Homotopy Method for Generalized Eigenvalue Problems $Ax = \lambda Bx^\pm$

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ABSTRACT

Generalized eigenvalue problems can be considered as a system of polynomials. The homotopy continuation method is used to find all the isolated zeros of the polynomial system which corresponds to the eigenpairs of the generalized eigenvalue problem. A special homotopy is constructed in such a way that there are exactly $n$ distinct smooth curves connecting trivial solutions to desired eigenpairs. Since the curves followed by general homotopy curve following scheme are computed independently of one another, the algorithm is a likely candidate for exploiting the advantages of parallel processing to the generalized eigenvalue problems.

1. INTRODUCTION

The generalized eigenvalue problem

$$Ax = \lambda Bx,$$  \hfill (1.1)
where \( x \in \mathbb{C}^n \) and \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are \( n \times n \) matrices, can be considered as a system of \( n+1 \) polynomials \( P = (p_1, p_2, \ldots, p_{n+1}) \) in \( n+1 \) variables \((\lambda, x_1, \ldots, x_n)\):

\[
P_1 = \lambda(b_{11}x_1 + \cdots + b_{1n}x_n) - (a_{11}x_1 + \cdots + a_{1n}x_n),
\]

\[
\vdots
\]

\[
P_n = \lambda(b_{n1}x_1 + \cdots + b_{nn}x_n) - (a_{n1}x_1 + \cdots + a_{nn}x_n),
\]

\[
P_{n+1} = c_1x_1 + \cdots + c_nx_n - 1,
\]

where \((c_1, \ldots, c_n) \in \mathbb{C}^n\) is chosen at random. An isolated zero \((\lambda, x_1, \ldots, x_n)\) of (1.2) corresponds to an eigenpair of (1.1) with eigenvector \((x_1, \ldots, x_n)\) normalized into the form \( P_{n+1} = 0 \).

Homotopy continuation method for finding all the isolated solutions of a system of polynomials was first given by Drexler [4,5] and Garcia and Zangwill [8,9]. It has attracted considerable attention recently. The basic idea of homotopy continuation is to construct a homotopy from a trivial map to the one of interest. Under suitable conditions, a smooth path starting from the solution of the trivial map will then lead us to a desired solution. For general reference of the underlying theory and some of its numerical treatments see [1,16].

Let \( p_1, \ldots, p_n \) be a system of polynomials in \( n \) variables with degree \( d_1, d_2, \ldots, d_n \) respectively. While the classical Bezout's theorem says that \( d = d_1d_2 \cdots d_n \) is the upper bound for the total number of isolated solutions (counting multiplicities) of this system, Garcia and Li [7] have proved that "generic" systems reach this upper bound. Therefore, the homotopies for general polynomial systems are usually constructed in such a way that \( d \) curves have to be followed in order to get all the isolated solutions [2,11,12,14]. For the situation we discuss here, those homotopies cannot be implemented when \( n \) is at all large, since there are at least \( 2^n - n \) curves wasted. In this paper a homotopy is constructed. We show that there are exactly \( n \) disjoint curves that have to be followed and all the eigenpairs can be reached by these curves. Since the curves followed by general homotopy curve following scheme are computed independently of one another, the algorithm is a likely candidate for exploiting the advantages of parallel processing for the generalized eigenvalue problems.

The homotopy method for symmetric eigenvalue problems was studied by Chu [3]. For the general eigenvalue problem, a homotopy was given by Li, Sauer, and Yorke [13]. The theory developed in [13] can be applied to the generalized eigenvalue problem \( Ax = \lambda Bx \) when \( B \) is nonsingular. In this
paper, we generalize the algorithm in [13] to include the case where $B$ is singular.

2. MAIN RESULTS

For the polynomial system $P = (p_1, p_2, \ldots, p_{n+1})$ in (1.2), consider the homotopy

$$H = (1 - t)Q + tP,$$  \hspace{1cm} (2.1)

where $H = (h_1, \ldots, h_{n+1})$, $Q = (q_1, \ldots, q_{n+1})$ with

$$q_1 = (\lambda + \alpha_1)(dx_1 + \beta_1),$$

$$\vdots$$

$$q_n = (\lambda + \alpha_n)(dx_n + \beta_n),$$

$$q_{n+1} = c_1x_1 + \cdots + c_n x_n + \alpha_{n+1},$$

and $(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n, d) \in C^{n+1} \times C^n \times C$.

