On the Zeros of Certain Linear Differential Polynomials

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Let \( f \) be meromorphic in the plane and let \( F \) be given by

\[
F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}.
\]

We show that there is no function \( f \) such that \( fF \) has no zeros, where \( k \geq 3 \), \( a_0, \ldots, a_{k-1} \) are constants, and \( a_0 \) is a polynomial of degree one. We also classify all functions \( f \) such that \( f^{(k)} F \) has no zeros, where \( F \) is as above with \( k \geq 3 \) and the \( a_j \) polynomials. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper is concerned with some aspects of the following problem: If \( k \geq 2 \), if \( f \) is meromorphic in the plane, and \( F \) is given by

\[
F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}
\]

with \( a_0, \ldots, a_{k-1} \) polynomials, then in what circumstances may \( f \) and \( F \) have only finitely many zeros?

For entire \( f \), this problem was settled by Frank and Hellerstein in [1]. With regard to the meromorphic case, they also proved in [1] the following theorem, which contains an earlier result of Frank, Hennekemper, and Polloczek [2]. Our notation throughout is that of [4].

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**THEOREM A.** If $k \geq 3$, if $f$ is meromorphic in the plane and $F$ is given by (1.1) with $a_{k-1} \equiv 0$, and $f$ and $F$ have only finitely many zeros, then

$$T(r, f'/f) = O(r^{\lambda}),$$

where

$$\lambda = 1 + \max\{(\deg(a_j))/(k-j) : j = 0, \ldots, k-2\}.$$ 

If all the $a_j$ are identically zero, then $r^\lambda$ may be replaced by $\log r$ in (1.2).

Note that eliminating $a_{k-1}$ in (1.1) amounts to multiplying both $f$ and $F$ by a factor $\exp(Q)$, with $Q$ a polynomial. Now if all the $a_j$ are constant in (1.1), Theorem A implies that $f'/f$ has order at most 1. The following was proved by Steinmetz in [9].

**THEOREM B.** Suppose that $k \geq 3$, that $f$ is meromorphic in the plane, and that $F$ is given by (1.1) with $a_0, \ldots, a_{k-1}$ constants. If $fF$ has no zeros then one of the following holds:

(i) $f = \exp(az + b + e^{cz+d})$;
(ii) $f = e^{az+b}(e^{cz+d}-1)^{-n}$;
(iii) $f = e^{az+b}(z-c)^{-n}$.

Here $a, b, c, d$ are constants and $n$ is a positive integer.

Now if $B$ is a polynomial and $H$ is given by $H''/H' = B$, then $f = (H')^{-k}H^{-L}$ satisfies

$$(D + B) \cdots (D + kB)f = cH^{-L-k}$$

with $c$ a constant. (Here $D$ denotes $d/dz$.) It seems possible that for non-constant coefficients $a_j$ this is essentially the only way to construct examples where $fF \neq 0$ and $f$ has infinitely many poles. Of course one may always adjust the coefficients as mentioned after Theorem A. Note also that with $H$ as above, $g = (H')^{-k}e^H$ satisfies

$$(D + B) \cdots (D + kB)g = e^H.$$ 

We prove the following theorem.

**THEOREM 1.** Suppose that $k \geq 3$ and that $a_1, \ldots, a_{k-1}$ are constants, and $a_0$ is a polynomial of degree 1. Then there is no function $f$ meromorphic in the plane such that

$$f \left( f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)} \right)$$

has no zeros.
The proof of Theorem 1 (as well as that of Theorem 2 below) depends on a system of linear differential equations in functions $g$ and $h$ given by $g^k = f/F$ and $h = (-f''/f)g$. These equations arise from Lemma 6 of [1] and in the context of Theorem B have constant coefficients. In the general case the coefficients are non-constant, but in the particular case of Theorem 1 the equations have a relatively simple form.

**Theorem 2.** Let $k \geq 3$ be an integer, and let $\alpha_0, ..., \alpha_{k-1}$ be polynomials. If $f$ is meromorphic in the plane, if $F$ is given by (1.1) and $ff'F$ has no zeros, then one of the following holds:

(i) \( f = \exp(\int_0^z e^{Q(t)} \, dt + c) \); 
(ii) \( f = c(\int_0^z e^{Q(t)} \, dt - b)^{-n}; \) 
(iii) \( f = (az + b)^{-n}; \) 
(iv) \( f = e^{az + b}. \) 

Here $a, b, c$ are constants, $Q$ is a polynomial, and $n$ is a positive integer.

It is apparent that all the forms (1.3) to (1.6) are possible (compare the examples above). It seems likely that Theorem 2 also holds for $k = 2$—see Section 8 for a partial result here. It also seems reasonable to conjecture that some sort of classification of $f$ is possible without any hypothesis on $f'$ but that is certainly beyond the methods of the present paper.

The proof of Theorem 1 is developed in Sections 3 to 5 while that of Theorem 2 is given in Sections 6 and 7.

**2. Preliminary Lemmas**

We make extensive use of results from [1] which for convenience we summarize as a lemma.

