MATHEMATICS

ON SOME SERIES OF DOUGALL

BY

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1. The well-known series of Dougall [1, 3.10 (6) (8) (9)] are the following:

$$P_k^{-m}(\cos\theta) = \frac{\sin k\pi}{\pi} \sum_{s=0}^{\infty} (-1)^s \left(\frac{1}{k-s} - \frac{1}{k+s+1}\right) P_s^{-m}(\cos\theta), \tag{1}$$

valid for arbitrary non-integral k, $-\pi<\theta<\pi$, and

$$\left\{\begin{array}{l} \theta\neq 0 \text{ and } \operatorname{Re}(m) > -\frac{1}{2} \\ \theta = 0 \text{ and } \operatorname{Re}(m) > 0 \text{ or } m=0 \end{array}\right\}, \tag{1a}$$

$$\left\{\begin{array}{l} P_k^{-l}(\cos\theta)\, P_k^{-m}(\cos\Psi) = \\ \qquad = \dfrac{\sin k\pi}{\pi} \displaystyle\sum_{s=0}^{\infty} (-1)^s \left(\frac{1}{k-s} - \frac{1}{k+s+1}\right) P_s^{-l}(\cos\theta)\, P_s^{-m}(\cos\Psi), \end{array}\right. \tag{2}$$

valid for arbitrary non-integral k, $-\pi<\theta+\Psi<\pi$, $-\pi<\theta-\Psi<\pi$ and the pair θ, m satisfying (1a), the pair Ψ, l satisfying the corresponding condition,

$$\left\{\begin{array}{l} P_k^{m}(\cos\theta)\, P_k^{-m}(\cos\Psi) = \\ \qquad = \dfrac{\sin k\pi}{\pi} \displaystyle\sum_{s=0}^{\infty} (-1)^s \left(\frac{1}{k-s} - \frac{1}{k+s+1}\right) P_s^{m}(\cos\theta)\, P_s^{-m}(\cos\Psi), \end{array}\right. \tag{3}$$

where m is a non-negative integer, k not an integer, $-\pi<\theta+\Psi<\pi$, $-\pi<\theta-\Psi<\pi$. In the above $P_k^m(x)$ denotes the associated Legendre function of the first kind.

In this paper we will extend these results to generalized associated Legendre functions of KUIPERS-MEULENBELD [2]. The definitions and some of the properties are found in [2] and [3].

2. We will prove the following theorems.

Theorem 1. (*First series of Dougall.*) *Let m, n and k be complex constants. Then for k not an integer*

$$\mathrm{P}_k^{-m,-n}(\cos\theta) = \frac{\sin k\pi}{\pi} \sum_{s=0}^{\infty} (-1)^s \left(\frac{1}{k-s} - \frac{1}{k+s+1}\right) P_s^{-m,-n}(\cos\theta), \tag{4}$$

where m and θ satisfy conditions (1a) *and* $-\pi<\theta<\pi$.

Theorem 2. (*Second series of Dougall.*) *Let m, n, μ, ν and k be complex constants. Then for k not an integer*

$$(5)\quad \begin{cases} P_k^{-m,-n}(\cos\theta)\, P_k^{-\mu,-\nu}(\cos\Psi) = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) \cdot \\ \qquad\qquad \cdot\, P_s^{-m,-n}(\cos\theta)\, P_s^{-\mu,-\nu}(\cos\Psi), \end{cases}$$

where $-\pi<\theta+\Psi<\pi$, $-\pi<\theta-\Psi<\pi$, and the pairs m, θ and μ, Ψ satisfy condition (1*a*).

Theorem 3. (*Third series of Dougall.*) *Let k be a complex constant not an integer, and m and n constants such that, m, $\frac{1}{2}(n-m)$* and *$\frac{1}{2}(m+n)$ are non-negative integers. Then*

$$(6)\quad \begin{cases} \dfrac{\Gamma\left(k-\dfrac{m-n}{2}+1\right)}{\Gamma\left(k+\dfrac{m-n}{2}+1\right)} P_k^{-m,-n}(\cos\theta)\, P_k^{m,n}(\cos\Psi) = \\ \qquad = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=\frac{1}{2}(m+n)}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) \cdot \\ \qquad\qquad \cdot \dfrac{\Gamma\left(s-\dfrac{m-n}{2}+1\right)}{\Gamma\left(s+\dfrac{m-n}{2}+1\right)} P_s^{-n,-m}(\cos\theta)\, P_s^{m,n}(\cos\Psi), \end{cases}$$

where $-\pi<\theta+\Psi<\pi$, $-\pi<\theta-\Psi<\pi$.

