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Gap Estimates of the Spectrum of the Zakharov-Shabat System

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Abstract—We prove new gap estimates for the Zakharov-Shabat systems with complex periodic potentials. Our method allows us to characterize in a precise way the decreasing properties of the gap length sequence in terms of the regularity of complex potentials in weighted Sobolev spaces. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND RESULTS

The nonlinear Schrödinger equation $(\mathrm{NLS})_\pm$ on the circle

$$i\partial_t\varphi = -\partial_x^2\varphi \pm 2|\varphi|^2\varphi \tag{1$_\pm$}$$

is a completely integrable Hamiltonian system of infinite dimension. $(\mathrm{NLS})_+$ is referred to as the defocusing nonlinear Schrödinger equation, whereas $(\mathrm{NLS})_-$ is referred to as the focusing one.

We choose as the phase space of this Hamiltonian system $H^{N,\omega}(\mathcal{S}^1;\mathbb{C})$ defined for $N \geq 0$ and $\omega \geq 0$ by

$$H^{N,\omega}\left(\mathcal{S}^1;\mathbb{C}\right) = \left\{\varphi(x) = \sum_{k\in\mathbb{Z}} e^{2i\pi kx}\hat{\varphi}(k); \|\varphi\|_{N,\omega} < \infty\right\},$$

where $\hat{\varphi}(k)$ denote the Fourier coefficients of φ and

$$\|\varphi\|_{N,\omega} = \left(\sum_{k\in\mathbb{Z}}(1+|k|)^{2N}e^{2\omega|k|}|\hat{\varphi}\ (k)|^2\right)^{1/2}.$$

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Equation $(1)_\pm$ admits a Lax pair representation

$$\frac{dM_\pm}{dt} = [M_\pm, A_\pm],$$

where $M_+ := L(\varphi, \overline{\varphi})$, $M_- := L(\varphi, -\overline{\varphi})$, L being the Zakharov-Shabat operator (see [1])

$$L(\psi_1, \psi_2) := i \begin{pmatrix} 1, & 0, \\ 0, & -1, \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0, & \psi_1, \\ \psi_2, & 0 \end{pmatrix}, \tag{2}$$

and $A_\pm$ are (rather complicated) operators, given in [2]. The periodic eigenvalues of the Zakharov-Shabat operator $L(\varphi, \overline{\varphi})$ (respectively, $L(\varphi, -\overline{\varphi})$) considered on the interval [0,2] are a complete set of conserved quantities for $(\mathrm{NLS})_+$ (respectively, $(\mathrm{NLS})_-$) on circle.

Motivated by this fact, we study in this paper the periodic spectrum of $L = L(\psi_1, \psi_2)$ for (ψ_1, ψ_2) in $H^{N,\omega}(S^1; \mathbb{C}) \times H^{N,\omega}(S^1; \mathbb{C})$.

We denote by $\sigma \equiv \sigma(\psi_1, \psi_2)$ the set of eigenvalues of $L(\psi_1, \psi_2)$ considered on the interval [0,2]. Recall that this set is discrete. Our principal result is the following theorem.

THEOREM. *Let $N > 0$, $\omega \geq 0$ and let $\gamma := 1/2$ if $N > 1/2$ and $0 \leq \gamma < N$, if $0 < N \leq 1/2$. Then, for any bounded subset $\mathcal{B}$ in $H^{N,\omega}(S^1; \mathbb{C}) \times H^{N,\omega}(S^1; \mathbb{C})$, there exist $n_0 \geq 1$ and $M \geq 1$, so that for any $|k| \geq n_0$ and $(\psi_1, \psi_2) \in \mathcal{B}$, the set $\sigma(\psi_1, \psi_2) \cap \{\lambda \in \mathbb{C}, |\lambda - k\pi| < \pi/2\}$ contains exactly one isolated pair of eigenvalues $\{\lambda_k^+, \lambda_k^-\}$. These eigenvalues satisfy the following estimates.*

(i) $\sum_{|k| \geq n_0} (1 + |k|)^{2N} e^{2\omega|k|} |\lambda_k^+ - \lambda_k^-|^2 \leq M$.

(ii) $\sum_{|k| \geq n_0} (1 + |k|)^{2N+2\gamma} e^{2\omega|k|} |\lambda_k^+ - \lambda_k^- - 2(\hat{\psi}_2(n)\hat{\psi}_1(-n))^{1/2}|^2 \leq M$.

Furthermore, $\sigma(\psi_1, \psi_2) \setminus \{\lambda_k^\pm, |k| \geq n_0\}$ is included in $\{\lambda \in \mathbb{C}, |\lambda| < n_0\pi - \pi/2\}$ and its cardinality is $4n_0 - 2$.

Notice that L is unitary equivalent to the well-known AKNS operator (see [3]). The operator $L(\psi_1, \psi_2)$ is self-adjoint iff $\overline{\psi}_1 = \psi_2$ and in this case, (ii) can be improved

$$\sum_{|k| \geq n_0} (1 + |k|)^{2N+4\gamma} e^{2\omega|k|} \left((\lambda_k^+ - \lambda_k^-) - 2\left|\hat{\psi}(n)\right|\right)^2 \leq M.$$

Thus, our theorem is a generalization of the gap estimates established by Marčenko [4] who considered only the self-adjoint case assuming that $\omega = 0$ and $N \in \mathbb{N} \setminus \{0\}$ (see also [5]).