**Lemma 1.** Suppose that $f$ is meromorphic in the plane and that $fF$ has no zeros, where

\[ F = f^{(k)} + \sum_{j=0}^{k-2} a_j f^{(j)} \]  

with $k \geq 2$ and $a_0, ..., a_{k-2}$ polynomials. Define entire functions $g$ and $h$ by

\[ g^k = f/F, \quad h = (-f''/f)g. \]

Let $f_1, ..., f_k$ be solutions of

\[ y^{(k)} + \sum_{j=0}^{k-2} a_j y^{(j)} = 0. \]
with $W(f_1, \ldots, f_k) = 1$. Then setting $w_j = f_j h + f_j^\prime g$, we have

$$W(w_1, \ldots, w_k) = (-1)^k$$

(2.4)

and $w_1, \ldots, w_k$ are linearly independent solutions of

$$M_k(w) = w^{(k)} + \sum_{j=0}^{k-2} A_j w^{(j)} = 0,$$

(2.5)

where $A_0, \ldots, A_{k-2}$ are entire, and are polynomials if $k \geq 3$. Also if $f$ is not entire,

$$m(r, f'/f) = O(\log T(r, f'/f) + \log r)$$

(2.6)

outside a set of $r$ of finite linear measure. Also if $k \geq 3$, then with the notation $b_j = A_j - a_j$, $A_{-1} = a_{-1} = 0$, and $A_{k-1} = a_{k-1} = 0$ we have the following equations for $g$ and $h$:

$$h' = -\frac{k-1}{2} g^\prime - \frac{b_{k-2}}{k} g;$$

(2.7)

$$b_{k-2} h = \frac{k(k^2 - 1)}{12} g^{(3)} + g^\prime \left( -\frac{k+1}{2} b_{k-2} + 2A_{k-2} \right) + g \left( \frac{k-1}{2} b_{k-2} + a_{k-2} - b_{k-3} \right);$$

(2.8)

$$\frac{2}{k-2} b_{k-3} h = \frac{k(k^2 - 1)}{12} g^{(4)} + g^\prime \left( \frac{k-1}{3} b_{k-2} + 2A_{k-2} \right) + g^\prime \left( 2 \frac{k-1}{3} b_{k-2} - \frac{2k}{k-2} b_{k-3} + \frac{6}{k-2} A_{k-3} \right) + g \left( \frac{k-1}{3} b_{k-2}^2 + \frac{2}{k} A_k b_{k-2} + \frac{2}{k-2} a_{k-3}^\prime - \frac{2}{k-2} b_{k-4} \right).$$

(2.9)

**Proof.** Equations (2.4) and (2.5) follow from (2.1), (2.2), (2.3), and Lemma 2 of [1]. The fact that the $A_j$ are polynomials if $k$ is at least 3 follows from Theorem A and Lemma 3 of [1]. Equation (2.6) is from Lemma 8 of [1], while (2.7), (2.8), and (2.9) arise from Lemma 6 of [1] as follows. With the notations

$$M_{k,\nu}(w) = \sum_{\mu = \nu}^{k} \binom{\mu}{\nu} A_{\mu} w^{(\mu - \nu)},$$

(2.10)
and $M_{k,0} = M_k$, $M_{k,-1} = 0$, then Lemma 6 of [1] gives

$$M_{k,v}(h) - a_v h = -M_{k,v-1}(g) + a_v M_{k,k-1}(g) + (a'_v + a_{v-1}) g$$

for $v = 0, ..., k - 1$. Now $v = k - 1$ gives (2.7). Setting $v = k - 2$ and using (2.7) we obtain (2.8) while (2.9) comes from $v = k - 3$ and (2.7). We omit the details.

The following lemma plays the same role as a result of Wittich [10] in [9]. Here the order of $f$ is

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

**Lemma 2.** Suppose that $a_0, ..., a_{k-1}$ are polynomials, and that all solutions of

$$y^{(k)} + \sum_{j=0}^{k-1} a_j y^{(j)} = 0$$

(2.10)

have order at most $\lambda$. Then for $j = 0, ..., k - 1$, we have

$$\deg(a_j) \leq (k - j)(\lambda - 1).$$

(2.11)

**Proof.** The lemma is an immediate consequence of the following claim, which we prove by induction.

**Claim.** If $a_0, ..., a_{k-1}$ are meromorphic, and if (2.10) has $k$ meromorphic linearly independent solutions all of order at most $\lambda$, then for any $\varepsilon > 0$ there exists a set $E$ of finite logarithmic measure such that if $r \notin E$ then for $|z| = r$,

$$a_j(z) = O(r^{(k-j)(\lambda - 1 + \varepsilon)}).$$

(2.12)

To establish the claim, we need the following estimate from [3]. If $f$ is meromorphic of order at most $\lambda$, and if $\varepsilon > 0$, then for $j = 1, ..., k$ we have

$$f^{(j)}(z)/f(z) = O(r^{(\lambda - 1 + \varepsilon)})$$

(2.13)

for all $z$ on $|z| = r$, provided $r$ lies outside a set of finite logarithmic measure. Now the claim is obvious for $k = 1$. If $k$ is 2 or greater, we take a non-trivial solution $f$ of (2.10) and set $y = uf$. Now $v = u'$ satisfies

$$v^{(k-1)} + v^{(k-2)}(kf''/f + a_{k-1})$$

$$+ v^{(k-3)} \left( \binom{k}{2} (f''/f) + (k-1)(f'/f) a_{k-1} + a_{k-2} \right) + \cdots$$

$$+ v(kf^{(k-1)}/f + (k-1)(f^{(k-2)}/f) a_{k-2} + \cdots + 2(f'/f) a_2 + a_1).$$
Now assuming the claim true for $k - 1$, then considering the coefficient of $v^{(k - 2)}$ and applying (2.13) gives (2.12) for $a_{k - 1}$. Now we obtain (2.12) for $a_{k - 2}$ by considering the coefficient of $v^{(k - 3)}$. Proceeding in this way we obtain (2.12) for $j = 1, \ldots, k - 1$. Now (2.12) for $j = 0$ follows from dividing (2.10) through by $y$ and putting $y = f$.

