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L^p Exponential Stability for the Equilibrium Solutions of the Navier–Stokes Equations

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The exponential stability of the solution of the Navier–Stokes equation in L^p, $p > 2, p \neq 3$, in bounded domain is considered in this paper. Under some assumptions on the external force, it can be shown that the bounded solution of the Navier–Stokes equation with initial and boundary conditions approaches the stationary solution of the system exponentially when time t goes to infinite. © 1995 Academic Press, Inc.

1. INTRODUCTION

About 30 years ago, Prodi studied in [11] the uniqueness of solutions of Navier–Stokes equations in L^p ($p > 3$) and in [12] the exponential stability of the stationary solutions of the Navier–Stokes systems in L^2 for bounded domain. Recently, for Navier–Stokes equations, Schonbek in [13] obtained the L^2 decay for weak solutions of the equations in the whole domain. A similar problem in R^n was considered by Kajikiya and Miyakawa [10]. For bounded domain, Foias and Saut 5, 6, 7, and 8 considered the L^2-exponential decay for solutions of the equations. The result of exponential stability of the equilibrium solutions in L^2 can also be found in Temam [14]. In the space of L^p, Da Veiga and Secchi [3] proved the L^p stability for the strong solutions in the whole domain. In this short

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note we will prove the L^p ($p > 2$) exponential stability of the equilibrium solutions of the Navier–Stokes equations in bounded domain, under some assumptions on the external forces.

The Navier–Stokes equations with initial-boundary conditions are as the following:

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla\pi &= f(x) && \text{in } Q = (0, \infty) \times \Omega; \\ \operatorname{div} u &= 0 && \text{in } Q; \\ u|_{t=0} &= u_0 && \text{in } \Omega; \\ u|_{\partial\Omega} &= 0 && \text{for } t \in (0, \infty), \end{aligned} \tag{1.1}$$

where $u(t, x)$, π, $f(x) = \nabla f_0(x)$, $u_0(x)$ stand for velocity, pressure, potential force, and initial velocity, respectively. The equilibrium solution satisfies the system

$$\begin{aligned} (v \cdot \nabla))v - \nu\nabla v + \nabla p &= f(x) && \text{in } \Omega; \\ \operatorname{div} v &= 0 && \text{in } \Omega; \\ v|_{\partial\Omega} &= 0. \end{aligned} \tag{1.2}$$

We will prove that under some conditions on $f(x)$ there exist constants C, c_0, β, independent of t, u, such that

$$|u - v|_p \leq C|u_0 - v|_p^{\beta} e^{-c_0 t} \tag{1.3}$$

for all $t > 0$, where $|\cdot|_p$ stands for the usual L^p norm.

This paper is arranged in the following order: We will introduce some definitions and notations in Section 2. A regularity result for the weak solution of (1.1) due to von Wahl [15] will be presented. We will establish some inequalities which will be used later in the article in deriving the exponential inequality. Finally, in Section 3, the main result of the L^p ($p > 2$, $p \neq 3$) exponential stability of the equilibrium solution of the Navier–Stokes equation will be established.

2. Lemmas and Inequalities

Assume that Ω is a bounded domain in R^3 with smooth boundary. Denote the standard Sobolev norm of L^p as $|\cdot|_p$, the L^2 norm as $\|\cdot\|$.

Definition 2.1. Let $u_0 \in H_2(\Omega) \cap L^p(\Omega)$, $f \in L^2((0, T), (H^{-1,2}(\Omega)^2)$. An element

$$u \in L^\infty((0, T), (L^2(\Omega))^2) \cap L^2((0, T), (H_0^{1,2}(\Omega))^2)$$

which fulfills div $u(t) = 0$ for almost all $t \in (0, T)$ and which is weakly continuous in $(L^2(\Omega))^2$ from $[0, T]$ to $(L^2(\Omega))^n$ is called a weak solution of (1.1) if

$$-\int_0^T (u, \phi')\,dt + \nu \int_0^T (\nabla u, \nabla\phi)\,dt - \int_0^T (u \cdot u, \nabla\phi)\,dt = (u_0, \phi(0)) + \int_0^T (f, \phi)\,dt \tag{2.1}$$

for all testing functions

$$\phi \in C^{0,1}([0, T], (L^2(\Omega))^2) \cap C^0([0, T], (H_0^{1,2}(\Omega)^2) \cap L^\infty((0, T), (H_0^{1,1}(\Omega)^2)$$

with div $\phi(t) = 0$ on $[0, T]$, $\phi(T) = 0$.