As an application of this theorem, we will prove in a subsequent paper that the defocusing nonlinear Schrödinger equation $i\partial_t\varphi = -\partial_x^2\varphi + 2|\varphi|^2\varphi$ admits globally defined, real analytic action angle variables on the phase space $H^{N,\omega}(S^1; \mathbb{C})$ which can be used to prove KAM-type theorems.

2. SKETCH OF THE PROOF

We consider the eigenvalue problem on the interval [0,2],

$$L(\psi_1, \psi_2) F = \lambda F. \tag{3}$$

In order to solve (3), we adapt a method used by Kappeler and Mityagin for the Schrödinger operator (see [6]). We decompose $F \in (L^2[0,2])^2$ with respect to the orthogonal basis $(1/\sqrt{2}\binom{1}{0}e^{ik\pi x}$, $1/\sqrt{2}\binom{0}{1}e^{ik\pi x})_{k\in\mathbb{Z}}$ (these functions correspond to the eigenfunctions associated to $\lambda = k\pi$ when $\psi_1 = \psi_2 = 0$) and we write $\lambda = n\pi + z$, $|z| \leq \pi/2$ and $n \in \mathbb{Z}$. Then, the couple (λ, F) is a solution of (3) if and only if there exists a nontrivial solution of the following system:

$$-zx + \widehat{\psi_2}(2n)y + \sum_{k \neq n} \widehat{\psi_2}(k+n)b_k = 0, \tag{4$_+$}$$

$$\widehat{\psi_1}(-2n)x - zy + \sum_{k \neq n} \widehat{\psi_1}(-k-n)a_k = 0, \tag{4$_-$}$$

$$(z - B_n)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y\left(\widehat{\hat{\psi}_2}(k+n)\right)_{k\neq n} \\ x\left(\widehat{\hat{\psi}_1}(-k-n)\right)_{k\neq n} \end{pmatrix}, \tag{$(4)_n$}$$

where $\widehat{\hat{\psi}_2}(k)$, $k \in \mathbb{Z}$, denote the Fourier coefficients of ψ when considered as a periodic function of period 2, x, and y belong to $\mathbb{C}$, $a := (a_k)_{k\neq n}$ and $b := (b_k)_{k\neq n}$ are sequences in $l^2(\mathbb{Z}\setminus\{n\})$, and

$$B_n := \left(\begin{array}{c|c} ((k-n)\pi\delta_{kj})_{k,j\in\mathbb{Z}\setminus\{n\}} & \left(\widehat{\hat{\psi}_{22}}(k+j)\right)_{k,j\in\mathbb{Z}\setminus\{n\}} \\ \hline \left(\widehat{\hat{\psi}_{21}}(-k-j)\right)_{k,j\in\mathbb{Z}\setminus\{n\}} & ((k-n)\pi\delta_{kj})_{k,j\in\mathbb{Z}\setminus\{n\}} \end{array}\right).$$

We first solve $(4)_n$, for n sufficiently large, inverting $(z - B_n)$ in an appropriate space. The system $(4)_+$, $(4)_-$, and $(4)_n$ is then equivalent to the 2×2 system

$$\begin{pmatrix} -z+\alpha(n,z) & \hat{\psi}_2(n)+\beta^+(n,z) \\ \hat{\psi}_1(-n)+\beta^-(n,z) & -z+\alpha(n,z) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5}$$

where the coefficients $\alpha(n,z)$ and $\beta^{\pm}(n,z)$ are easily determined by $(4)_+$, $(4)_-$, and $(4)_n$ (see [7]). The existence of a nontrivial solution of (5) follows from the vanishing of the determinant

$$\begin{vmatrix} -z+\alpha(n,z) & \hat{\psi}_2(n)+\beta^+(n,z) \\ \hat{\psi}_1(-n)+\beta^-(n,z) & -z+\alpha(n,z) \end{vmatrix},$$

which leads to the following analytic equation for z:

$$(z-\alpha(n,z))^2 = \left(\hat{\psi}_2(n)+\beta^+(n,z)\right)\left(\hat{\psi}_1(-n)+\beta^-(n,z)\right). \tag{6}$$

Precise estimates of $\alpha(n,z)$ and $\beta^{\pm}(n,z)$ allow us to prove that equation (6) has exactly two solutions z_n^+ and z_n^- for $|n|$ sufficiently large and

$$\sum_{|n|\geq n_0} (1+|n|)^{2N+2\gamma}\, e^{2\omega|n|}\left|(z_n^+ - z_n^-) - 2\left(\hat{\psi}_2(n)\hat{\psi}_1(-n)\right)^{1/2}\right|^2 < \infty,$$

This formula leads to the statements (i) and (ii) of the theorem above. Furthermore, a counting lemma, using Rouche's theorem, proves that there are exactly $4n_0 - 2$ eigenvalues in the disc $\{\lambda \in \mathbb{C},\ |\lambda| < n_0\pi - \pi/2\}$.

The details of the proof are contained in [7].

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