The following lemma simplifies some cases in the proof of Theorem 1. Here $\deg(R)$ is defined for a rational function $R$ by

$$\deg(R) = \lim_{z \to \infty} \frac{\log |R(z)|}{\log |z|}$$

with $\deg(0) = -\infty$.

**Lemma 3.** Suppose that $R_1, R_0$ are rational, with $\deg(R_0) \leq \deg(R_1)$ and $\deg(R_1) > 0$. If $g$ is meromorphic in the plane and satisfies

$$g'' + R_1 g' + R_0 g = 0,$$

and if $\sigma(g) < 1 + \deg(R_1)$, then $g$ has only finitely many zeros and $g'/g$ is rational.

**Proof.** This is essentially just a special case of Pöschl's Theorem [8] but with rational instead of polynomial coefficients. Obviously $g$ has only finitely many poles. Suppose now that the half-plane $S$ given by

$$\alpha < \arg z \leq \alpha + \pi$$

contains infinitely many zeros of $g$. Define $u$ in

$$\alpha - \pi/2 < \arg z < \alpha + 3\pi/2, \quad |z| > r_0$$

by $2u'/u = -R_1$. Then we write

$$u = z^\beta \phi(z) e^{P(z)}$$

with $P$ a polynomial, $\beta$ a constant, and $\phi$ having a removable singularity at infinity. Now $v = g/u$ satisfies

$$v'' + Rv = 0$$

with

$$R = R_0 - R_1^2/4 - R_1/2$$

and $\deg(R) = 2 \deg(R_1)$. Now the result follows from Hille's asymptotic method. A suitable reference is Theorem C of [5]—see also [6, Chap 7] or
[7]. For if \( v \) has infinitely many zeros in \( S \), the number \( n(r, 0) \) of zeros satisfies

\[
\frac{\log n(r, 0)}{\log r} > 1 + \deg(R_1) - o(1).
\]

3. **Proof of Theorem 1, First Part**

Suppose then that \( f \) is meromorphic in the plane and that \( fF \) has no zeros, where

\[
F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}
\]

and \( k \geq 3 \), while \( a_1, ..., a_{k-1} \) are constants and \( a_0 \) is a polynomial of degree one. There is no loss of generality in assuming that \( a_{k-1} \equiv 0 \).

Now we may easily dispose of the case where \( f'/f \) is rational, for then we may write

\[
f'(z)/f(z) = P_1(z) + O(|z|^{-1}) \quad \text{as} \quad z \to \infty,
\]

with \( P_1 \) a polynomial, and calculating \( F/f \) we see that by a degree argument (using the non-constancy of \( a_0 \) if \( P_1 \) is constant) this function has at least one zero. Also \( F/f \) has a pole at any pole of \( f \), and so \( F \) has at least one zero. So we may assume henceforth that \( f'/f \) is transcendental. As in Lemma 1 we define entire functions \( g \) and \( h \) by \( g^k = f^k f \) and \( h = (-f'/f) g \) and find that \( w_j = f_j h + f_j' g \) form a fundamental set of solutions of an equation

\[
w^{(k)} + \sum_{j=0}^{k-2} A_j w^{(j)} = 0 \tag{3.1}
\]

with the \( A_j \) polynomials. Here \( f_1, ..., f_k \) are linearly independent solutions of the homogeneous equation (2.3). Now Theorem A and well-known estimates yield

\[
T(r, f'/f) + T(r, f_j) = O(r^{(1 + k)/k})
\]

so that each \( w_j \) has order at most \((1 + k)/k\). Applying Lemma 2 to (3.1) we see that \( A_1, ..., A_{k-2} \) are constants while \( A_0 \) has degree at most one. Now with the notation \( b_j = A_j - a_j \) we obtain the Eqs. (2.7), (2.8), and (2.9) for \( g \) and \( h \).

We consider separately the cases \( k \geq 4 \) and \( k = 3 \).
4. CONCLUSION OF THE PROOF OF THEOREM 1, IF $k \geq 4$

In this case, (2.7), (2.8), and (2.9) become, using the fact that $a_j' \equiv 0$, $A_j' \equiv 0$ for $j \geq 1$, and $a_0^j \equiv 0$, $A_0^j \equiv 0$,

\[ h' = -\frac{k-1}{2} g'' - \frac{b_{k-2}}{k} g, \]  
\[ \text{(4.1)} \]

\[ b_{k-2} h = \frac{k(k^2-1)}{12} g^{(3)} + g' \left( -\frac{k+1}{2} b_{k-2} + 2A_{k-2} \right) + g(-b_{k-3}), \]
\[ \text{(4.2)} \]

and
\[ \frac{2}{k-2} b_{k-3} h = \frac{k(k^2-1)}{12} g^{(3)} + g' \left( \frac{k-1}{3} b_{k-2} + 2A_{k-2} \right) + g' \left( -\frac{2k}{k-2} b_{k-3} + \frac{6}{k-2} A_{k-3} \right) + g \left( \frac{2}{k} A_{k-2} b_{k-2} - \frac{2}{k-2} b_{k-4} \right). \]
\[ \text{(4.3)} \]

Now differentiate (4.2) and use (4.1). Note that all coefficients here are constants. We obtain
\[ \frac{k(k^2-1)}{12} g^{(4)} + g''(2A_{k-2} - b_{k-2}) - g'b_{k-3} + g \frac{b_{k-2}^2}{k} = 0 \]  
\[ \text{(4.4)} \]