The following lemma was proved in [13].

**LEMMA 2.1.** For any fixed $d$, there exists an open dense set $U^o \subset C^{n+1} \times C^n$ such that $(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n) \in U^o$ implies that if $H(\lambda^o, x_1^o, \ldots, x_n^o) = 0$ then

$$\text{rank } \frac{\partial (h_1, \ldots, h_{n+1})}{\partial (\lambda, x_1, \ldots, x_n)}|_{(\lambda^o, x_1^o, \ldots, x_n^o)} = n + 1.$$

With this lemma and the implicit function theorem, the homotopy curves of (2.1) are smooth and can be parameterized by $t$.

Denote $n$-dimensional projective space by $CP^n \sim \{(x_0, \ldots, x_n) | x_i \subset C, x_i \not= 0 \}$ where $(sx_0, \ldots, sx_n) \sim (x_0, \ldots, x_n)$ for $0 \neq s \subset C$. We are interested in solving equations in $C^n = \{(x_0, \ldots, x_n) \subset CP^n | x_0 = 1\}$. A point $x = (x_0, \ldots, x_n)$ with $x_0 = 0$ is called a point at infinity.

A complex polynomial $p(x_1, \ldots, x_n)$ can be written as a sum $p = p^0 + \cdots + p^d$, where $p^i$ consists of the $i$th degree terms of $p$ for each $i$. Each $p^i$ is homogeneous of degree $i$ in the sense that $p^i(tx_1, \ldots, tx_n) = \lambda^i p^i(x_1, \ldots, x_n)$.
\(t'p'(x_1,\ldots, x_n)\) for all \(t \in \mathbb{C}\). Define

\[\tilde{p}(x_1,\ldots, x_n) = p^d(x_1,\ldots, x_n)\]

and

\[\tilde{p}(x_0, x_1,\ldots, x_n) = x_0^d p\left(\frac{x_1}{x_0},\ldots, \frac{x_n}{x_0}\right)\].

Both \(\tilde{p}\) and \(\tilde{p}\) are homogeneous polynomials of the same degree as \(p\). We further denote \(\tilde{P} = (\tilde{p}_1,\ldots, \tilde{p}_n)\), \(\tilde{P} = (\tilde{p}_1,\ldots, \tilde{p}_n)\), and \(\tilde{P} = (\tilde{p}_1,\ldots, \tilde{p}_n)\).

The connection between \(P, \tilde{P}\), and \(\tilde{P}\) is the following. If \(x_0 = 1\), then \(\tilde{p}(x_0,\ldots, x_n) = P(x_1,\ldots, x_n)\). If \(x_0 = 0\), then \(\tilde{p}(x_0,\ldots, x_n) = \tilde{P}(x_1,\ldots, x_n)\).

In geometric terms, if \(Z\) is the common zero set in \(n\)-dimensional complex projective space \(\mathbb{CP}^n\) of the polynomial system \(\tilde{p}(x_0,\ldots, x_n)\), then \(Z\) is the disjoint union

\[\{(1, x_1,\ldots, x_n) | P(x_1,\ldots, x_n) = 0\} \cup \{(0, x_1,\ldots, x_n) | \tilde{P}(x_1,\ldots, x_n) = 0\}\]

of the zeros of \(P\) in \(\mathbb{C}^n\) and the zeros in the "hyperplane at infinity" defined by \(x_0 = 0\).