The existence of such a weak solution is standard (cf. [9]). When $n = 2$ von Wahl proved in [15, Theorem IV.5.7] the following regularity result:

LEMMA 2.1. *Let $u_0 \in H_2(\Omega) \cap L^p(\Omega)$ and $f \in C^{1/2p}([0, T], (L^p(\Omega))^2) \cap C^0((0, T], (H^{1/p,p}(\Omega))^2)$ for $p > 2$ and all $T > 0$. Then the solution of* (1.1), *in the sense of Definition* 2.1 *over* $(0, T) \times \Omega$ *for all* $T > 0$, $u \in C^0((0, \infty), H^{2\ p}(\Omega))^2)$, *applies.*

The following result can be found in Temam [14].

LEMMA 2.2. *If the L^2 norm of f is small enough* (*see Theorem* 10.2 *in* [14] *for the exact condition, for instance*) *and* $v \in \{u, \Delta u \in H^{0,2}(\Omega)$, div $u = 0$ *in* $\Omega\}$ *solves* (1.2), *then*

$$\|u(t) - v\| \leq \|u_0 - v\| e^{-ct}$$

where c is a positive constant independent of u.

To establish the exponential stability of the equilibrium solutions of the Navier–Stokes equations in L^p, we need the following results.

LEMMA 2.3. *If* $u \in (L^p(0, T, W^{1,p}))^2 \cap (L^\infty(0, T, L^p))^2$, div $u = 0$, *then* $|u|^{p-2}u \in (L^q(0, T, W^{1,q}))^2 \cap (L^\infty(0, T, L^q))^2$ *where* $1/p + 1/q = 1$.

Proof. The simple identity $|\,|u|^{p-2}u|_q = |u|_p^{p/q}$ implies that

$$|u|^{p-2}u \in (L^q(0, T, L^q))^2 \cap (L^\infty(0, T, L^q))^2. \tag{2.2}$$

Next, by using the condition div $u = 0$, it can be shown that

$$|\nabla \cdot |u|^{p-2}u|_q^q \leq c \int_\Omega \sum_{i,j=1}^2 |u|^{(p-2)q}|u_{j,i}|^q\,dx \tag{2.3}$$

for some positive constant c. Applying the Hölder inequality, we obtain

$$|\nabla \cdot |u|^{p-2} u|_q^q \leq c|u|_p^{p-q}|u|_{W^{1,p}}^q \leq c(|u|_p^p + |u|_{W^{1,p}}^p). \tag{2.4}$$

Equations (2.2) and (2.4) thus imply that $|u|^{p-2} u \in (L^q(0, T, W^{1,q}))^2$. ∎

Integration by parts shows that

$$\int_\Omega \nabla u \cdot \nabla (|u|^{p-2} u)\, dx = \int_\Omega |\nabla u|^2 |u|^{p-2}\, dx + \frac{4(p-2)}{p^2} \int_\Omega (\nabla |u|^{p/2})^2\, dx. \tag{2.5}$$

If $u(\cdot) \in H_0^{1,p}$, by applying Hölder's inequality it can be shown that the integrals on the right side of (2.5) are defined. For all u satisfying div $u = 0$, let us denote $[u]_p = \int_\Omega |\nabla u|^2 |u|^{p-2}\, dx$. The integral is well defined according to (2.5). We have the following result:

LEMMA 2.4. *If* $u \in (L^p(0, T, W_0^{1,p}))^2 \cap (L^\infty[0, T], L^p))^2$, div $u = 0$, $v \in L^\infty(0, T, L^p))^2$ *then*

$$\int_\Omega (u \cdot \nabla) v \cdot |u|^{p-2} u\, dx \leq c|v|_{p+2}[u]_p^{(p+5)/2(p+2)} |u|_p^{p(p-1)/2(p+2)} \tag{2.6}$$

for all $t \in [0, T]$.