We now claim that $g$ satisfies a homogeneous linear differential equation of order at most 3 with constant coefficients. This is obvious if $b_{k-2} = 0$, from (4.2). If $b_{k-2} \neq 0$, we may substitute for $h$ in (4.3) using (4.2) to obtain an equation
\[ \frac{k(k^2-1)}{12} g^{(4)} + c_1 g^{(3)} + c_2 g'' + c_3 g' + g \left( c_4 - \frac{2b_{k-4}}{k-2} \right) = 0, \]
\[ \text{(4.5)} \]

where $c_1, \ldots, c_4$ are constants. Now (4.4) implies that $g$ has order at most 1, and thus by (4.5), $b_{k-4}$ is constant. Now subtract (4.4) from (4.3). This gives
\[ \frac{2}{k-2} b_{k-3} h = g'' \left( \frac{k+2}{3} b_{k-2} \right) + c_5 g' + c_6 g, \]
\[ \text{(4.6)} \]
where $c_5$, $c_6$ are constants. Now we can differentiate (4.6) and using (4.1) obtain an equation

$$g^{(3)} + B_2 g'' + B_1 g' + B_0 g = 0,$$

with $B_0$, $B_1$, $B_2$ constants. Consider the characteristic equation

$$d^3 + B_2 d^2 + B_1 d + B_0 = 0. \tag{4.7}$$

Suppose first that (4.7) has just one root $d$. Then if $d = 0$, $g$ is a polynomial. By (4.1) $h$ is a polynomial also and $f'/f = -h/g$ is rational. If $d \neq 0$, then $g$ has form $g = P_2(z) e^{dz}$ with $P_2$ a polynomial. Now if $b_{k-2} \neq 0$, (4.2) implies that $f'/f$ is rational again. On the other hand, if $b_{k-2} = 0$, then by (4.1),

$$h = -\frac{k-1}{2} g' + C$$

for some constant $C$. This gives

$$f'/f = ((k-1)/2)(g'/g) - C/g. \tag{4.8}$$

This implies that there are sectors in which $g$ is large and $f'/f = O(1)$. Using Cauchy's estimate, the equation $F/f = g^{-k}$, and the fact that $a_0$ is non-constant we obtain a contradiction.

Suppose now that (4.7) has 2 roots $d_1$, $d_2$ say. Then we can write

$$g(z) = (c_1 z + c_8) e^{d_1 z} + c_9 e^{d_2 z}, \tag{4.9}$$

with $c_j$ constants. Again if $b_{k-2} = 0$, we obtain (4.8) and a contradiction as in the previous case. On the other hand if $b_{k-2} \neq 0$, we observe that by (4.9) $g$ satisfies an equation

$$g'' + s_1 g' + s_0 g = 0. \tag{4.10}$$

with $s_1$, $s_0$ rational and $s_j(\infty)$ finite. Now (4.2) and (4.10) yield

$$h = s_2 g' + s_3 g$$

or

$$-f'/f = s_2(g'/g) + s_3$$

with $s_2$, $s_3$ rational and $s_j(\infty)$ finite again. As before we obtain $f'/f = O(1)$ in sectors where $g$ is large and we have a contradiction.

Suppose finally that (4.7) has 3 distinct roots $d_1$, $d_2$, and $d_3$, say. Then (4.1) and the equation $F/f = g^{-k}$ imply that $a_0$ is a rational function of $\exp(d_1 z)$, $\exp(d_2 z)$, and $\exp(d_3 z)$ which is clearly impossible.
5. Conclusion of the Proof of Theorem 1, if $k = 3$

In this case Eqs. (2.7), (2.8), and (2.9) become

\begin{equation}
    h' = -g'' - \frac{b_1}{3} g, \quad (5.1)
\end{equation}

\begin{equation}
    b_1 h = 2g^{(3)} + g'(2a_1) + g(-b_0). \quad (5.2)
\end{equation}

and

\begin{equation}
    b_0 h = g^{(4)} + g'' \left( \frac{b_1}{3} + A_1 \right) + g'(3a_0) + g \left( \frac{1}{3} A_1 b_1 + a_0 \right). \quad (5.3)
\end{equation}

Suppose first that $b_1 = 0$. Then (5.1) gives, for some constant $C$,

\begin{equation}
    h = -g' + C. \quad (5.4)
\end{equation}

Now if $f$ is entire, since $g$ has order at most $\frac{4}{3}$, we obtain $g(z) = \exp(c_{10} z + c_{11})$ with the $c_j$ constants and from (5.4) we obtain

\begin{equation*}
    f'/f = c_{10} - C/g.
\end{equation*}

Now as in Section 4, whether or not $c_{10} = 0$, we obtain sectors in which $1/g = O(1)$ and $f'/f = O(1)$; these estimates together with Cauchy's estimate, the equation $F/f = g^{-3}$ and the fact that $a_0$ is non-constant, yield a contradiction.

Now suppose that $f$ is not entire, still assuming that $b_1 = 0$. At a pole of $f$ of order $m$, we have $h = mg'$. From (5.4) we thus have $g' = C(m+1)^{-1}$. But we also have $(g')^{-3} = -m(m+1)(m+2)$ at such a pole. We conclude that all poles of $f$ have a fixed multiplicity which we continue to call $m$. But then we can write $f = \exp(v^*)/g''$, where $v^*$ is entire, and setting $v = (v^*)'$,

\begin{equation*}
    f'/f = v - mg'/g = g'/g - C/g.
\end{equation*}

We observe that, by (2.6) of Lemma 1, $v$ is a polynomial, and since $g$ has order at most $\frac{4}{3}$, $v$ is a constant. Now solving for $g$ gives

\begin{equation*}
    f'/f = c_{12} + c_{13}/g
\end{equation*}

with the $c_j$ constants and we obtain a contradiction as before.

