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Test ideals in diagonal hypersurface rings, II

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Abstract

Let $R = k[x_1, \ldots, x_n]/(x_1^d + \cdots + x_n^d)$, where k is a field of characteristic p, p does not divide d and $n \geqslant 3$. We describe a method for computing the test ideal for these diagonal hypersurface rings. This method involves using a characterization of test ideals in Gorenstein rings as well as developing a way to compute tight closures of certain ideals despite the lack of a general algorithm. In addition, we compute examples of test ideals in diagonal hypersurface rings of small characteristic (relative to d) including several that are not integrally closed. These examples provide a negative answer to Smith's question [K.E. Smith, The multiplier ideal is universal test ideal, Comm. Algebra 28 (12) (2000) 5912–5929] of whether the test ideal in general is always integrally closed.

Keywords: Test ideal; Tight closure

Introduction

Test ideals play an important role in the theory of tight closure developed by Melvin Hochster and Craig Huneke. Unfortunately, both tight closure and test ideals are difficult to compute in general. In this paper we describe a method for computing the test ideal for diagonal hypersurfaces $k[x_1, \ldots, x_n]/(x_1^d + \cdots + x_n^d)$, where k is a field of characteristic p, p does not divide d and $n \geqslant 3$. This method involves using a characterization of test ideals in Gorenstein rings as well as developing a way to compute tight closures of certain ideals despite the lack of a general algorithm.

It is worth noting that the test ideals for these diagonal hypersurface rings are very different depending on the magnitude of p (usually relative to d). If $d \geqslant n$ and the

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dimension of the ring is two, then results of Huneke and Smith [10] show that the test ideal is $(x_1, x_2, x_3)^{d-3+1}$ for $p > d$. In this case the test ideal is essentially all elements of degree greater than the a-invariant. This follows from their tight closure interpretation of the Strong Kodaira Vanishing Theorem. Huneke and Smith also point out that the Vanishing Theorem is true for hypersurface rings with a similar restriction on the characteristic. Huneke gives a direct proof of the Strong Vanishing Theorem for hypersurfaces [12, (6.4)] using ideas found in earlier work of Fedder [2]. There are also similar results in [5, Cor 3] using the idea of F-injectivity in negative degree. For the diagonal hypersurfaces, the Strong Vanishing Theorem implies that if $d \geqslant n$, then the test ideal is $(x_1, \dots, x_n)^{d-n+1}$ for sufficiently large p, and if $d < n$, then the test ideal is the unit ideal [12, (6.3)]. Fedder and Watanabe also have results that show that if $d < n$, then the ring is F-regular [3, (2.11)] and hence the test ideal is the unit ideal, again for sufficiently large p. On the other hand, we show in [14] that if $p < d$, then the test ideal is contained in $(x_1, \dots, x_n)^{p-1}$, which is much smaller than would be expected in many cases. This result does not depend on n. We also show in [14] that if $p = d - 1$, then the test ideal is in fact equal to $(x_1, \dots, x_n)^{p-1}$.

We are interested in computing test ideals in these rings when p is less than the bounds required in [2,5,12] and [3]. For $k[x_1, \dots, x_n]/(x_1^d + \cdots + x_n^d)$, where k is a field of characteristic p, the bound in [2] is $p > n(d-1) - d$. The bound in [12] is $p > n(d-1) - 2d + 1$ and the bound in [5] is $p > n(d-1) - 2d$. The bound in [3] comes from the bound in [2] and so is also $p > n(d-1) - d$. It is quite likely that these are not the best possible bounds. The bound in the two dimensional case, $p > d$ [10], is quite a bit better than these. In many examples when the dimension is greater than two, the bound of $p > d$ is sufficient. For this reason, and the fact that the case when $p = d - 1$ is known [14], we are particularly interested in computing test ideals when $p < d - 1$. We do have one example (see Example 13) where p is greater than d but less than the bound in [2,3,12], and [5], and the ring is not F-regular as predicted.