A \textit{projective variety} is the zero set \(Z\) in \(\mathbb{CP}^n\) of a set \(\{p_1,\ldots, p_r\}\) of homogeneous polynomials in the variables \(x_0,\ldots, x_n\). A point \(y \in Z\) is called nonsingular if

\[\text{rank}_C \frac{\partial (p_1,\ldots, p_r)}{\partial (x_0,\ldots, x_n)} \bigg|_y = \text{codim}_y (Z, \mathbb{CP}^n),\]

where \(\text{codim}_y (Z, \mathbb{CP}^n) = n - \dim_y Z\), and where \(\dim\) denotes complex dimension. Here, the notion of dimension of an arbitrary variety is defined as the largest of the dimensions of its irreducible components considered as differential manifold; see Kendig [10]. A variety is \textit{nonsingular} if each point of the variety is nonsingular.

The following theorem is a more powerful version of Bézout’s Theorem.

\textbf{Theorem 2.2} [6, 9.1.1, 9.1.2]. Let \(p_1,\ldots, p_n\) be polynomials in \(\mathbb{C}[x_1,\ldots, x_n]\) of degree \(d_1,\ldots, d_n\), and suppose the zero set of \(\{p_1,\ldots, p_n\}\) in \(\mathbb{CP}^n\) is a disjoint union of nonsingular points \(Y_1,\ldots, Y_r\) and nonsingular
linear spaces \( Z_1, \ldots, Z_s \). Then

\[
d_1 \cdots d_n = r + \sum_{i=1}^{s} [Z_i],
\]

where for a linear space \( Z \equiv \mathbb{C}^n \subseteq \mathbb{C}^n \), the equivalence \([Z]\) is given by the coefficient of \( t^e \) in the power series

\[
(1 + t)^{-n} \prod_{i=1}^{n} (1 + d_i t).
\]

From Lemma 2.1 and inverse function theorem, if \((a_1, \ldots, a_n, \beta_1, \ldots, \beta_n) \in U^0\), then for each \( t \in [0, 1) \) the zero set \( Z^1(t) \) of \( \{ h_1(t), \ldots, h_n(t) \} \) in \( \mathbb{C}^n \) consists of isolated simple zeros, since the Jacobians of the zeros are nonsingular. Thus, \( Z^1(t) \) is of dimension less than 1. In order to use (2.2) to calculate the number of isolated zeros in \( \mathbb{C}^n \), we only have to confine our attention to the zero set of (2.1) at infinity, i.e., the zero set of \( \vec{H} = (\vec{h}_1, \ldots, \vec{h}_{n+1}) \) for each \( t \).

**Lemma 2.3.** For a matrix \( B = (b_{ij}) \in \mathbb{C}^{n \times n} \), there exists an open dense set \( V \subseteq \mathbb{C} \) such that if \( d \in V \) then

\[
B'(t) = (1 - t)dI + tB
\]

is nonsingular for each \( t \in [0, 1) \).

**Proof.** For each fixed \( t \in [0, 1) \), \( B'(t) \) is singular for \( n \) values of \( d \), say \( \beta_1(t), \ldots, \beta_n(t) \). Since \( \beta_i(t) \) are continuous, the union \( T \) of these curves is a set of one real dimension. Therefore, the complement \( V \) of \( T \) in \( \mathbb{C} \) is open and dense. \( \square \)

Let

\[
Z_1 = \left\{ (0, \lambda, x_1, \ldots, x_n) \in \mathbb{C}P^{n+1} \mid \lambda = 0, c_1 x_1 + \cdots + c_n x_n = 0 \right\} \approx \mathbb{C}P^{n-2},
\]

\[
Z_2 = \left\{ (0, \lambda, x_1, \ldots, x_n) \in \mathbb{C}P^{n+1} \mid x_1 = x_2 = \cdots = x_n = 0 \right\} \approx \mathbb{C}P^0.
\]

Let \( V \) be the open dense set in Lemma 2.3, and \( V_1 = V \setminus \{0\} \).
**LEMMA 2.4.** If \( d \in V_1 \), then the zero set of \( \vec{H} = (\vec{h}_1, \ldots, \vec{h}_{n+1}) \) in (2.1) is \( Z_1 \cup Z_2 \) for all \( t \in [0, 1) \).