Proof. It takes only simple calculus and Cauchy's inequality to show that

$$\left| \int_\Omega (u \cdot \nabla) v \cdot |u|^{p-2} u\, dx \right| \leq (p-1) \int_\Omega |u|^{p-1} |v|\, |\nabla u|\, dx. \tag{2.7}$$

Applying Cauchy's inequality and then Hölder's inequality on the right side of (2.7), we have

$$\begin{aligned} \left| \int_\Omega (u \cdot \nabla) v \cdot |u|^{p-2} u\, dx \right| &\leq (p-1) \left(\int_\Omega |\nabla u|^2 |u|^{p-2}\, dx \right)^{1/2} \left(\int_\Omega |v|^2 |u|^p\, dx \right)^{1/2} \\ &\leq (p-1) \left(\int_\Omega |\nabla u|^2 |u|^{p-2}\, dx \right)^{1/2} \left(\int_\Omega |v|^{p+2}\, dx \right)^{1/(p+2)} \\ &\quad \left(\int_\Omega |u|^{p+2}\, dx) \right)^{p/2(p+2)} \\ &= (p-1) \left(\int_\Omega |\nabla u|^2 |u|^{p-2}\, dx \right)^{1/2} |v|_{p+2} |u|_{p+2}^{p/2}. \end{aligned} \tag{2.8}$$

To estimate (2.8), let us observe that, similar to Eq. (1.14) in [2], we can show

$$|u|_{p+2}^{p+2} \leq c|u|_p^{p-1}[u]_p^{3/p} \tag{2.9}$$

where in (2.9) $[u]_p = \int_\Omega |\nabla u|^2|u|^{p-2}\,dx$. Then (2.6) is derived from (2.8) and (2.9). ∎

LEMMA 2.5. *If* $u \in (L^p(0, T, W_0^{1,p}))^2 \cap (L^\infty(0, T, L^p))^2$, $w, v \in (L^\infty(0, T, L^p))^2$, *then*

$$\int_\Omega (w \cdot \nabla)v \cdot |u|^{p-2}u\,dx \leq c[u]_p^{1/2+3(p-2)/2(p+2)}|u|_p^{(p-1)(p-2)/2(p+2)}|v|_{p+2}|w|_{p+2}. \tag{2.10}$$

Proof. It can be shown that

$$\left|\int_\Omega (w \cdot \nabla)v \cdot |u|^{p-2}u\,dx\right| \leq (p-1)\int_\Omega |u|^{p-2}|w|\,|v|\,|\nabla u|\,dx. \tag{2.11}$$

Using Cauchy's inequality, Hölder's inequality, and the definition of $[\cdot]_p$ we get

$$\begin{aligned}\left|\int_\Omega (w \cdot \nabla)v \cdot |u|^{p-2}u\,dx\right| &\leq (p-1)\left(\int_\Omega |\nabla u|^2|u|^{p-2}\,dx\right)^{1/2}\\ \left(\int_\Omega |v|^2|w|^2|u|^{p-2}\,dx\right)^{1/2} &\leq (p-1)[u]_p^{1/2}\left(\int_\Omega |u|^{p+2}\,dx\right)^{(p-2)/2(p+2)}\\ \left(\int_\Omega |v|^{(p+2)/2}|w|^{(p+2)/2}\,dx\right)^{2/(p+2)} &\leq (p-1)[u]_p^{1/2}|u|_{p+2}^{(p-2)/2}|v|_{p+2}|w|_{p+2}.\end{aligned} \tag{2.12}$$

Substituting (2.9) into (2.12) yields

$$\int_\Omega (w \cdot \nabla)v \cdot |u|^{p-2}u\,dx \leq c[u]_p^{1/2+3(p-2)/2(p+2)}|u|_p^{(p-1)(p-2)/2(p+2)}|v|_{p+2}|w|_{p+2}. \tag{2.13}$$ ∎

Last result we will prove in this section is the following:

LEMMA 2.6. *Let us make the following assumptions:*

1. *There exists a constant* $M > 0$ *such that for all* $t > 0$ *the function* $y(t) < M$.

