text{ even}, \end{cases}$$

with the real quadratic form S of the signature $(\mathbf{p}, \mathbf{q}) = (2q, 2p-2)$

$$S = -x_2\bar{x}_2 - \cdots - x_p\bar{x}_p + x_{p+1}\bar{x}_{p+1} + \cdots + x_{p+q}\bar{x}_{p+q}$$

since our space $X_1 = G/H_1$ can be identified with the set $[x,x] = 1$, see (1.1) and used the result from [4], p. 257:

$$\operatorname*{Res}_{\lambda = -\mathbf{n}/2-k} P_+^\lambda = \frac{(-1)^{\mathbf{n}/2+k-1}}{\Gamma(\mathbf{n}/2+k)} \delta_1^{(\mathbf{n}/2+k-1)}(P) + \frac{(-1)^{\mathbf{q}/2}\pi^{\mathbf{n}/2}}{2^{2k}k!\,\Gamma(\mathbf{n}/2+k)} L^k \delta(y)$$

with

$$L = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{\mathbf{p}}^2} - \frac{\partial^2}{\partial y_{\mathbf{p}+1}^2} - \cdots - \frac{\partial^2}{\partial y_{\mathbf{p}+\mathbf{q}}^2}$$

and the fact that $\delta_1^{(k)}(P) = \delta_2^{(k)}(P)$ for even $\mathbf{n}$ and $k < \mathbf{n}/2 - 1$, see [4], p. 267.

Let D_0 be the following domain: $D_0 = \{|\operatorname{Re}\alpha| < 1/2,\ \alpha \neq 0,\ |\operatorname{Re}\beta| < 1/2,\ \operatorname{Re}(\alpha \pm \beta) < 1/2\}$. According to (3.9) within D_0 we have the Parseval equality. We are going to write that equality for the functions $N_{\sigma,n}$.

LEMMA 4.1. – *In the domain*

$$D_0 \cap \{\mathrm{Re}(\sigma+\alpha+\beta) < -1/2\} \cap \{\mathrm{Re}\,\sigma > -1\}$$

we can write for $h \in \mathcal{H}$

$$(N_{\sigma,n} \mid h) = \int_{-\infty}^{\infty} \Psi\big(T^{(1)}, \bar{h}\big)\big|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu + \sum_{k=0}^{\infty} \Psi_1\big(T^{(2)}, \bar{h}\big)\big|_{\tau=-\alpha+\beta+k}, \tag{4.3}$$

where

$$\begin{aligned} \Psi &= \frac{2^{\alpha+\beta+\sigma-2}}{\pi^2}(2\tau+1)\sin 2\tau\pi \frac{\Gamma(\sigma+1)}{\Gamma(-2\alpha-\sigma)}\Gamma(-\alpha-\beta-\sigma+\tau) \\ &\quad \times \Gamma(-\alpha-\beta-\sigma-\tau-1)\Gamma(-\alpha+\beta+\tau+1)\Gamma(-\alpha+\beta-\tau), \\ \Psi_1 &= (-1)^{n+1}2^{\alpha+\beta+\sigma}(-2\alpha+2\beta+2k+1) \\ &\quad \times \frac{\Gamma(-2\alpha+2\beta+k+1)\Gamma(\sigma+1)\Gamma(-2\alpha-\sigma+k)}{k!\,\Gamma(-2\alpha-\sigma)\Gamma(2\beta+\sigma+2+k)}. \end{aligned} \tag{4.4}$$

Proof. – We follow the scheme in the proof of Lemma 7.3, [1]. First consider the case when n is odd. Then $N_{\sigma,n} = |c-1|^\sigma$. Since $N_{\sigma,n}(c) \notin L^2(\mathbb{R}, \mathrm{d}t)$ for any σ, we represent it as a sum:

$$N_{\sigma,n}(c) = N^+_{\sigma,n}(c) + N^-_{\sigma,n}(c), \quad N^\pm_{\sigma,n}(c) = |c-1|^\sigma \cdot Y\big(\pm(2-|c|)\big),$$

where Y is the Heaviside function.

One can see that the function $GN^+_{\sigma,n} = |c-1|^{\sigma+\alpha} \cdot |c+1|^\beta \cdot Y(2-|c|)$ belongs to $L^2((-1,\infty))$ if $\mathrm{Re}(\sigma+\alpha) > -1/2$ and $\mathrm{Re}\,\beta > -1/2$. And $GN^-_{\sigma,n} \in L^2((-1,\infty))$ if $\mathrm{Re}(\sigma+\alpha+\beta) < -1/2$. Here G is as in (3.1).

We are going to write two Parseval equalities: first for the pair $\varphi = GN^+_{\sigma,n},\ \psi = \overline{G}^{-1}h$ and then for the pair $\varphi = GN^-_{\sigma,n},\ \psi = \overline{G}^{-1}h$, where h belongs to the image of our averaging map, see (1.5) (taking in account the substitution $t = (c+1)/2$).