We may henceforth assume that $b_1 \neq 0$, and now show that there exist rational functions $R_1$, $R_0$ such that

\begin{equation}
    g'' + R_1 g' + R_0 g = 0. \quad (5.5)
\end{equation}
To prove (5.5), (5.1) and (5.2) yield

\[ 2g^{(4)} + g''(2a_1 + b_1) + g'(-b_0) + g\left(\frac{b_1^2}{3} - b_0^2\right) = 0. \]  
(5.6)

Also (5.2) and (5.3) yield

\[
g^{(4)} + g^{(3)}\left(-\frac{2b_0}{b_1}\right) + g''\left(\frac{1}{3} b_1 + A_1\right) + g'\left(3a_0 - \frac{2a_1 b_0}{b_1}\right) \\
+ g\left(\frac{1}{3} A_1 b_1 + a_1' + \frac{b_0^2}{b_1}\right) = 0. 
\]  
(5.7)

Now (5.6) and (5.7) yield

\[
g^{(3)}\left(\frac{2b_0}{b_1}\right) + g''\left(-\frac{5b_1}{6}\right) + g'\left(-\frac{b_0}{2} - 3a_0 + \frac{2a_1 b_0}{b_1}\right) \\
+ g\left(\frac{b_1^2}{6} - \frac{1}{2} b_1 a_1' - \frac{1}{3} A_1 b_1 - a_1' - \frac{b_0^2}{b_1}\right) = 0. 
\]  
(5.8)

Now if \( b_0 \equiv 0 \), (5.5) follows at once, since \( b_1 \neq 0 \). From (5.8) we obtain, using \( r_j \) henceforth to denote rational functions and assuming that \( b_0 \neq 0 \),

\[
g^{(4)} + g^{(3)}\left(\frac{b_0'}{b_0} - \frac{5b_1^2}{12b_0}\right) + g''\left(-\frac{1}{4} b_1 - \frac{3a_0 b_1}{2b_0} + a_1\right) \\
+ r_1 g' + g(-b_0') = 0. 
\]  
(5.9)

Dividing (5.6) by 2, and subtracting from (5.9) we obtain

\[
g^{(3)}\left(\frac{b_0'}{b_0} - \frac{5b_1^2}{12b_0}\right) + g''\left(-\frac{3}{4} b_1 - \frac{3a_0 b_1}{2b_0}\right) + r_2 g' \\
+ g\left(-\frac{1}{2} b_0' - \frac{b_1^2}{6}\right) = 0. 
\]  
(5.10)

Now if the coefficient of \( g^{(3)} \) in (5.10) vanishes identically, we have \( b_0' = \frac{5b_1^2}{12} \neq 0 \), which makes the coefficient of \( g \) in (5.10) non-zero, so that (5.5) is trivial in this case. Thus if (5.5) fails, (5.10) must be a rational multiple of (5.8). But this gives

\[
\frac{5b_1^2}{12} \left(\frac{b_0'}{b_0} - \frac{5b_1^2}{12b_0}\right) = \frac{3b_1 b_0}{4} + \frac{3a_0 b_1}{2}. 
\]
This equation forces $b_0$ to be constant. But then we have a contradiction, since $b_1 \neq 0$ and $a_0$ is non-constant.

We may therefore suppose henceforth that $g$ satisfies (5.5). Since $\sigma(g) \leq \frac{4}{3}$, we obtain $\deg(R_0) \leq \max\{\deg(R_1), 0\}$. Now if $\deg(R_1)$ is positive then by Lemma 3, $g'/g$ is rational and so by (5.2), $f'/f = -h/g$ is rational. Therefore we may assume that $\deg(R_0) \leq 0$, $\deg(R_1) \leq 0$. Now if $b_0$ is constant we obtain from (5.2) and (5.5) an equation

$$h = s_1 g' + s_0 g$$

(5.11)

with $s_1$ rational and $s_j(\infty)$ finite. On the other hand if $b_0$ is non-constant we obtain (5.11) from (5.3) and (5.5). Now (5.11) gives

$$-f'/f = s_1(g'/g) + s_0.$$  

(5.12)

We now know (from (5.5)) that $\sigma(g) \leq 1$. At arbitrarily large maximum modulus points of $g$ (we may assume $g$ transcendental since otherwise $f'/f$ is rational by (5.12)) we have $g^{(j)}(z)/g(z) = O(|z|^\varepsilon)$ for any $\varepsilon > 0$ and $j \leq 3$. Now calculating $f^{(j)}/f$ from (5.12) and using the equation $F/f = g^{-3}$ we obtain a contradiction as before, since $a_0$ is non-constant.

This concludes the proof of Theorem 1, a contradiction having been obtained in all possible cases.