In this paper we describe a method for computing test ideals in diagonal hypersurface rings. We then use this method to compute many examples of test ideals when $p < d - 1$ and when p is less than the previously mentioned bounds.

Recently Karen Smith has shown that test ideals are closely related to certain multiplier ideals that arise in vanishing theorems in algebraic geometry. In [17] Smith established that for normal local Cohen Macaulay $\mathbb{Q}$-Gorenstein rings essentially of finite type over a field of characteristic zero, the multiplier ideal is a *universal test ideal* (see [15] and [1] for more information about the multiplier idea). Similar results were also obtained independently by Hara in [6]. The multiplier ideal is integrally closed. This lead Smith to ask whether the test ideal in general is integrally closed. Several of the examples we are able to compute are particularly interesting because they are not integrally closed. These examples provide a partial negative answer to Smith's question.

1. Notation and definitions

Throughout this paper R is a commutative Noetherian ring of prime characteristic $p > 0$. The letter q will always stand for a power p^e of p, where $e \in \mathbb{N}$.

We review the definition of tight closure for ideals of rings of characteristic $p > 0$. Tight closure is defined more generally for modules and also for rings containing fields of arbitrary characteristic. See [7] or [11] for more details.

Definition 1. Let R be a ring of characteristic p and I be an ideal in a Noetherian ring R of characteristic $p > 0$. An element $u \in R$ is in the tight closure of I, denoted I^*, if there exists an element $c \in R$, not in any minimal prime of R, such that for all large $q = p^e$, $cx^q \in I^{[q]}$ where $I^{[q]}$ is the ideal generated by the qth powers of all elements of I.

In many applications one would like to be able to choose the element c in the definition of tight closure independent of x or I. It is very useful when a single choice of c, a test element, can be used for all tight closure tests in a given ring.

Definition 2. The ideal of all $c \in R$ such that for any ideal $I \subseteq R$, we have $cu^q \in I^{[q]}$ for all q whenever $u \in I^*$ is called the test ideal for R and is denoted by τ. An element of the test ideal that is not in any minimal prime is called a test element.

2. Determining the test ideal

We will make use of the following grading in our calculation of the test ideal. We denote by $\mathbb{Z}_n$ the ring $\mathbb{Z}/n\mathbb{Z}$. Next we describe a $\mathbb{Z}_n$-grading of rings of the form $R = A[z]/(z^n - a)$ where $a \in A$. The ring R has the following decomposition as an A-module:

$$R = A \oplus Az \oplus \cdots \oplus Az^{n-1}.$$

This is true because every element of R can be uniquely expressed as an element of $A \oplus Az \oplus \cdots \oplus Az^{n-1}$ by replacing every occurrence of z^n by a. R is $\mathbb{Z}_n$-graded, where the ith piece of R, denoted by R_i, is Az^i, $0 \leqslant i < n$, since $Az^i Az^j \subseteq Az^{i+j}$ if $i + j < n$ and $Az^i Az^j \subseteq Az^{i+j-n}$ if $i + j \geqslant n$.

We use this idea to obtain multiple $\mathbb{Z}_n$-gradings of $R = k[x_1, \ldots, x_n]/(x_1^d + \cdots + x_n^d)$, where k is a field of characteristic p. Let $z = x_i$ and $A = k[x_1, \ldots, \hat{x}_i, \ldots, x_n]$.

It is not difficult to show that if I is a graded ideal, then so is I^*.

Lemma 3 [14]. *Let R be a finitely generated k-algebra that is $\mathbb{Z}_n$-graded and of characteristic p, where p is not a prime factor of n ($p = 0$ is allowed). Then the tight closure of a homogeneous ideal of R is homogeneous.*

We will use the following result about the test ideal in a Gorenstein ring to compute the test ideal for the diagonal hypersurfaces.