The first equality can be written for the domain $D_1 = D_0 \cap \{\mathrm{Re}(\sigma+\alpha) > -1/2\}$. The second equality can be written for the domain $D_2 = D_0 \cap \{\mathrm{Re}(\sigma+\alpha+\beta) < -1/2\}$. We can see that $D_1 \cap D_2 \neq \emptyset$ at least for negative β.

We notice that the first equality can be written within the wider domain $D_3 = D_0 \cap \{\mathrm{Re}\,\sigma > -1\}$. Indeed, the left-hand side exists for $\mathrm{Re}\,\sigma > -1$, see [4]. According to 7.512(5), p. 863, [5] the conditions $\mathrm{Re}\,\sigma > -1$ and $\mathrm{Re}\,\beta > -1/2$ give the existence of the integral $(GT^{(2)}, GN^+_{\sigma,n})$ in the right-hand side. Also the condition $\mathrm{Re}\,\sigma > -1$ gives the existence of the integral $(GT^{(1)}, GN^+_{\sigma,n})$ in the right-hand side. Thus, $D_2 \cap D_3 \neq \emptyset$ even for some positive β.

Within $D_2 \cap D_3$ both equalities are true. Taking the sum we get the Parseval equality for $\varphi = GN_{\sigma,n}$, $\psi = \overline{G}^{-1}h$.

$$\big(GN_{\sigma,n} \mid \overline{G}^{-1}h\big) = \int_{-\infty}^{\infty} \Omega B\big(T^{(1)}, \bar{h}\big)\big|_{\tau=-1/2+i\nu}\,\mathrm{d}\nu + \sum_{k=0}^{\infty} \Omega_1 B_1\big(T^{(2)}, \bar{h}\big)\big|_{\tau=-\alpha+\beta+k}, \tag{4.5}$$

where

$$B = \left(GT^{(1)}, GN^{+}_{\sigma,n}\right) + \left(GT^{(1)}, GN^{-}_{\sigma,n}\right) = \int_1^{\infty} (c-1)^{\sigma+2\alpha}(c+1)^{2\beta} T(c)\,\mathrm{d}c,$$

$$B_1 = \left(GT^{(2)}, GN^{+}_{\sigma,n}\right) + \left(GT^{(2)}, GN^{-}_{\sigma,n}\right) = \int_{-1}^{1} (1-c)^{\sigma+2\alpha}(c+1)^{2\beta} T(c)\,\mathrm{d}c.$$

It remains to compute B and B_1. First of all rewrite $T(c)$, using formula 2.9(1,3) from [2], p. 105. Then by means of 7.512(3), p. 863, [5]:

$$B = 2^{\alpha+\beta+\sigma+1} \frac{\Gamma(\sigma+1)\Gamma(-\alpha-\beta-\sigma+\tau)\Gamma(-\alpha-\beta-\sigma-\tau-1)}{\Gamma(-\alpha-\beta+\tau+1)\Gamma(-\alpha-\beta-\tau)\Gamma(-2\alpha-\sigma)}.$$

Here we make the change of variable $t = (c-1)/(c+1)$ and recall that $\tau = -\frac{1}{2} + \mathrm{i}\nu$.

To calculate B_1 we use 7.512(5), p. 863, [5]:

$$\begin{aligned} B_1 = 2^{\alpha+\beta+\sigma+1} &\frac{\Gamma(\sigma+1)\Gamma(1+2\beta)}{\Gamma(\sigma+2\beta+2)\Gamma(1-2\alpha)} \\ &\times {}_3F_2(-\alpha+\beta+\tau+1, -\alpha+\beta-\tau, \sigma+1; 1-2\alpha, \sigma+2\beta+2; 1). \end{aligned}$$

Considering the case n is even, i.e. $N_{\sigma,n}(c) = |c-1|^{\sigma} \cdot \operatorname{sgn}(c-1)$, we see that the only difference is that in (4.5) B_1 appears with a minus sign. Thus for any n we multiply the above expression for B_1 by $(-1)^{n+1}$.

Finally, putting B and B_1 into (4.5) we have (4.3). To get Ψ and Ψ_1 we just used the expressions for Ω, Ω_1, see (3.10). Moreover, substituting $\tau = -\alpha+\beta+k$ in the hypergeometric function ${}_3F_2$ in B_1 we are able to use formula 4.4(3), p. 188, [2]. □

Both sides of (4.3) can be analytically continued in $\alpha,\ \beta,\ \sigma$, to the point $\alpha = \alpha_0,\ \beta = \beta_0,\ \sigma = \sigma_0$, where

$$\alpha_0 = \frac{n-2}{2}, \qquad \beta_0 = \frac{l}{2}, \qquad \sigma_0 = 1-n.$$