3. Proof of Theorem 1. Taking $m=0$ in (1), and putting $\cos\theta = 1-2x\sin^2\frac{1}{2}\varphi$, where $0 \leqq x \leqq 1$, $-\pi<\varphi<\pi$, we obtain:

$$(7)\quad \begin{cases} P_k(1-2x\sin^2\frac{1}{2}\varphi) = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) \cdot \\ \qquad\qquad \cdot\, P_s(1-2x\sin^2\frac{1}{2}\varphi). \end{cases}$$

Putting $u_s(x) = (-1)^s P_s(1-2x\sin^2\frac{1}{2}\varphi)$ and $v_s = \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right)$, it is clear that $\sum\limits_{s=0}^{\infty} |v_s - v_{s-1}|$ converges. Furthermore, it can easily be shown that

$$\left|\sum_{s=0}^{N} u_s(x)\right| < K,$$

where K is independent of x and N. Hence, from a modified Abel theorem, the series in (7) converges uniformly in x, $0 \leqq x \leqq 1$ for each φ, $-\pi<\varphi<\pi$.

Writing (7) in terms of hypergeometric functions we find:

$$(8)\quad \begin{cases} F(-k,\, k+1;\, 1;\, x\sin^2 \tfrac{1}{2}\varphi) = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) \cdot \\ \qquad\qquad\qquad \cdot\, F(-s,\, s+1;\, 1;\, x\sin^2 \tfrac{1}{2}\varphi). \end{cases}$$

Suppose Re $(m) \geqq 1$ and then multiply both sides of (8) by

$$(9)\quad (1-x)^{m-1}(1-x\sin^2 \tfrac{1}{2}\varphi)^{-\frac{1}{2}(m-n)}\, F\left(\frac{m-n}{2}, \frac{m-n}{2};\, m;\, \frac{(1-x)\sin^2\frac{1}{2}\varphi}{1-x\sin^2\frac{1}{2}\varphi}\right).$$

Since the function in (9) is bounded in x, $0 \leqq x \leqq 1$ for each φ, $-\pi < \varphi < \pi$, the resulting series is still uniformly convergent and we can integrate term by term over $[0, 1]$. Thus we get:

$$(10)\quad \begin{cases} \int\limits_0^1 (1-x)^{m-1}(1-x\sin^2\tfrac{1}{2}\varphi)^{-\frac{1}{2}(m-n)}\, F(-k,\, k+1;\, 1;\, x\sin^2\tfrac{1}{2}\varphi)\cdot \\ \qquad\qquad \cdot\, F\left(\dfrac{m-n}{2}, \dfrac{m-n}{2};\, m;\, \dfrac{(1-x)\sin^2\frac{1}{2}\varphi}{1-x\sin^2\frac{1}{2}\varphi}\right) dx \\ \qquad = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right)\cdot \\ \quad \cdot \int\limits_0^1 (1-x)^{m-1}(1-x\sin^2\tfrac{1}{2}\varphi)^{-\frac{1}{2}(m-n)}\, F(-s,\, s+1;\, 1;\, x\sin^2\tfrac{1}{2}\varphi)\cdot \\ \qquad\qquad \cdot\, F\left(\dfrac{m-n}{2}, \dfrac{m-n}{2};\, m;\, \dfrac{(1-x)\sin^2\frac{1}{2}\varphi}{1-x\sin^2\frac{1}{2}\varphi}\right) dx. \end{cases}$$