Lemma 4 ([11, (2.14)], [12, (4.3)]). *Let (R, m) be a Gorenstein local ring with an m-primary test ideal. Let I be generated by a system of parameters consisting of test elements. Then the test ideal is $I : I^*$.*

We also know that the elements of the Jacobian ideal are always test elements by Hochster and Huneke's "test elements via Lipman Sathaye" result [8, (1.5.5)], [12, (3.12)]. In practice, it is preferable to use $x_1^d, \dots, x_{n-1}^d$ as the sequence of test elements. This allows us to capitalize on certain symmetries that arise in the diagonal hypersurface rings. Because we have the defining relation $x_1^d + \cdots + x_n^d = 0$, the ideals generated by any $n-1$ of the dth powers of the variables are the same.

Let $R = k[x_1, \dots, x_n]/(x_1^d + \cdots + x_n^d)$ where k is a field of characteristic p, p does not divide d, and $n \geqslant 3$. Then $\tau = (x_1^d, \dots, x_{n-1}^d) : (x_1^d, \dots, x_{n-1}^d)^*$ and the problem of computing the test ideal reduces to determining $(x_1^d, \dots, x_{n-1}^d)^*$.

Remark 5. It is also worth noting that the test ideal for R can be generated by monomials. Recall that there is a $\mathbb{Z}_n$-grading of R associated with each x_i, $1 \leqslant i \leqslant n$. We also know that if I is a homogeneous ideal, then so are I^* (Lemma 3) and $I : I^*$. Since $(x_1^d, \dots, x_{n-1}^d)$ is homogeneous with respect to each of the gradings, so is $(x_1^d, \dots, x_{n-1}^d) :_R (x_1^d, \dots, x_{n-1}^d)^*$ and hence so is the test ideal. Using the grading with respect to each x_i, the multigrading, we see that the test ideal can be generated by monomials. Essentially, this is because only monomials are homogeneous with respect to all n gradings simultaneously.

3. Computing $(x_1^d, \dots, x_{n-1}^d)^*$

Let $R = k[x_1, \dots, x_n]/(x_1^d + \cdots + x_n^d)$ where k is a field of characteristic p, p does not divide d, and $n \geqslant 3$. First we describe a general method for computing $(x_1^d, \dots, x_{n-1}^d)^*$. Let $m = (x_1, \dots, x_n)$ and $I = (x_1^d, \dots, x_{n-1}^d)$.

(1) Let J be a candidate for I^*. (Begin with $J = I$.)
(2) Compute $\operatorname{Soc}(R/J) = J :_R m = (u_1, \dots, u_m)$.
(3) Determine whether $u_i \in I^*$, $1 \leqslant i \leqslant m$.
(4) Form a new candidate for I^* by adding all u_i's that are in I^* to J and repeat.

When no generators of $\operatorname{Soc}(R/J)$ are in I^*, the process is complete and $J = I^*$. Since R is Noetherian, the process must eventually end. Next we explain why it is sufficient to check elements of $\operatorname{Soc}(R/J)$.

Lemma 6. *Let (R, m, k) be a Noetherian local ring and $J \subseteq I$ ideals of R. If $J \subset I$, then I contains an element of* $\operatorname{Soc}(R/J)$.

Proof. Let $u \in I \backslash J$. We know that $\operatorname{Soc}(R/J)$ contains all simple submodules of R/J and therefore meets every submodule of R/J. Consider $N = Ru/J$, the submodule generated by u. If $N \neq (0)$, then $N \cap \operatorname{Soc}(R/J) \neq (0)$. Let $\bar{u}$ be the image of u in R/J. There exists $r \in R$ with $r\bar{u} \in \operatorname{Soc}(R/J)$. Then ru is the desired element of I.

Suppose J is a candidate for I^*. By Lemma 6, if I^* strictly contains J, then I^* must contain an element of $\operatorname{Soc}(R/J)$. Therefore, to show $J = I^*$, it is sufficient to show that $\sum k u_i \cap I^* = 0$ where $\operatorname{Soc}(R/J)$ is generated by the u_i.