The following lemma is obtained from Lemma 4.1 by this analytic continuation. Moreover, to pass from the decomposition of $N_{\sigma,n}$ to the decomposition of δ we take the pull back $\mathcal{M}'$ of both sides of the equality. Finally, we pass to χ-spherical distributions multiplying both parts by x_1^l. We introduce the following notation:

$$(\delta_l \mid h) = \left(x_1^l \cdot \delta \mid h\right).$$

LEMMA 4.2. – *Let $h \in \mathcal{D}(X_1)$. We have the following decomposition of the function δ_l:*

$$E \cdot (\delta_l \mid h) = \begin{cases} \mathrm{I} + \mathrm{II} + \sum_{i=0}^{\frac{l-n+1}{2}} A_2 \cdot \left(\mathcal{A}'T^{(1)}, \bar{h}\right)\big|_{\tau=(l-n)/2-i}, & \textit{for } l \geqslant n-1, \\ \mathrm{I} + \mathrm{II}, & \textit{for } l = n-2, \\ \mathrm{I} + \mathrm{II} + \sum_{j=0}^{\frac{n-3-l}{2}} A_3 \cdot \left(\mathcal{A}'T^{(1)}, \bar{h}\right)\big|_{\tau=(n-l)/2-2-j}, & \textit{for } l \leqslant n-3 \end{cases} \tag{4.6}$$

with E as in (4.2)*, A_j as in* (4.17) *and the following notations*:

$$\mathrm{I} = \int_{-\infty}^{\infty} A \cdot \left(\mathcal{A}'T^{(1)}, \bar{h}\right)\big|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu,$$

$$\mathrm{II} = \begin{cases} \sum_{k=0}^{\infty} A_1 \cdot \left(\mathcal{A}' T^{(2)}, \bar{h}\right)\big|_{\tau=(l-n)/2+k+1}, & l > n-3, \\ \sum_{k=n-l-2}^{\infty} A_1 \cdot \left(\mathcal{A}' T^{(2)}, \bar{h}\right)\big|_{\tau=(l-n)/2+k+1}, & l \leqslant n-3. \end{cases}$$

Proof. – The scheme of the proof follows the proof of Lemma 7.4, [1]. Since in our case $\beta \neq 0$, we have to make an analytic continuation twice for big and for small l. When we do an analytic continuation some additional terms can appear on the right-hand side of (4.3), since poles of the integrand in (4.3) pass the integration line C: $\operatorname{Re}\tau = -1/2$. The integrand has two series of poles:

$$\begin{aligned} &(1) \quad \tau = \alpha+\beta+\sigma-i, \qquad \tau = -\alpha-\beta-\sigma-1+i,\ i = 0,1,2,\ldots \\ &\qquad\ \text{coming from the factor } \Gamma(-\alpha-\beta-\sigma+\tau)\Gamma(-\alpha-\beta-\sigma-\tau-1); \\ &(2) \quad \tau = \alpha-\beta-1-j, \qquad \tau = -\alpha+\beta+j,\ j = 0,1,2,\ldots \\ &\qquad\ \text{coming from the factor } \Gamma(-\alpha+\beta+\tau+1)\Gamma(-\alpha+\beta-\tau). \end{aligned} \tag{4.7}$$

In the domain $D_2 \cap D_3$ we have for any $i, j = 0,1,2,\ldots$

Fig. 2.

First let us move α, β and σ in such a way that the inequality $\operatorname{Re}(\alpha-\beta) < 1/2$ is maintained. Then poles in (2) do not pass the integration line C. On the contrary, some poles in (1) pass C, namely, those with $i \leqslant (l-n+1)/2$. Since $\Psi T^{(1)}$ is invariant with respect to $\tau \to -\tau-1$, every pair of them with the same i passing C gives by Cauchy theorem the following additional term in the right-hand side of (4.3):

$$D_i = 4\pi \cdot \operatorname{Res}_{\tau=\alpha+\beta+\sigma-i} \Psi T^{(1)} = \Psi_2 T^{(1)}\big|_{\tau=\alpha+\beta+\sigma-i},$$

with

$$\Psi_2 = 2^{\alpha+\beta+\sigma}(2\alpha+2\beta+2\sigma-2i+1)\frac{\Gamma(\sigma+1)\Gamma(2\beta+\sigma-i+1)\Gamma(-2\alpha-\sigma+i)}{\Gamma(2+2\alpha+2\beta+2\sigma-i)\Gamma(-2\alpha-\sigma)i!}. \tag{4.8}$$

Thus, after the continuation in α, β and σ, when (α,β,σ) belong to a sufficiently small neighbourhood of $(\frac{n-2}{2}, \frac{l}{2}, 1-n)$, and the inequality $\operatorname{Re}(\alpha-\beta) < 1/2$ is satisfied, in the decomposition (4.3) the following additional sum appears:

$$\sum_{i=0}^{\frac{l-n+1}{2}} \Psi_2 \cdot \left(T^{(1)}, \bar{h}\right)\big|_{\tau=\alpha+\beta+\sigma-i}. \tag{4.9}$$

Notice that after the continuation the inequality $\operatorname{Re}(\alpha-\beta)<1/2$ becomes $l>n-3$, but from (4.9) we see that the additional sum appears for $l\geqslant n-1$. Thus, in a neighbourhood of point $(\alpha_0,\beta_0,\sigma_0)$ we can write the decomposition (4.3) in the same form for $l=n-2$ and with additional term (4.9) for $l\geqslant n-1$.