**Proof.** For \( t \in [0, 1) \),

\[
\begin{align*}
\vec{h}_1 &= (1-t)\lambda(dx_1) + t\lambda(b_1x_1 + \cdots + b_1n x_n), \\
\vdots \\
\vec{h}_n &= (1-t)\lambda(dx_n) + t\lambda(b_nx_1 + \cdots + b_n n x_n), \\
\vec{h}_{n+1} &= c_1x_1 + \cdots + c_n x_n.
\end{align*}
\](2.4)

For \( \lambda = 0 \), the zero set is \( Z_1 \), and for \( \lambda \neq 0 \) the zeros of (2.4) must satisfy

\[
((1-t)dI + tB)x = 0,
\](2.5)

where \( x = (x_1, \ldots, x_n) \). For \( d \in V_1 \), from Lemma 2.3, the solution of (2.5) is

\[x_1 = x_2 = \cdots = x_n = 0.\]

That is, when \( \lambda \neq 0 \), the zero set of (2.4) is \( Z_2 \).

Given a set of homogeneous polynomials in \( C[x_0, \ldots, x_n] \), we will use the concept of a resultant system. Suppose \( p_1(x_0, \ldots, x_n), \ldots, p_r(x_0, \ldots, x_n) \) are homogeneous polynomials in the variables \( x_0, \ldots, x_n \) with indeterminate coefficients \( c_1, \ldots, c_N \). Then there is a set of homogeneous polynomials \( q_1(c_1, \ldots, c_N), \ldots, q_r(c_1, \ldots, c_N) \) in the variables \( c_i \), with integer coefficients, with the following properties: for any set of special values of the \( c_i \in C \), a necessary and sufficient condition for the set \( \{ p_1, \ldots, p_r \} \) to have a common zero different from \( (0, \ldots, 0) \) is that the \( c_i \) are a common zero of the polynomials \( \{ q_1, \ldots, q_r \} \) [15, Section 16.5].

**LEMMA 2.5.** For \( d \in V_1 \), there exists an open dense set \( U^1 \subset C^n \times C^n \) such that if \( (\alpha, \beta) \in U^1 \) with \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) then the zero sets \( Z_1 \) and \( Z_2 \) of \( \vec{H} \) in (2.4) are nonsingular for each \( t \in [0, 1) \).

**Proof.** For \( z = (0, \lambda, x_1, \ldots, x_n) \in Z_2 \), the Jacobian matrix is

\[
J(z) = \frac{\partial(\vec{h}_1, \ldots, \vec{h}_{n+1})}{\partial(x_0, \lambda, x_1, \ldots, x_n)} = \\
\begin{bmatrix} r_1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
r_n & 0 \\
0 & c_1 & \cdots & c_n \\
\end{bmatrix}
\]
where $r_i = (1 - t)\beta_i \lambda$ for $i = 1, \ldots, n$, and $r_{n+1} = (1 - t)\alpha_{n+1} - t$. For $\lambda \neq 0$, by Lemma 2.3, the matrix $\lambda[(1 - t)dI + tB]$ is nonsingular for $t \in [0, 1)$. Therefore, for $\alpha_{n+1} \neq 0$ the rank of $J(z)$ is $n + 1$ for $t \in (0, 1)$, which is the codimension of $Z_2$. Hence, $Z_2$ is nonsingular for all $t \in [0, 1)$. For $z = (0, \lambda, x_1, \ldots, x_n) \in Z_1$, the Jacobian matrix is

$$J(z) = \frac{\partial (\tilde{h}_1, \ldots, \tilde{h}_n)}{\partial (x_0, \lambda, x_1, \ldots, x_n)} = \begin{bmatrix} w_1 x & w_2 x & 0 \\ r & 0 & c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

where $w_1 x$ and $w_2 x$ are $n \times 1$ vectors with

$$(w_1 x)_i = (1 - t)a_i(dx_i) - t(a_{i1} x_1 + \cdots + a_{in} x_n), \quad i = 1, \ldots, n,$$

$$(w_2 x)_i = (1 - t)dx_i + t(b_{i1} x_1 + \cdots + b_{in} x_n), \quad i = 1, \ldots, n,$$

and $r = (1 - t)a_{n+1} - t$. We now construct an open dense set $U^1 \subset C^{n+1} \times C^n$ in such a way that the $n \times 2$ matrix $W = [w_1 x, w_2 x]$ is of rank 2 for all $(\alpha, \beta) \in U^1$. This would imply that for $z \in Z_1$ and $t \in [0, 1)$ the rank of $J(z)$ is 3, which is the codimension of $Z_1$. Therefore $Z_1$ is nonsingular.