The final step in the “algorithm” to be justified is step 3. The method described above is not a true algorithm since there is no known algorithm for computing tight closure, except in some special cases. *A priori*, one might have to test infinitely many exponents in order to determine whether one element is in the tight closure of a given ideal. Another way to describe this problem is in terms of test exponents as in [9].

Definition 7. Let R be a reduced Noetherian ring of positive prime characteristic p. Let c be a fixed test element for R. We shall say that $q = p^e$ is a *test exponent for* c, I, R if whenever $cu^Q \in I^{[Q]}$ and $Q \geqslant q$, then $u \in I^*$.

Whenever one can compute what the test exponent is, one obtains an effective test for tight closure. Despite the lack of test exponents, in practice, one can compute tight closure in diagonal hypersurface rings in many cases. The following two observations are helpful in determining whether a specific element is in the tight closure of a given ideal. Even though in general there is no bound on the power of an element needed to test tight closure, there are two situations where one exponent is enough.

Remark 8. Note that the Frobenius closure I^F of an ideal I is the set of elements u such that $u^q \in I^{[q]}$ for $q \gg 0$. It follows from elementary properties of the Frobenius map that if $u^{q'} \in I^{[q']}$ for one value of $q' = p^e$, then $u^q \in I^{[q]}$ for all higher powers $q \geqslant q'$. In fact, I^F is often defined to be the set of elements u such that $u^q \in I^{[q]}$ for some $q = p^e$. This means that if $u^q \in I^{[q]}$ for just one value of q, then $u \in I^F \subseteq I^*$.

Remark 9. Recall that if c is a test element, then $cu^q \in I^{[q]}$ for all $u \in I^*$ and for all $q = p^e$. This means that if $cu^q \notin I^{[q]}$ for even one q, then $u \notin I^*$.

In principle, this method would only work if $I^F = I^*$. There is some evidence that $I^F = I^*$ when $n = d = 3$ and $p \equiv 2 \bmod 3$ [13], however, we are not conjecturing that $I^F = I^*$ for all diagonal hypersurface rings. In practice, however, the potential gap between I^F and I^* has never prevented us from computing a test ideal. Instead, the current limitations are memory and monomial bounds in Macaulay 2 [4]. In almost every example, we show that an element is in the tight closure of an ideal by using the observation in (8) and showing that the element is actually in the Frobenius closure of the ideal. In a few cases, we have used the following result of Hara. Smith has a similar result [16, Lemma 3.2].

Lemma 10 [5, Lemma 2]. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian* **N***-graded ring defined over a perfect field $k = R_0$ of characteristic $p > 0$. Assume that R is Cohen–Macaulay. Let $x_1, \ldots, x_d$ be a homogeneous system of parameters of R, and assume that $d = \dim R \geqslant 1$. If a homogeneous element z satisfies $\deg(z) \geqslant \sum_{i=1}^{d} \deg(x_i)$, then $z \in (x_1, \ldots, x_d)^*$.*

It is interesting to note that every instance where we could not show that an element was in the Frobenius closure of an ideal by direct computation was an instance where Hara’s lemma applied.

4. Examples

Example 11. Let R be the localization at $(x_1,\dots,x_5)$ of the ring

$$\frac{k[x_1,\dots,x_5]}{(x_1^5+x_2^5+x_3^5+x_4^5+x_5^5)},$$

where k is a field of characteristic two. In this case the test ideal for R is generated by the elements $x_i^2x_j$ for all $1\leqslant i,j\leqslant 5$.