Secondly, we can move $\alpha,\ \beta$ and σ from $D_2\cap D_3$ to the point $(\alpha_0,\beta_0,\sigma_0)$ in such a way that the inequality $\operatorname{Re}(\alpha+\beta+\sigma)<-1/2$ is maintained. Then poles in (1), see (4.7), do not pass the integration line C, but poles in (2) do, namely, those with $j\leqslant(n-3-l)/2$. In the right-hand side of decomposition (4.3) the following additional sum appears:

$$\sum_{j=0}^{\frac{n-3-l}{2}}\Psi_3\cdot\bigl(T^{(1)},\bar h\bigr)\big|_{\tau=\alpha-\beta-1-j},\tag{4.10}$$

where

$$\Psi_3=2^{\alpha+\beta+\sigma}\cdot(2\alpha-2\beta-2j-1)\frac{\Gamma(\sigma+1)\Gamma(-2\beta-\sigma-j-1)\Gamma(-2\alpha-\sigma+j)}{\Gamma(-2\alpha-\sigma)\Gamma(2\alpha-2\beta-j)j!}.\tag{4.11}$$

After the continuation the inequality $\operatorname{Re}(\alpha+\beta+\sigma)<-1/2$ becomes $l<n-1$, but the additional term (4.10) appears for $l\leqslant n-3$.

Thus, in a sufficiently small neighbourhood of $(\alpha_0,\beta_0,\sigma_0)$ we have the decomposition of the form:

$$(N_{\sigma,n}\mid h)=\begin{cases}\cdots+\sum_{i=0}^{\frac{l-n+1}{2}}\Psi_2\cdot\bigl(T^{(1)},\bar h\bigr)\big|_{\tau=\alpha+\beta+\sigma-i}, & \text{for } l\geqslant n-1;\\ \cdots, & \text{for } l=n-2;\\ \cdots+\sum_{j=0}^{\frac{n-3-l}{2}}\Psi_3\cdot\bigl(T^{(1)},\bar h\bigr)\big|_{\tau=\alpha-\beta-1-j}, & \text{for } l\leqslant n-3,\end{cases}\tag{4.12}$$

where … denotes the right-hand side of (4.3).

As the right hand sides of (4.12) are regular at $\alpha=\alpha_0$, see (4.4), (4.8), (4.11), we can easily set $\alpha=\alpha_0$. After that we see that $\Psi,\ \Psi_2,\ \Psi_3$ are regular at $\beta=\beta_0$, so just setting $\beta=\beta_0$ we have:

$$\begin{aligned}\Psi&=\frac{2^{\frac{n+l}{2}+\sigma-3}}{\pi^2}(2\tau+1)\sin 2\tau\pi\frac{\Gamma(\sigma+1)}{\Gamma(-n+2-\sigma)}\Gamma\left(1-\frac{n+l}{2}-\sigma+\tau\right)\\&\quad\times\Gamma\left(1-\frac{n+l}{2}-\sigma-\tau-1\right)\Gamma\left(\frac{l-n}{2}+\tau+2\right)\Gamma\left(\frac{l-n}{2}-\tau+1\right),\\ \Psi_2&=2^{\frac{n+l}{2}+\sigma-1}(n+l+2\sigma-2i-1)\frac{\Gamma(\sigma+1)\Gamma(l+\sigma-i+1)\Gamma(-n-\sigma+i+2)}{\Gamma(n+l+2\sigma-i)\Gamma(-n-\sigma+2)i!},\\ \Psi_3&=2^{\frac{n+l}{2}+\sigma-1}\cdot(n-l-2j-3)\frac{\Gamma(\sigma+1)\Gamma(-l-\sigma-j-1)\Gamma(-n-\sigma+j+2)}{\Gamma(-n-\sigma+2)\Gamma(n-l-2-j)j!}.\end{aligned}\tag{4.13}$$

For Ψ_1 the matter is more complicated. For $l\leqslant n-3$ the first $n-3-l$ terms $\Psi_1T^{(2)}$ of the series in (4.12) have at $\beta=\beta_0$ poles (of the first order). Fortunately, the poles of the k-th term and of the $(n-3-l-k)$-th term are annihilated. (This situation differs from the one in [1], where the poles of the discrete series were annihilated by the poles of the additional sum.) So we have to compute the following limit:

$$H_k=\lim_{\beta\to\beta_0}\bigl\{\Psi_1T^{(2)}\big|_{k=k}+\Psi_1T^{(2)}\big|_{k=n-3-l-k}\bigr\}.$$