2. *There exist differentiable functions* $g(t) > 0$ *and continuous functions* $f_1(t), f_2(t), \ldots, f_n(t)$ *for* $t > 0$ *such that*

$$\lim_{t\to\infty}\frac{g'}{g} = l\,(>0); \qquad \lim_{t\to\infty}\frac{f_i(t)}{g(t)} = 0,\ i = 1, \ldots, n.$$

3. *There are constants* $a_1 > a_2 > \cdots > a_n > a_0 > 1$.
4. *Function* $y(t)$ *satisfies the differential inequality*

$$y' + c\,g\,y^{a_0} \le b_0 y + f_1 y^{a_1} + \cdots + f_n y^{a_n} \tag{2.14}$$

where c, $b_0 > 0$. *Then the estimate*

$$y(t)^{a_0-1} \le \frac{b_0}{cl}\, g(t)^{-1} \tag{2.15}$$

holds for all $t \in [0, \infty)$.

Proof. From (2.14) we have

$$\frac{d}{dt}(e^{-b_0 t} y) \le -e^{-b_0 t} cgy^{a_0} + e^{-b_0 t} f_1 y^{a_1} + \cdots + e^{-b_0 t} f_n y^{a_n}. \tag{2.16}$$

Therefore, by dividing both sides by $(e^{-b_0 t} y)^{a_0}$ and integrating from t_0 to t, we obtain

$$\begin{aligned} e^{-b_0 t(1-a_0)} y^{1-a_0}(t) \ge y^{1-a_0} + (1-a_0)\int_{t_0}^{t} e^{-b_0(1-a_0)s}\,(-cg(s) \\ + f_1(s)\, y^{a_1-a_0} + \cdots + f_n(s)\, y^{a_n-a_0})\, ds. \end{aligned} \tag{2.17}$$

Equation (2.17) implies that

$$y^{a_0-1} \le \frac{e^{b_0(a_0-1)t}}{y^{1-a_0}(t_0) + (a_0-1)\int_{t_0}^{t} e^{-b_0(1-a_0)s} (cg(s) - f_1(s) y^{a_1-a_0} - \cdots - f_n(s) y^{a_n-a_0})\, ds}. \tag{2.18}$$

Choose t_0 large enough so that the integrand of the integral on the right of (2.18) is positive. By dividing top and bottom of the right side fraction by $e^{b_0(a_0-1)t} g(t)$ and using the assumptions 1 and 2 in the lemma and L'Hopital's rule, we can easily show that (2.15) holds. ∎

3. Main Results

In this section we will prove our main result: the equilibrium solutions of (1.1) are exponentially stable in L^p topology.

It is well known that the solution of the steady state problem (1.2) can be very smooth if $f(x)$ and the boundary of Ω are smooth enough (cf.

Constantin and Foias [4]). Let us denote the bounded solution of (1.1) in L^p by u, the solution of (1.2) by v. If $f(x) = \nabla f_0(x)$ satisfies conditions of Lemma 2.1, and $f(x) \in L^p(\Omega)$, then $u \in C^0((0, \infty), H^{2,p}(\Omega))^n)$, $v \in L^{p+2}(\Omega)$. The difference $w = u - v$ obviously solves the system

$$\begin{aligned} &\frac{dw}{dt} + (u \cdot \nabla) w + (w \cdot \nabla) v - \nu \Delta w + \nabla(p - \pi) = 0 \qquad \text{in } Q; \\ &\operatorname{div} w = 0 \qquad \text{in } Q; \\ &w(0, x) = u_0 - v \qquad \text{in } \Omega; \\ &w|_{\partial\Omega} = 0 \qquad \text{for } t \in (0, \infty), \end{aligned} \tag{3.1}$$

where $Q = (0, \infty) \times \Omega$. Let us denote $P = p - \pi$. We see from (3.1) that

$$\frac{dw}{dt} + (u \cdot \nabla) w + (w \cdot \nabla) v - v \Delta w + \nabla P = 0. \tag{3.2}$$

Similar to [3], we can show that

$$|P|^2_{(p+2)/2} \leq C|w|^2_{p+2} (|v|^2_{p+2} + |w|^2_{p+2}). \tag{3.3}$$

Multiplying both sides of (3.1) by $|w|^{p-2} w$, integrating the resulting equation over Ω, and applying (2.5) implies

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} |w|_p^p + [w]_p \leq &- \int_\Omega (u \cdot \nabla) w \cdot |w|^{p-2} w \, dx - \int_\Omega (w \cdot \nabla) v \cdot |w|^{p-2} w \, dx \\ &- \int_\Omega (\nabla P) \cdot |w|^{p-2} w \, dx. \end{aligned} \tag{3.4}$$