6. PROOF OF THEOREM 2, FIRST PART

Suppose then that $F$ is given by (1.1) with $k \geq 3$, the $x_j$ polynomials, and that $ff'F$ has no zeros. Set $u = f/f'$. Now $u$ is entire. If $u$ is a polynomial then $u$ must have degree at most 1 and either (1.5) or (1.6) must hold. Now suppose that $u$ is transcendental. There exist polynomials $a, a_0, \ldots, a_{k-2}$ such that writing

$$f = \phi e^{a^*}, \quad F = \Phi e^{a^*},$$

(6.1)

where $(a^*)' = a$, we have

$$\Phi = \phi^{(k)} + \sum_{j=0}^{k-2} a_j \phi^{(j)}$$

(6.2)

and

$$\phi'(a\phi + \phi a) \Phi \neq 0.$$  

(6.3)

By Theorem A, $\phi'/\phi$ has finite order and therefore so has $u$. If $f$ is entire we
therefore have \( u = \exp(-P_1) \) with \( P_1 \) a polynomial and (1.3) holds. Suppose now that \( f \) is not entire. By (2.6) of Lemma 1 we have

\[
m(r, 1/u) = O(\log r)
\]

so that we may assume henceforth that \( f \) has infinitely many poles, since otherwise \( u \) is a polynomial. Define \( g \) by

\[
g^k = f/F = \phi/\Phi
\]

and write

\[
H = (-f'/f) g, \quad h = (-\phi'/\phi) g.
\]

Thus

\[
h = H + ag = e' + ag,
\]

say, with \( P \) a polynomial, since \( H \) has finite order and has no zeros or poles. Now let \( f_1, \ldots, f_k \) be a fundamental solution set of

\[
y^{(k)} + a_{k-2} y^{(k-2)} + \cdots + a_0 y = 0
\]

with \( W(f_1, \ldots, f_k) = 1 \). As in Lemma 1, the functions \( w_j = f_j h + f'_j g \) form a fundamental solution set of an equation

\[
w^{(k)} + A_{k-2} w^{(k-2)} + \cdots + A_0 w = 0
\]

with the \( A_j \) polynomials. We make the following claim. Here, as before, \( b_j \) is given by \( b_j = A_j - a_j \), and we have Eqs. (2.7), (2.8), and (2.9) for \( g \) and \( h \).

**Claim.** If \( b_{k-2} = 0 \), all poles of \( f \) have a fixed multiplicity \( m \). If \( b_{k-2} \neq 0 \), there exist rational functions \( R_1, R_2 \) such that

\[
g'' + R_1 g' + R_2 g = 0.
\]

To prove the claim, recall (6.7). This representation for \( h \) yields, on substitution in (2.7),

\[
P' e^p = -\frac{k-1}{2} g'' - ag' - \left( \frac{b_{k-2}}{k} + a' \right) g.
\]

Assume for the time being that \( b_{k-2} \) does not vanish identically. We may assume also that \( P \) is non-constant, for otherwise (6.10) follows.
immediately from (6.11). We also have, from (2.8) and (2.9) of Lemma 1 and (6.7),
\[ b_{k-2}e^p = \frac{k(k^2 - 1)}{12} g^{(3)} + g\left( -\frac{k+1}{2} b_{k-2} + 2A_{k-2} \right) \]
\[ + g\left( \frac{k-1}{2} b'_{k-2} - b_{k-3} + a'_{k-2} - ab_{k-2} \right) \]  \hspace{1cm} (6.12)

and
\[ \left( \frac{2}{k-2} b_{k-3} - b'_{k-2} \right) e^p = \frac{k+2}{3} b_{k-2} g'' + p_1 g' + p_0 g \]  \hspace{1cm} (6.13)

with \( p_1, p_0 \) polynomials. Here (6.13) comes from differentiating (2.8), using (2.7), and subtracting from (2.9), with finally (6.7) used to substitute for \( h \). Comparing the coefficients of \( g'' \) and \( e^p \) in (6.11) and (6.13), the conclusion (6.10) follows unless
\[ b_{k-3} = \frac{k-2}{2} b'_{k-2} - \frac{k^2 - 4}{3(k-1)} b_{k-2} P'. \]  \hspace{1cm} (6.14)

From (6.11) we obtain
\[ -\frac{k-1}{2} g^{(3)} + g'' \left( \frac{k-1}{2} \left( \frac{P''}{P'} + P' \right) - a \right) \]
\[ + g' \left( a \left( \frac{P''}{P'} + P' \right) - \frac{b_{k-2}}{k} - 2a' \right) \]
\[ + g \left( \left( \frac{P''}{P'} + P' \right) \left( \frac{b_{k-2}}{k} + a' \right) - \frac{b'_{k-2}}{k} - a'' \right) = 0. \]  \hspace{1cm} (6.15)

Also from (6.11), (6.12), and (6.14) we obtain
\[ \frac{k(k^2 - 1)}{12} g^{(3)} + g'' \left( \frac{k-1}{2} b_{k-2} \right) \]
\[ + g' \left( -\frac{k+1}{2} b_{k-2} + 2A_{k-2} + \frac{ab_{k-2}}{P'} \right) \]
\[ + g \left( \frac{b'_{k-2}}{2} + a'_{k-2} + \frac{k^2 - 4}{3(k-1)} b_{k-2} P' \right) \]
\[ - ab_{k-2} + \frac{b_{k-2}}{P'} \left( \frac{b_{k-2}}{k} + a' \right) \right) = 0. \]  \hspace{1cm} (6.16)
Thus (6.10) follows unless (6.15) and (6.16) are the same equation up to multiplication by a constant. If this is the case, comparison of the coefficients of \( g'' \) yields

\[
\frac{P''}{P'} + p' = \frac{2a}{k-1} - \frac{6}{k(k+1)} \frac{b_{k-2}}{P'}. \tag{6.17}
\]

Also, comparing the coefficients of \( g' \) and using (6.17) we obtain

\[
A_{k-2} = \frac{k+1}{3} b_{k-2} + \frac{k(k+1)}{6} a' - \frac{k(k+1)}{6(k-1)} a^2
\]

so that

\[
a_{k-2} = A_{k-2} - b_{k-2} = \frac{k-2}{3} b_{k-2} + \frac{k(k+1)}{6} a' - \frac{k(k+1)}{6(k-1)} a^2. \tag{6.18}
\]