After long manupulations with gamma-functions and their derivatives, using formulae 1.7.1(10) and 1.7.1(11), p. 16, [2] and the fact that in the limit $T^{(2)}|_{k=k} = T^{(2)}|_{k=n-3-l-k}$ we have:

$$\begin{aligned} H_k \;=\; & (-1)^{n+1} 2^{\frac{n+l}{2}+\sigma} \frac{\Gamma(\sigma+1)\Gamma(-n-\sigma+k+2)}{k!\Gamma(-n+2-\sigma)\Gamma(l+\sigma+2+k)\Gamma(n-l-k-2)} \\ & \times \biggl\{ 2-(l-n+2k+3)\cdot\biggl[\biggl(\frac{1}{-\sigma-n+k+2}-\frac{1}{k+1}\biggr) \\ & \quad + \biggl(\frac{1}{-\sigma-n+k+3}-\frac{1}{k+2}\biggr)+\cdots \\ & \quad + \biggl(\frac{1}{-\sigma-k-l-2}-\frac{1}{n-l-k-3}\biggr)\biggr]\biggr\}\cdot T^{(2)}|_{k=k}. \end{aligned} \tag{4.14}$$

Remark 4.1. – Later on we shall take the limit when $\sigma \to 1-n$. As we see H_k will tend to zero. Thus, for $l \leqslant n-3$ the first $n-l-3$ terms of the discrete part in (4.12) will disappear. Notice that if $n-l-3$ is even, then there is one term which is not involved in any H_k, namely $\Psi_1 T^{(2)}$ with $k=(n-l-3)/2$. But this one is zero because of the factor $(-2\alpha+2\beta+2k+1)$.

For $l > n-3$ and for $l \leqslant n-3$ with $k \geqslant n-l-2$ one can easily see that Ψ_1 is regular at $\beta=\beta_0$, so that just setting $\beta=\beta_0$ we have:

$$\begin{aligned} \Psi_1 \;=\; & (-1)^{n+1} 2^{\frac{n+l}{2}+\sigma-1}(-n+l+2k+3) \\ & \times \frac{\Gamma(-n+l+k+3)\Gamma(\sigma+1)\Gamma(-n-\sigma+k+2)}{k!\Gamma(-n-\sigma+2)\Gamma(l+\sigma+2+k)}. \end{aligned} \tag{4.15}$$

Thus for $\alpha=\alpha_0$ and $\beta=\beta_0$ we can rewrite (4.12) as follows:

$$(N_{\sigma,n} \mid h) = \begin{cases} \mathrm{I}+\mathrm{II}+\sum_{i=0}^{\frac{l-n+1}{2}} \Psi_2\cdot\bigl(T^{(1)},\bar{h}\bigr)\big|_{\tau=\frac{n+l}{2}+\sigma-i-1}, & \text{for } l \geqslant n-1, \\ \mathrm{I}+\mathrm{II}, & \text{for } l=n-2, \\ \mathrm{I}+\mathrm{II}+\sum_{j=0}^{\frac{n-3-l}{2}} \Psi_3\cdot\bigl(T^{(1)},\bar{h}\bigr)\big|_{\tau=(n-l)/2-2-j}, & \text{for } l \leqslant n-3 \end{cases} \tag{4.16}$$

with

$$\mathrm{I} = \int_{-\infty}^{\infty} \Psi\cdot\bigl(T^{(1)},\bar{h}\bigr)\big|_{\tau=-1/2+\mathrm{i}\nu}\,\mathrm{d}\nu,$$

$$\mathrm{II} = \begin{cases} \sum_{k=0}^{\infty} \Psi_1\cdot\bigl(T^{(2)},\bar{h}\bigr)\big|_{\tau=(l-n)/2+k+1}, & l > n-3, \\ \sum_{k=0}^{\frac{n-l-3}{2}} \bigl(H_k,\bar{h}\bigr) + \sum_{k=n-l-2}^{\infty} \Psi_1\cdot\bigl(T^{(2)},\bar{h}\bigr)\big|_{\tau=(l-n)/2+k+1}, & l \leqslant n-3 \end{cases}$$

and Ψ, Ψ_1, Ψ_2, Ψ_3, H_k as in (4.13), (4.14), (4.15).

Now introduce the following notations:

$$
\begin{aligned}
A &= \lim_{\sigma\to 1-n} \Psi = \frac{(-1)^n 2^{(l-n)/2-2}(2\tau+1)\sin 2\tau\pi}{\pi^2\Gamma(n-1)} \cdot \Gamma\big((n-l)/2+\tau\big) \\
&\quad \times \Gamma\big((n-l)/2-\tau-1\big)\Gamma\big((l-n)/2+\tau+2\big)\Gamma\big((l-n)/2-\tau+1\big), \\
A_1 &= \lim_{\sigma\to 1-n} \Psi_1 = \frac{2^{(l-n)/2}}{\Gamma(n-1)}(n-l-2k-3), \\
A_2 &= \lim_{\sigma\to 1-n} \Psi_2 = \frac{(-1)^n 2^{(l-n)/2}}{\Gamma(n-1)}(l-n-2i+1), \\
A_3 &= \lim_{\sigma\to 1-n} \Psi_3 = \frac{(-1)^n 2^{(l-n)/2}}{\Gamma(n-1)}(n-l-2j-3).
\end{aligned} \tag{4.17}
$$

Here we used the limit $\lim_{\sigma\to 1-n}(\sigma+n-1)\Gamma(\sigma+1) = (-1)^n/\Gamma(n-1)$.