Let $M = (m_{ij})$ be an $n \times 2$ matrix with

$$m_{1i} = a_i(dx_i) - s(a_{i1} x_1 + \cdots + a_{in} x_n), \quad i = 1, \ldots, n,$$

$$m_{12} = dx_i + s(b_{i1} x_1 + \cdots + b_{in} x_n), \quad i = 1, \ldots, n,$$

with $s \in C$. The determinants of $2 \times 2$ minors of $M$ form a system of polynomials in $C[x_1, \ldots, x_n]$ with indeterminate coefficients $\alpha_i$, $i = 1, \ldots, n$, and $s$. Let $\gamma_1, \ldots, \gamma_l$ be the resultant system. The $\gamma_l(\alpha_i, \ldots, \alpha_n, s)$, $1 \leq l \leq t$, are homogeneous polynomials in $\alpha_i, \ldots, \alpha_n, s$ such that there exists a common zero $x = (x_1, \ldots, x_n) \neq 0$ of all the determinants of $2 \times 2$ minors of $M$ if and only if $0 = \gamma_1(\alpha_1, \ldots, \alpha_n, s) = \gamma_2(\alpha_1, \ldots, \alpha_n, s) = \cdots = \gamma_t(\alpha_1, \ldots, \alpha_n, s)$. For $s = 0$, $M$ is of rank 2 for any $x = (x_1, \ldots, x_n) \in Z_1$ with $\alpha_i \neq \alpha_j$ if $i \neq j$. Therefore for any of these $(\alpha_1, \ldots, \alpha_n) \in C^n$, there exists some $l$, $\gamma_l(\alpha_1, \ldots, \alpha_n) \neq 0$. Hence $\gamma_l(\alpha_1, \ldots, \alpha_n) = \gamma_l(\alpha_1, \ldots, \alpha_n, 0)$ is not the identically zero polynomial, and vanishes on a complex subvariety $T_1$ of $C^n$ of codimension 1. The complement $U_0$ of $T_1$ is an open dense subset of $C^n$ such that if $(\alpha_1, \ldots, \alpha_n) \in U_0$, then the matrix $W$ is of rank 2 for $t = 0$. Further, since $\gamma_l$ is homogeneous, $\gamma^2(\alpha_1, \ldots, \alpha_n) = \gamma_l(\alpha_1, \ldots, \alpha_n, 1)$ is not the identically zero polynomial. Let $T_2$ be the complex codimension-one subvariety of $C^n$ on which $\gamma^2$ vanishes.
Now let $0 \neq t \in \mathbb{R}$. The rank of $W$ is 2 for all $0 \neq x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ as long as
\[
\gamma \left( \frac{1-t}{t} (\alpha_1, \ldots, \alpha_n) \right) \neq 0,
\]
i.e., as long as
\[
(\alpha_1, \ldots, \alpha_n) \notin \frac{t}{1-t} T_2.
\]
If we choose $\alpha \in T^0 = RT_2 = \{ r\alpha | r \in \mathbb{R}, \alpha \in T_2 \}$, then the rank of $W$ is 2 for $t \in [0, 1)$ and $0 \neq x \in \mathbb{C}^n$. The variety $T^0$ is real codimension one in $\mathbb{C}^n$; therefore the complement $U_1$ of $T^0$ is open and dense. Let $U_0 \cap U_1$; then $U_2$ is open and dense. For $(\alpha_1, \ldots, \alpha_n) \in U_2$, the rank of $W$ is 2 for $t \in [0, 1)$ and $0 \neq x \in \mathbb{C}^n$.