Finally, comparing coefficients of \( g \) and using (6.17) and (6.18) we find that

\[
\frac{2ab_{k-2} - 2k + 4}{k} + \frac{b'_{k-2}}{k^2 - 1} k - 2 + \frac{2(k^2 - 4)}{k(k^2 - 1)} b_{k-2} P' = 0.
\]

From this equation we see that \( b_{k-2} \) is constant and

\[
a = \frac{k+2}{2} P'. \tag{6.19}
\]

From (6.17) and (6.19),

\[
\frac{k-1}{2} b_{k-2} = \frac{k(k+1)(k+2)}{12} P'^2 - \frac{k(k^2 - 1)}{12} (P'' + P'^2)
\]

which shows that \( P' \) is constant and (say \( P' = \alpha \))

\[
b_{k-2} = \frac{k(k+1)}{2(k-1)} \alpha^2. \tag{6.20}
\]

We substitute (6.19) and (6.20) into (6.11). For some nonzero constant \( \beta \),

\[
\beta e^{\alpha z} = -\frac{k-1}{2} g'' - \frac{k + 2}{2} \alpha g' - g \left( \frac{(k + 1)}{2(k - 1)} \alpha^2 \right).
\]

This gives

\[
g = e^{\alpha z} (C + Be^{\alpha z} + Ae^{2\alpha z}),
\]
where $A, B, C$ are constants, with $C \neq 0$, and $\omega = -k\alpha/(k-1)$. If $AB = 0$, the conclusion (6.10) follows immediately. Suppose now that $A \neq 0$. In this case we can write, for some constants $C_0, C_1, C_2, U,$ and $V$,

$$g = C_0 e^{(1 - k)\omega z/k} (e^{\omega z} - U)(e^{\omega z} - V)$$

and since

$$f'/f = -H/g = -\frac{e^{\omega z} + C_1}{g}$$

we get

$$f'/f = C_1 \left( \frac{1}{e^{\omega z} - U} - \frac{1}{e^{\omega z} - V} \right).$$

Calculating $f$ from the above representation we see that $f$ has poles where $e^{\omega z} = U$, $e^{\omega z} = V$, with multiplicities $m_1$, $m_2$, respectively, and $m_1/m_2 = -V/U$. When $e^{\omega z} = U$, $g = 0$ and

$$g' = C_0 U^{(1 - k)/k}(\omega U)(U - V)$$

so that $U \neq V$ and

$$(g')^k = C_0^k \omega^k U (U - V)^k.$$  

Similarly $e^{\omega z} = V$ gives

$$(g')^k = C_0^k \omega^k V (V - U)^k.$$  

But when $e^{\omega z} = V$,

$$(g')^k = (m_2(m_2 + 1) \cdots (m_2 + k - 1))^{-1}(-1)^k.$$  

So

$$\frac{m_2(m_2 + 1) \cdots (m_2 + k - 1)}{m_1(m_1 + 1) \cdots (m_1 + k - 1)} = (-1)^k U/V = (-1)^{k+1} m_2/m_1.$$  

This forces $k$ to be odd and $m_1 = m_2$ and $U = -V$ so that

$$g = C_0 e^{2z}(e^{2\omega z} - U^2)$$

and both conclusions of the claim hold.

We now consider the case $b_{k-2} \equiv 0$. Then from (2.7),

$$h = -\frac{k-1}{2} g' + C_3.$$
for some constant $C_3$. At a pole of $f$ of order $m$, we have $h = mg'$. Thus $g' = C_3(m + (k - 1)/2)^{-1}$ at such a pole. But we also have $(g')^{-k} = (-1)^k m(m + 1) \cdots (m + k - 1)$ at such a pole of $f$. This gives

$$\psi(m) = m(m + 1) \cdots (m + k - 1)(m + (k - 1)/2)^{-k} = C_4$$

for some constant $C_4$. Now by the Cauchy–Schwarz inequality, $\psi'/\psi > 0$ for $m > 0$, so that $\psi' > 0$ on $(0, +\infty)$. Thus all poles of $f$ have a fixed multiplicity and the claim is proved in full.

7. Conclusion of the Proof of Theorem 2

Suppose first that the following holds:

all poles of $f$ have a fixed multiplicity $m$. \hspace{1cm} (7.1)

Recall that we are assuming that $f$ has infinitely many poles, the contrary case having been dealt with. Now (7.1) implies that $v = (u' + 1/m)/u$ is entire. By (6.4) and the fact that $u$ has finite order, $v$ must in fact be a polynomial, and this leads to (1.4).

Therefore to conclude the proof of Theorem 2 we need only show that (6.10) implies (7.1). Of course we may assume that $b_{k-2}$ is not identically zero.