At last, taking pull back $\mathcal{M}'$ of distributions in (4.16), then multiplying by $(\sigma+n-1)$ and taking the limit when $\sigma\to 1-n$ and finally multiplying by x_1^l we get (4.6). We also set $\alpha=\alpha_0$ and $\beta=\beta_0$ everywhere and use (4.1) for the left-hand sides. □

Now we are going to express $\mathcal{A}'T^{(j)}$ which occurs in the right-hand side of (4.6) by means of spherical distributions associated with representations of our group G, see Section 1. Then Lemma 4.2 gives:

LEMMA 4.3. – *Let* $l=0,1,2,\ldots,\ \tau_\nu=-\frac{1}{2}+\mathrm{i}\nu$ *and* $\tau_k=(l-n)/2+k+1$. *For* $h\in\mathcal{D}(X_1)$ *the function* δ_l *has a decomposition into spherical distributions as follows:*

$$
\begin{aligned}
E\cdot(\delta_l \mid h) &= \int_{-\infty}^{\infty} E_1\cdot(\zeta_{2\tau_\nu+1,l},\bar h)\,\mathrm{d}\nu \\
&\quad + \sum_{k=\frac{n-l-3}{2}}^{n-3} E_2\cdot(\zeta_{2\tau_k+1,l},\bar h) + \sum_{k=n-2}^{\infty} E_3\cdot(\theta_{k-n+2,l},\bar h)|_{\tau_k},
\end{aligned} \tag{4.18}
$$

with $A,\ A_1$ *from* (4.17) *and*

$$
\begin{aligned}
E_1 &= A\cdot\Upsilon|_{\tau=\tau_\nu}, \\
E_2 &= (-1)^{n+1}\cdot A_1\cdot\Upsilon|_{\tau=\tau_k}, \\
E_3 &= A_1\cdot\Upsilon_1, \\
\Upsilon &= \frac{(-1)^{p+1}2^{-(n+l+2)/2}}{\pi^{n-1}\sin\pi((n-l)/2+\tau)}\cdot\frac{\Gamma(\frac{n-2+l}{2}+\tau+1)}{\Gamma(\frac{2-n-l}{2}+\tau+1)}, \\
\Upsilon_1 &= \frac{(-1)^{k+q+1}}{2^{(n+l)/2}\pi^n}\frac{\Gamma(l+k+1)}{\Gamma(k-n+3)}.
\end{aligned} \tag{4.19}
$$

Proof. – Fortunately, we already have the expressions of spherical distributions by means of hypergeometric functions, see Propositions 1.3 and 1.4.

First we consider $T^{(1)}$ which occurs in the continuous part of the decomposition (4.6). Here $\tau=-\frac{1}{2}+\mathrm{i}\nu$. Thus, we can use formula (1.14), because it holds for $\lambda\neq(l+2r)(l+2r+2n-2)$, i.e. $s\neq\pm(l+n-1+2r),\ r=0,1,2,\ldots$.

From now we always keep in mind the notations $t=(c+1)/2,\ \tau=(s-1)/2,\ \mu=n-2$, moreover, everywhere we set $\alpha=(n-2)/2,\ \beta=l/2$.

For $\tau = -\frac{1}{2} + \mathrm{i}\nu$ we know by Theorem 1.1 that $\dim \mathcal{D}'_{\lambda,l}(X_1) = 1$. Thus, with a constant V

$$\mathcal{A}' T_{\lambda,l} = V \cdot \mathcal{A}' T^{(1)}. \tag{4.20}$$

We are going to calculate V. First recall for $T^{(1)}$, see (2.6)

$$T^{(1)} = W \cdot P^{(1)} + Z. \tag{4.21}$$

The relation (4.20) is true for any $f \in \mathcal{D}(X_1)$. We can choose f with support on a regular set in X_1, then all terms concentrated on the cone, for example Z in (4.21), are equal to zero, and $T_{\lambda,l}$ and $T^{(1)}$ are given by regular functions. We indeed, can make such a choice, because if $\tau = -\frac{1}{2} + \mathrm{i}\nu$ then the regular parts of $T_{\lambda,l}$ and $T^{(1)}$ are nonzero. (Neither $G(t, \lambda, \mu, l)$ in (1.8) nor $P(c, \tau, \alpha, \beta)$ in (4.21) are zero.)

Thus, $T_{\lambda,l}$ can be expressed by means of Φ, see (1.8), (1.7); Φ can be expressed by means of P, see (1.4) and (2.2), hence $T_{\lambda,l}$ can be expressed by means of $T^{(1)}$ with the coefficient:

$$V = (-1)^n l! 2^{(n+l)/2-1} \sin\pi\big((n-l)/2 + \tau\big) \frac{\Gamma(\frac{2-n-l}{2} + \tau + 1)}{\Gamma(\frac{n-2+l}{2} + \tau + 1)}.$$

Finally, it follows from (1.14) and (4.20) with Υ as in (4.19)

$$\mathcal{A}' T^{(1)} = \Upsilon \cdot \zeta_{2\tau+1,l}. \tag{4.22}$$

We see that $T^{(1)}$ also occurs in some additional terms of the decomposition (4.6) with $\tau_i = \frac{l-n}{2} - i$ and $\tau_j = \frac{n-l}{2} - 2 - j$ respectively. In both cases $s = 2\tau + 1$ can not be written in the form $s = \pm(l + n - 1 + 2r)$, $r = 0, 1, 2, \ldots$.