To verify this, we use the observation that in a Gorenstein ring with isolated singularity, the test ideal is $J:J^*$ where J is an ideal generated by a system of parameters that are test elements (Lemma 4). We also know that elements in the Jacobian ideal are test elements [8, (1.5.5)], so in this ring $x_1^4,\dots,x_5^4$ are test elements. Some of our calculations will be easier if we use the fact that $x_1^5,\dots,x_5^5$ are also test elements. Thus we use $J=(x_1^5,\dots,x_4^5)$. One can calculate directly that

$$J^*=\big(x_1^5,x_2^5,x_3^5,x_4^5,x_1^3x_2^3x_3^3x_4^3x_5^3,x_1^4x_2^4x_3^4x_4^4x_5^2,x_1^4x_2^4x_3^4x_4^2x_5^4,\\ x_1^4x_2^4x_3^2x_4^4x_5^4,x_1^4x_2^2x_3^4x_4^4x_5^4,x_1^2x_2^4x_3^4x_4^4x_5^4\big).$$

Using the $\mathbb{Z}_5$ grading we can assume that J^* is generated by elements of the form $u=x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$. Also any monomial of that form in $J^*\backslash J$ must have all $0<a_i<5$. Clearly, we must have all $a_i<5$ in order to have $u\notin J$. The fact that all $a_i>0$ follows from "tight closure from contractions" [11, (1.7)] since $k[x_1,\dots,x_5]/(x_1^5+\dots+x_5^5)$ is a module finite extension of $k[x_1,\dots,x_4]$. To verify that the monomials listed above are in J^* we use the observation in Remark 8, namely if $u^{q'}\in I^{[q']}$, then $u^q\in I^{[q]}$, $q\geqslant q'$, and hence $u\in I^*$. One easily checks that $u^4\in J^{[4]}$ for all monomials u listed above. It is also easy to check that no generators of the socle modulo the candidate for J^* are in J^*. Since J^* and m are both monomial ideals, it is routine to compute $J^*:m$, the socle modulo J^*. We compute $J^*:m$ and see that the socle modulo J^* has 25 generators that are not in J^*. Those generators are as follows:

$$u_1=x_1x_2^4x_3^4x_4^4x_5^4,\ u_2=x_1^4x_2x_3^4x_4^4x_5^4,\ \dots,\ u_5=x_1^4x_2^4x_3^4x_4^4x_5,\\ u_6=x_1^2x_2^3x_3^4x_4^4x_5^4,\ u_7=x_1^2x_2^4x_3^3x_4^4x_5^4,\ \dots,\ u_{25}=x_1^4x_2^4x_3^4x_4^3x_5^2.$$

We use $c=x_1^4$ as a test element and see that, for example, $cu_1^{32}\notin J^{[32]}$ and $cu_6^{16}\notin J^{[16]}$. Similar calculations and the observation in Remark 9 show that the remaining monomials are not in J^*. Computing $J:J^*$ gives the desired result.

Example 12. Let R be the localization at $(x_1,\dots,x_4)$ of the ring

$$\frac{k[x_1,\dots,x_4]}{(x_1^7+x_2^7+x_3^7+x_4^7)},$$

where k is a field of characteristic three. In this case the test ideal for R is generated by the elements $x_i^2x_j^2$ for all $1\leqslant i,j\leqslant 4$.

In this example we let $J=(x_1^7,x_2^7,x_3^7)$. One can calculate directly that

$$J^*=\left(x_1^7,x_2^7,x_3^7,x_1^3x_2^5x_3^5x_4^5,x_1^5x_2^3x_3^5x_4^5,x_1^5x_2^5x_3^3x_4^5,x_1^5x_2^5x_3^5x_4^3\right).$$

As in the previous example, we use the observation in Remark 8 and check that $u^3\in J^{[3]}$ for all monomials u listed above. We compute $J^*:m$ and see that the socle modulo J^* has 10 generators that are not in J^*. Those generators are as follows:

$$u_1=x_1^2x_2^6x_3^6x_4^6,\ u_2=x_1^6x_2^2x_3^6x_4^6,\ u_2=x_1^6x_2^6x_3^2x_4^6,\ u_4=x_1^6x_2^6x_3^6x_4^2,$$
$$u_5=x_1^4x_2^4x_3^6x_4^6,\ u_6=x_1^4x_2^6x_3^4x_4^6,\ \ldots,\ u_{10}=x_1^6x_2^6x_3^4x_4^4.$$

We use $c=x_1^6$ as a test element and see that, for example, $cu_1^9\notin J^{[9]}$ and $cu_5^9\notin J^{[9]}$. Similar calculations and the observation in Remark 9 show that the remaining monomials are not in J^*. Computing $J:J^*$ gives the desired result.