Now with $R_3, \ldots$ denoting rational functions, (6.10) and (6.12) imply that

$$e^p = R_3 g' + R_4 g.$$ 

But by (6.6), $g = -(f/f')H = -ue^p$ so that

$$u' + R_5 u = R_6.$$ \hspace{1cm} (7.2)

Now $f' = f/u$ so that $f'' = f(1-u')/u^2$ and $f^{(3)} = f(1-3u'+2u'^2)/u^3 - fuu''/u^3$. We claim that for $j \geq 3$,

$$f^{(j)} = f(Q_j(-u') + uS_j(u))/u^j,$$ \hspace{1cm} (7.3)

where $Q_j$ is a polynomial with positive coefficients and degree $(j - 1)$, and $S_j$ is a differential polynomial in $u$, with constant coefficients and degree at most $(j - 2)$. We prove this by induction. Now (7.3) gives

$$f^{(j+1)} = f(Q_j + uS_j)/u^{j+1} + f(-ju')(Q_j + uS_j)/u^{j+1}$$

$$+ f \left( \frac{d}{dz} (Q_j(-u') + uS_j) \right) / u^{j+1}.$$
Thus to prove (7.3) we need only set

\[ Q_{j+1} = (-ju' + 1) Q_j \]

and

\[ S_{j+1} = S_j - ju'S_j + \frac{d}{dz} (Q_j(-u') + uS_j). \]

Now substituting (7.3) into the definition of \( F \),

\[ \frac{F}{f} u^k = (-e^{-P})^k = Q_k(-u') + uS(u), \quad (7.4) \]

where \( S(u) \) is a differential polynomial in \( u \) of degree at most \( (k - 1) \), with polynomial coefficients. Using (7.2) and (7.4) we can also write

\[ \frac{F}{f} u^k = \lambda_n T(u) = \lambda_n u^n + \ldots + \lambda_0, \quad (7.5) \]

where \( n \leq k \) and \( \lambda_0, \ldots, \lambda_n \) are rational. If \( n = 0 \) (and \( \lambda_n T(u) \equiv \lambda_0 \)) then \( \lambda_0 \) must be a polynomial. From (7.4) and (7.5), if \( \zeta \) is a pole of \( f \) of order \( m \), we get

\[ Q_k(1/m) = \lambda_0(\zeta) \]

and conclude that (7.1) holds. It is clear from (7.4) that (7.1) also holds if \( P \) is constant. To finish the proof we suppose, therefore, that \( P \) is non-constant and \( 1 \leq n \leq k \). Since \( Fu^k/f \) has no zeros or poles the Turnura-Clunie theorem [4, p. 69] gives

\[ \lambda_n T(u) = \lambda_n(u + \lambda)^n, \quad (7.6) \]

where

\[ T(r, \lambda) = S(r, u). \]

Since \( u \) is transcendental over the field of functions \( b^* \) satisfying \( T(r, b^*) = S(r, u) \), \( \lambda \) must be rational. So (7.4), (7.5), and (7.6) give

\[ u = -\lambda + \mu e^{-P/k/n}, \quad (7.7) \]

where \( \lambda \) and \( \mu \) are rational, and \( \mu^n = (-1)^k/\lambda_n \). Thus

\[ g = -ue^P = \lambda e^P - \mu e^{(n-k)P/n}. \quad (7.8) \]
Setting \( v = \mu e^{(n-k)p/n} \) and substituting (7.8) in (6.11) we see that
\[
- \frac{k-1}{2} v'' - av' - \left( \frac{b_{k-2}}{k} + a' \right) v = 0.
\]
Thus \( v \) is entire, and \( \mu \) is a polynomial. From (7.7), since \( u \) is entire, \( \lambda \) is a polynomial. Thus we can write
\[
\lambda = -\lambda + \mu e^u,
\]
with \( \lambda, \mu, q \) polynomials, and \( q \) non-constant. Now \( u = 0 \) gives
\[
u' = -\lambda' + (\mu' + q'\mu) e^u = (q' + o(1))\lambda. \tag{7.9}
\]
(Note that \( \lambda \not\equiv 0 \), since \( f \) has infinitely many poles.) Now (7.9) implies that \( q' \) and \( \lambda \) are constant, and we have (7.1). This completes the proof of Theorem 2.

8. The Case \( k = 2 \) in Theorem A

The estimate (1.2) of Theorem A is not known in the case \( k = 2 \) except where \( f \) is entire. However, we can handle the following special case.

Theorem 3. Suppose that \( f \) is meromorphic in the plane and that
\[
F = f'' + a_1 f' + a_0 f,
\]
where \( a_1, a_0 \) are polynomials and \( a_0 \) is not identically zero. If \( ff'F \) has no zeros, then \( f \) satisfies (1.3), (1.4), (1.5), or (1.6).

Proof. As before, write \( u = f/f' \) so that \( u \) is entire and
\[
u^2 F/f = 1 - u' + a_1 u + a_0 u^2. \tag{8.1}
\]
Now if \( u \) is a polynomial then as in Section 6, \( u \) must have degree at most 1 and (1.5) or (1.6) holds. If \( u \) is transcendental then by [4, p. 69] again we have
\[
u^2 F/f = a_0 (u - v)^2, \tag{8.2}
\]
where \( T(r, v) = S(r, u) \). Writing \( U = u - v \), we obtain
\[
UD = v' - a_1 v - a_0 u^2 - 1, \tag{8.3}
\]
where
\[
D = a_1 + 2va_0 - U'/U.
\]
Since \( T(r, D) + T(r, v) = S(r, U) \), both the right-hand side of (8.3) and \( D \) must vanish identically. Thus \( v \) is rational and \( U \) (and hence \( u \) also) has finite order.

It now follows that if \( f \) is entire then (1.3) holds. Finally if \( f \) is not entire, \[
1 - u' + a_1 u + a_0 u^2 = a_0 u^2 - 2va_0 u + v^2 a_0
\]
and, using (8.2), \( a_0 v^2 \) is a polynomial. As before, all poles of \( f \) have a fixed multiplicity \( m \), say, so that \( (u' + 1/m)/u = Q \) is entire. Since \( u \) has finite order, (2.6) of Lemma 1 implies that \( Q \) is a polynomial, so that (1.4) holds.

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