Thus, we can use formula (1.14) and the fact (4.20). In the case $\tau = \tau_i$ we can proceed as above. Namely, we can apply both parts of (4.20) to a function f_0 with support on a regular set in X_1, because for such a function $\mathcal{A}' T_{\lambda,l}$ and $\mathcal{A}' T^{(1)}$ are given by regular nonzero functions (G in (1.7) and W in (2.7) are not equal to zero). Thus, proceeding as for the continuous part we get

$$\mathcal{A}' T^{(1)}|_{\tau_i} = \Upsilon|_{\tau_i} \cdot \zeta_{2\tau_i+1,l} \tag{4.23}$$

with Υ as in (4.19).

In the case $\tau = \tau_j$ one has $G = 0$ in (1.7) and $W = 0$ in (2.7). It follows from (1.8) and (4.21) that:

$$\begin{aligned} T_{\lambda,l}(\varphi) &= A(\lambda, \mu, l) \sum_{k=0}^{\mu-1} (-1)^{\mu-k-1} \pi \cdot a_k(\lambda, -\mu, l) \frac{\varphi^{(\mu-k-1)}(t)|_{t=1}}{(\mu-k-1)!}, \\ T^{(1)}(\varphi) &= (Z, \varphi) \end{aligned} \tag{4.24}$$

with Z as in (2.7). One can see that (4.24) gives, with Υ as in (4.19),

$$\mathcal{A}' T^{(1)}|_{\tau_j} = \Upsilon|_{\tau_j} \cdot \zeta_{2\tau_j+1,l}. \tag{4.25}$$

Now we are going to express the distribution $T^{(2)}$ which occurs in the discrete part of the decomposition (4.6) by means of spherical distributions.

In this case $\tau_k = (l-n)/2 + k + 1$. We see that if $0 \leqslant k < n-2$ then $s = 2\tau + 1$ can not be written in the form $s = \pm(l+n-1+2r),\ r = 0, 1, 2, \ldots$. On the contrary, for $k \geqslant n-2$ there exist $r = 0, 1, 2, \ldots$ such that $s = l+n-1+2r$.

First we consider the case $k < n-2$. In this case we can use formula (1.14) with $T_{\lambda,l}$ as in (1.8) and formula (2.6)

$$(-1)^n T^{(2)} = W \cdot P^{(2)} - Z.$$

As G in (1.7) and W are zero for $\tau_k = (l-n)/2 + k + 1$ and $k < n-2$, we can take a function f_0 with support on the cone in X_1 and proceeding as in (4.24)–(4.25) (with $(-1)^{n+1}T^{(2)}$ instead of $T^{(1)}$) we get

$$\mathcal{A}'T^{(2)}\big|_{\tau_k} = (-1)^{n+1}\Upsilon|_{\tau_k} \cdot \zeta_{2\tau_k+1,l} \tag{4.26}$$

with Υ as in (4.19).

Now we consider the case $k \geqslant n-2$. In this case the basis of spherical distributions is given by Proposition 1.4. We will show that $\mathcal{A}'T^{(2)}$ is proportional to $\theta_{r,l}$, see (1.16).

Fortunately, we see that $\mathcal{A}'T^{(2)}$ and $\theta_{r,l}$ are given by nonzero functions for a test function f_0 with support on a regular set in X_1. All terms concentrated on the cone are zero for such f_0. Thus, we can use any relation for the hypergeometric function $T^{(2)}$, see [1]. For example, using the notations $\mu = n-2,\ t = (c+1)/2,\ \tau = \beta - \alpha + k,\ \alpha = (n-2)/2,\ \beta = l/2$ and relation $r = k-n+2$ we can combine formula (3.29) from [1] with (1.16) such that

$$\mathcal{A}'T^{(2)}\big|_{\tau_k} = \Upsilon_1 \cdot \theta_{k-n+2,l} \tag{4.27}$$

with Υ_1 as in (4.19). Here we also used formula (1.9) for $T'_{\lambda_r,l}$ and $\mathcal{A}'S_{\lambda_r,l}$, formula (1.7) for G, formula (1.6) for $A(\lambda,\mu,l)$, formula 1.2(4) from [2], p. 3, formula (1.4) for $\Phi(t,\lambda,\mu,l),\ F(t,\lambda,\mu,l)$ and formula (2.2) for P.