Example 13. Let R be the localization at $(x_1,\ldots,x_5)$ of the ring

$$\frac{k[x_1,\ldots,x_5]}{(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4)},$$

where k is a field of characteristic seven. In this case the test ideal for R is $(x_1,\ldots,x_5)$, the maximal ideal.

In this example we let $J=(x_1^4,x_2^4,x_3^4,x_4^4)$. One can calculate directly that

$$J^*=\left(x_1^4,x_2^4,x_3^4,x_1^3x_2^3x_3^3x_4^3x_5^3\right).$$

As in the previous example, we use the observation in Remark 8 and check that $(x_1^3x_2^3x_3^3x_4^3x_5^3)^7\in J^{[7]}$. We compute $J^*:m$ and see that the socle modulo J^* has 5 generators that are not in J^*. Those generators are as follows:

$$u_1=x_1^2x_2^3x_3^3x_4^3x_5^3,\ u_2=x_1^3x_2^2x_3^3x_4^3x_5^3,\ u_3=x_1^3x_2^3x_3^2x_4^3x_5^3,$$
$$u_4=x_1^3x_2^3x_3^3x_4^2x_5^3,\ u_5=x_1^3x_2^3x_3^3x_4^3x_5^2.$$

We use $c=x_1^3$ as a test element and see that, for example, $cu_1^7\notin J^{[7]}$. Similar calculations and the observation in Remark 9 show that the remaining monomials are not in J^*. Computing $J:J^*$ gives the desired result.

Remark 14. Using our notation, the previous example is the case where $n=5$, $d=4$ and $p=7$. Since $d<n$, the results of Fedder and Watanabe [3], Huneke [12] and Hara [5] would predict that the ring is F-regular if $p>11$, $p>8$ or $p>7$, respectively. Note that in the previous example $p>d$, but p is less than each of the bounds and the ring is not F-regular.

We have been able to compute the test ideal in the following cases:

d	p	n
4	7	5
5	2,3	3,4,5,6,7,8,9
7	2,3,5	3,4,5,6,7,8,9
8	3,5	3,4,5,6,7,8
9	2	3,4,5,6,7,8,9
9	5	3,4,5,6
9	7	3,4
10	3	3

We have not included any examples where $p > d$ and we obtained the predicted result, although we can compute many examples in those cases. Our computations tend to be limited by the degrees of the monomials involved and the number of generators of the ideals involved. As p and n grow, the degrees of the monomials grow, and as n grows, the number of generators of the ideals involved grows. Also, our examples do not represent the absolute limits of current computation. Computing further examples is incredibly time consuming and for a fixed p and d, the pattern as n increases tends to stabilize. We expect that future results will eventually make further computations unnecessary.

5. Non-integrally closed test ideals

Many of the test ideals that we can compute are not integrally closed. In this section we confirm that two of the examples of test ideals in the previous section are not integrally closed.

Example 15. The test ideal computed in Example 11 is not integrally closed. Let τ be the test ideal. The integral closure of τ is $(x_1, \dots, x_5)^3 \neq \tau$. For example, $x_1x_2x_3 \in (x_1, \dots, x_5)^3 \setminus \tau$.

Example 16. The test ideal computed in Example 12 is not integrally closed. Let τ be the test ideal. The integral closure of τ is $(x_1, \dots, x_4)^4 \neq \tau$. For example, $x_1x_2^3 \in (x_1, \dots, x_4)^4 \setminus \tau$.

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