From [1, 2.4 (3)] we deduce the relation

$$(11)\quad \begin{cases} \dfrac{1}{m}\, F\left(k + \dfrac{m-n}{2} + 1,\, -k + \dfrac{m-n}{2};\, 1+m;\, z\right) = \\ \qquad = \int\limits_0^1 (1-x)^{m-1}(1-xz)^{-\frac{1}{2}(m-n)}\, F(-k,\, k+1;\, 1;\, xz)\cdot \\ \qquad\qquad \cdot\, F\left(\dfrac{m-n}{2}, \dfrac{m-n}{2};\, m;\, \dfrac{(1-x)z}{1-xz}\right) dx, \end{cases}$$

valid for Re $(m) > 0$, $|\arg(1-z)| < \pi$. Thus (10) becomes:

$$(12)\quad \begin{cases} F\left(k + \dfrac{m-n}{2} + 1,\, -k + \dfrac{m-n}{2};\, 1+m;\, \sin^2\tfrac{1}{2}\varphi\right) = \\ \qquad = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right)\cdot \\ \qquad\qquad \cdot\, F\left(s + \dfrac{m-n}{2} + 1,\, -s + \dfrac{m-n}{2};\, 1+m;\, \sin^2\tfrac{1}{2}\varphi\right) \end{cases}$$

Since

$$P_k^{-m,-n}(\cos\varphi) = \frac{1}{\Gamma(1+m)} \frac{(1-\cos\varphi)^{\frac{1}{2}m}}{(1+\cos\varphi)^{\frac{1}{2}n}} \cdot$$

$$\cdot F\left(k+\frac{m-n}{2}+1,\ -k+\frac{m-n}{2};\ 1+m;\ \sin^2 \tfrac{1}{2}\varphi\right)$$

we have (4), valid for $-\pi<\varphi<\pi$ and Re $(m)\geqq 1$.

By uniform convergence and analytic continuation, it can be shown that (4) holds under the conditions of (1a).

4. Starting from (2) and using the same method twice, with the necessary modifications to show uniform convergence of the series involved, Theorem 2 is easily proved.

5. To prove Theorem 3, we set $\mu=\nu=0$ in (5) and interchange m and n. Since n is a non-negative integer, condition (1a) is satisfied for $-\pi<\theta<\pi$. Thus

$$(13)\quad \begin{cases} P_k^{-n,-m}(\cos\theta)\, P_k(\cos\Psi) = \\ \quad = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) P_s^{-n,-m}(\cos\theta)\, P_s(\cos\Psi). \end{cases}$$

As $\frac{1}{2}(m+n)$ is a non-negative integer, we have:

$$P_k^{\frac{1}{2}(m+n)}(x) = (-1)^{\frac{1}{2}(m+n)} (1-x^2)^{\frac{1}{2}(m+n)} \left(\frac{d}{dx}\right)^{\frac{1}{2}(m+n)} P_k(x).$$

Applied to (13) we obtain:

$$(14)\quad \begin{cases} P_k^{-n,-m}(\cos\theta)\, P_k^{\frac{1}{2}(m+n)}(\cos\Psi) = \\ \quad = \dfrac{\sin k\pi}{\pi} \sum\limits_{s=0}^{\infty} (-1)^s \left(\dfrac{1}{k-s} - \dfrac{1}{k+s+1}\right) \cdot \\ \quad\quad \cdot P_s^{-n,-m}(\cos\theta)\, P_s^{\frac{1}{2}(m+n)}(\cos\Psi). \end{cases}$$

The term-by-term differentiation is justified for $\Psi\neq 0$ because of the uniform convergence of (14). However, it is evident that (14) also holds for $\Psi=0$.

Since $\frac{1}{2}(n-m)$ is a non-negative integer, we can apply the relation

$$(15)\quad \begin{cases} P_k^{m,n}(x) = \dfrac{(1+x)^{\frac{1}{2}n}}{(1-x)^{\frac{1}{2}m}} \dfrac{\Gamma\left(k+\frac{m-n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)} \cdot \\ \quad \cdot 2^{\frac{1}{2}(n-m)} \left(\dfrac{d}{dx}\right)^{\frac{1}{2}(n-m)} \left\{\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{2}(m+n)} P_k^{\frac{1}{2}(m+n)}(x)\right\}, \end{cases}$$

(see [4, sec. 2]), to (14). Thus we obtain (6). The justification for the term by term differentiation is the same as for (14). However, because of the appearence of the gamma functions in formula (6), the interchange of m and n in (5) is sufficient to assure the uniform convergence of (6).

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