Now substitute (4.22), (4.23), (4.25), (4.26), (4.27) into (4.6). Noticing that $A_2 \cdot \Upsilon|_{\tau_i} = A_3 \cdot \Upsilon|_{\tau_j} = (-1)^{n+1} \cdot A_1 \cdot \Upsilon|_{\tau_k}$ we introduce the following notations $E_1,\ E_2,\ E_3$, see (4.19). Then we have from (4.6):

$$E \cdot (\delta \mid h) = \begin{cases} \mathrm{I} + \mathrm{II} + \sum_{i=0}^{\frac{l-n+1}{2}} E_2 \cdot (\zeta_{2\tau+1,l}, \bar{h})|_{\tau=(l-n)/2-i}, & \text{for } l \geqslant n-1, \\ \mathrm{I} + \mathrm{II}, & \text{for } l = n-2, \\ \mathrm{I} + \mathrm{II} + \sum_{j=0}^{\frac{n-3-l}{2}} E_2 \cdot (\zeta_{2\tau+1,l}, \bar{h})|_{\tau=(n-l)/2-2-j}, & \text{for } l \leqslant n-3 \end{cases} \tag{4.28}$$

with the following notations

$$\mathrm{I} = \int_{-\infty}^{\infty} E_1 \cdot (\zeta_{2\tau+1,l}, \bar{h})|_{\tau=-1/2+\mathrm{i}\nu}\, \mathrm{d}\nu,$$

$$\mathrm{II} = \sum E_2 \cdot (\zeta_{2\tau+1,l}, \bar{h})|_{\tau=\tau_k} + \sum_{k=n-2}^{\infty} E_3 \cdot (\theta_{k-n+2,l}, \bar{h})|_{\tau=\tau_k},$$

where the first sum in II is taken over $k = 0, 1, \ldots, n-3$ for $l > n-3$ and over $k = n-l-2, n-l-1, \ldots, n-3$ for $l \leqslant n-3$.

Finally, combining the additional sums in (4.28) with the first sum in II, we get (4.18). $\square$

Now we express the coefficients in (4.18) by means of c-function, see (1.12).

THEOREM 4.4. – *For $h \in \mathcal{D}(X_1)$ the decomposition of the function δ_l into spherical distributions is given by the following formula*:

$$
\begin{aligned}
(\delta_l \mid h) = 4^{n-2} \cdot 2\pi \cdot \Bigg\{ & \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{c(\mathrm{i}\nu, l)c(-\mathrm{i}\nu, l)} \cdot (\zeta_{i\nu,l}, \bar{h})\, \mathrm{d}\nu \\
& + \sum_{r=\frac{-n-l+1}{2}}^{-1} \operatorname{Res}\left(\frac{1}{c(s,l)c(-s,l)}; s = \rho + l + 2r\right) \cdot (\zeta_{\rho+l+2r,l}, \bar{h}) \\
& + \sum_{r=0}^{\infty} c_{-2}\left(\frac{1}{c(s,l)c(-s,l)}; s = \rho + l + 2r\right) \cdot (\theta_{r,l}, \bar{h}). \Bigg\}.
\end{aligned} \tag{4.29}
$$

Proof. – Using formulae (1.13) one can see that with notations $\tau = (s-1)/2$ and $r = -n + k + 2$

$$
\begin{aligned}
E_1 &= E \cdot 4^{n-2} \cdot \frac{2}{c(\mathrm{i}\nu, l)c(-\mathrm{i}\nu, l)}, \\
E_2 &= E \cdot 4^{n-2} \cdot 2\pi \cdot \operatorname{Res}\left(\frac{1}{c(s,l)c(-s,l)}; s = \rho + l + 2r\right), \\
E_3 &= E \cdot 4^{n-2} \cdot 2\pi \cdot c_{-2}\left(\frac{1}{c(s,l)c(-s,l)}; s = \rho + l + 2r\right).
\end{aligned}
$$

Then, we write

$$
\begin{aligned}
\int_{-\infty}^{\infty} E_1 \cdot (\zeta_{2\tau_\nu+1,l}, \bar{h})\, \mathrm{d}\nu &= \frac{1}{2} \cdot \int_{-\infty}^{\infty} E_1 \cdot (\zeta_{\mathrm{i}\nu,l}, \bar{h})\, \mathrm{d}\nu, \\
\sum_{k=\frac{n-l-3}{2}}^{n-3} E_2 \cdot (\zeta_{2\tau_k+1,l}, \bar{h}) &= \sum_{r=\frac{-n-l+1}{2}}^{-1} E_2 \cdot (\zeta_{\rho+l+2r,l}, \bar{h}), \\
\sum_{k=n-2}^{\infty} E_3 \cdot (\theta_{k-n+2,l}, \bar{h})|_{\tau_k} &= \sum_{r=0}^{\infty} E_3 \cdot (\theta_{r,l}, \bar{h}).
\end{aligned}
$$

Now we rewrite (4.18) as (4.29). □

The formula (4.29) was obtained in [13] by a different method, namely as in [3], “première démonstration de la formule de Plancherel”, p. 424.

Notice that the spherical distributions which occur in the second term of formula (4.29) are “real” distributions, i.e. they are concentrated on a cone, i.e. on a singular orbit of the group H_1, cf. [7].

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