In summary, for the total requirements of the result, we let $U' = U_2 \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n$.

Remark. By the way we choose $U_0$, it is easy to conclude that for $(\alpha_1, \ldots, \alpha_n) \in U'$, $\alpha_i \neq \alpha_j$ if $i \neq j$.

Theorem 2.6. Let $d \in V_1$ be fixed. There exists an open dense set $U \subset \mathbb{C}^{n+1} \times \mathbb{C}^n$ such that if $(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n) \in U$, then the zero set of the homotopy $H = 0$ in (2.1) consists of $n$ smooth paths, parametrized by $t \in [0, 1)$, which lead to all the isolated solutions of (1.2).

Proof. Let $U = U^0 \times U^1$, where $U^0$ is an open dense set in Lemma 2.1 and $U^1$ is an open dense set in Lemma 2.5. Then, for $(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n) \in U$, the solution set of $H = 0$ in (2.1) consists of smooth paths which can be parametrized by $t$.

The disjoint zero sets of $H = 0$ at infinity are $Z_1$ and $Z_2$ in (2.3) for all $t \in [0, 1)$, and by Lemma 2.5 they are nonsingular for all $t \in [0, 1)$. By Lemma 2.1, the isolated zeros of $H$ are nonsingular for any $t \in [0, 1)$. Hence, by Theorem 2.2, the number of isolated zeros of $H(t)$ is independent of $t$. Let this number be $k$. It follows that none of the homotopy paths will diverge to infinity as $t \to t_0 < 1$. For if a homotopy path $c(t) \to \infty$ as $t \to t_0 < 1$, then, at $t_0$, there are $k$ isolated zeros $c_1(t_0), \ldots, c_k(t_0)$ of $H(t_0)$. The homotopy paths of $H = 0$ pass through these points are bounded for $t \in (t_0 - \delta, t_0]$, $\delta > 0$. This implies that for each fixed $t$ in $(t_0 - \delta, t_0)$, $H(t) = 0$ has $k + 1$
isolated solutions, which is a contradiction. By the degree argument in [2], all isolated zeros of $H(1)$ are reached by one of these smooth paths.

The number of the paths $k$ may be calculated by Theorem 2.2. Since $d_1 = d_2 = \cdots = d_n = 2$, $d_{n+1} = 1$, and $Z_1 \cong \mathbb{C}P^{n-2}$, the equivalence $[Z_1]$ is given by the coefficient of $t^{n-2}$ in the Maclaurin series

$$(1 + t)^{n-2} - (n+1)(1+2t)^n(1+t) = (1+2t)^n(1+t)^{-2} = \frac{(1+2t)^n}{(1+t)^2}$$

$$= \frac{(1+t+t)^n}{(1+t)^2} = \sum_{i=0}^{n} \frac{(1+t)^{n-i} t^i \binom{n}{i}}{(1+t)^2},$$

$$= \sum_{i=0}^{n} (1+t)^{n-2-i} t^i \binom{n}{i}.$$ 

Hence,

$$[Z_1] = \sum_{i=0}^{n-2} \binom{n}{i}.$$ 

The equivalent of $Z_2 \cong \mathbb{C}P^0$ is the constant coefficient of

$$(1 + t)^{-n}(1+2t)^n(1+t),$$

so $[Z_2] = 1$. Therefore the number of the paths is

$$k = 2^n - \sum_{i=0}^{n-2} \binom{n}{i} - 1 = n. \quad \blacksquare$$

**Remark.** The starting points of the paths are easily found. From the remark follows Lemma 2.5, $\alpha_i \neq \alpha_j$ for $i \neq j$. Thus, the $n$ starting points in $\mathbb{C}P^{n-1}$ are

$$\left( -\alpha_i, \frac{-\beta_1}{d}, \ldots, \frac{-\beta_{i-1}}{d}, \alpha_{n+1} - \sum_{j \neq i} c_j \frac{\beta_j}{d}, \frac{-\beta_{i+1}}{d}, \ldots, \frac{-\beta_n}{d} \right),$$

$$i = 1, \ldots, n.$$
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