The first integral on the right side is zero because u is divergence free. Applying Lemma 2.4 to the second integral we find that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} |w|_p^p + [w]_p &\leq (p - 1) \int_\Omega |w|^{p-1} |\nabla w| \, |v| \, dx + (p - 2) \int_\Omega |P| \, |w|^{p-2} |\nabla w| \, dx \\ &\leq C|v|_{p+2} [w]_p^{(p+5)/2(p+2)} |w|_p^{p(p-1)/2(p+2)} \\ &\quad + C[w]_p^{1/2} |w|_{p+2}^{(p-2)/2} |P|_{(p+2)/2}. \end{aligned} \tag{3.5}$$

For the terms on the right side of (3.5), let us apply (3.3), Hölder's inequality ($a \cdot b \leq a^p/p + b^q/q$, where $1/p + 1/q = 1$) and Cauchy's inequalities to obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} |w|_p^p + [w]_p \leq \frac{1}{4} [w]_p &+ C|v|_{p+2}^{2(p+2)/(p-1)} |w|_p^p + C|w|_{p+2}^p |v|_{p+2}^2 \\ &+ C|w|_{p+2}^{p+2}. \end{aligned} \tag{3.6}$$

Applying (2.9) to the third and fourth terms on the right side of (3.6) and then Hölder's inequality to the resulting inequality we have, if $p \neq 3$,

$$\frac{1}{p}\frac{d}{dt}|w|_p^p + [w]_p \leq \frac{1}{2}[w]_p + C|v|_{p+2}^{2(p+2)/(p-1)}|w|_p^p + C|w|_p^{p(p+2)/(p-3)}. \quad (3.7)$$

Therefore we arrive at

$$\frac{1}{p}\frac{d}{dt}|w|_p^p + \frac{1}{2}[w]_p \leq C|v|_{p+2}^{2(p+2)/(p-1)}|w|_p^p + C|w|_p^{p(p+2)/(p-3)}. \quad (3.8)$$

By applying the interpolation result we find that

$$[w]_p \geq C\,\big|\,|w|^{p/2}\big|_6^2 = C|w|_{3p}^p. \quad (3.9)$$

Sobolev inequality further implies that

$$[w]_p \geq C|w|_2^{-4p/3(p-2)}|w|_p^{p+4p/3(p-2)}. \quad (3.10)$$

Denote $k = 4p/3(p-2)$. By applying Lemma 2.2 to the right side of (3.10), we have from (3.8) that

$$\frac{1}{p}\frac{d}{dt}|w|_p^p + C|w_0|_2^{-k}e^{ckt}|w|_p^{p+k} \leq C|w|_p^{p+k+r_1} + C|w|_p^{p+k+r_2} \quad (3.11)$$

where $r_1 = 2p^2/3(p-2)(p-3)$, $r_2 = p(11p-18)/3(p-2)(p-3)$, and $w_0 = u_0 - v$. Applying Lemma 2.6 to $y = |w|_p^p$ for $p > 3$ we see then it is easy to prove that there exists constants C, c_0 such that

$$|w|_p \leq C|w_0|_2^{\beta}e^{-c_0t} \qquad \text{for all } t > 0, \quad (3.12)$$

where $\beta = 4/3(p-2)$. So we finally have proven the following asymptotic stability theorem:

THEOREM 3.1. *Denote the bounded solution of* (1.1) *in* L^p *by* u, *the solution of* (1.2) *by* v. *Suppose that* $f(t,x) = \nabla f_0(x)$ *satisfies conditions of Lemma* 2.1, *the* L^2 *norm of* f *is small enough, and* $u_0 \in H_2 \cap L^p$, $p > 3$, *then there exists constants* C, c_0, *independent of* t, u, *such that*

$$|u(t,x) - v(x)|_p \leq C|u_0(x) - v(x)|_p^{\beta}e^{-c_0t} \qquad \textit{for all } t > 0. \quad (3.13)$$

Combining Lemma 2.2, Theorem 3.1, and the embedding theorems for L^p spaces when $p \geq 2$, we obtain a more general theorem:

THEOREM 3.2. *The result of Theorem* 3.1 *holds for all* $p \geq 2$, $p \neq 3$.

Note that the result of Theorem 3.2 in L^p, $p \geq 2$, is the generalization of the asymptotic stability result in L^2 that Prodi obtained in [12].

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