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Positive Solutions of $2m^{\text{th}}$-Order Boundary Value Problems

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Abstract—We study the existence of positive solutions of the differential equation $(-1)^m y^{(2m)}(t) = f(t, y(t), y''(t), \dots, y^{(2(m-1))}(t))$ with the boundary condition $y^{(2i)}(0) = 0 = y^{(2i)}(1)$, $0 \le i \le m-1$, and $y^{(2i)}(0) = 0 = y^{(2i+1)}(1)$, $0 \le i \le m-1$. We show the existence of at least one positive solution if f is either superlinear or sublinear by an application of a fixed-point theorem in a cone.

Keywords—Positive solution, Boundary value problem, Cone.

1. INTRODUCTION

Boundary value problems for even-order differential equations can arise, especially for fourth-order equations, in applications such as

(a) modeling a number of axially loaded beams fastened together with boundary conditions involving displacement (or deflection at ends), velocity (or vibration at ends), bending moments, and shear forces; see [1];
(b) modeling behavior of a compressed beam subjected to a load causing buckling with the stipulation that the ends are constrained to remain straight and there is zero end shear stress (such as deflection of girders in multilevel buildings as well as deflection of flat-bed trailers in tractor-trailer trucks); see [2]; and
(c) modeling the effects of soil settlement on elastically bedded building girders loaded by concentrated forces; see [3].

For the second-order case, boundary value problems for nonlinear ordinary differential equations have received much attention in determining conditions on the nonlinearity for which there are either at least one, at least two, or at least three positive solutions. Some of those results, along with excellent lists of references are contained in [4–11].

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The techniques in this work will follow along the lines of those introduced by Ma [12] for fourth-order problems. By adapting the techniques of [12], it is easier to apply some of the fixed-point theorems from the cone theory than in many of the papers cited above. To some extent, this method involves our reducing higher even-order boundary value problems in a sense to second-order considerations. In turn, this then allows us to define cones in terms of lower bounds on a single higher-order derivative as opposed to requiring lower bounds on all lower derivatives.

In this paper, we consider the existence of positive solutions of the equation

$$(-1)^m y^{(2m)}(t) = f\left(t, y(t), y''(t), \dots, y^{(2(m-1))}(t)\right), \qquad m \geq 2, \tag{1.1}$$

with either the Lidstone boundary conditions,

$$y^{(2i)}(0) = 0 = y^{(2i)}(1), \qquad 0 \leq i \leq m-1, \tag{1.2}$$

or the focal boundary conditions,

$$y^{(2i)}(0) = 0 = y^{(2i+1)}(1), \qquad 0 \leq i \leq m-1. \tag{1.3}$$

We will mention a couple of examples. The solution of the boundary value problem,

$$y^{(4)} = -Py'' + p(t,y), \qquad 0 \leq t \leq 1,$$

with boundary conditions (1.2), represents the deflection of a hinged beam column. P represents the axial loading and p represents the nonconservative force. We refer the reader to [1] for discussion. If $P > 0$, then the axial load is said to be applying compression. If $P < 0$, then the axial force is said to be applying tension.

Meirovitch [13] used higher even-order boundary value problems in studying the open-loop control of a distributed structure whose undamped behavior is governed by

$$Hw(x) + m(x)w(x) = f(x), \qquad 0 < x < L,$$

$w(x)$ is displacement at a point x in the structure, H is a homogeneous differential stiffness operator of order $2p$, $m(x)$ is the mass density, and $f(x)$ is a distributed control force. The solution $w(x)$ is subject to boundary conditions,

$$B_i w(x) = 0, \qquad x = 0, L, \quad 1 \leq i \leq p,$$

where the B_i are differential operators of maximum order $2p-1$.

We assume $f : [0,1] \times I_1 \times I_2 \times \cdots \times I_m \to [0,\infty)$ is continuous where $I_i = [0,\infty)$ if i is an odd integer and $I_i = (-\infty, 0]$ if i is an even integer and set

$$f_0 := \lim_{p\to 0^+} \min_{t\in[0,1]} \frac{f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right)}{p},$$

$$f_\infty := \lim_{p\to \infty} \min_{t\in[0,1]} \frac{f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right)}{p},$$

$$f^0 := \lim_{p\to 0^+} \max_{t\in[0,1]} \frac{f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right)}{p},$$

$$f^\infty := \lim_{p\to \infty} \max_{t\in[0,1]} \frac{f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right)}{p},$$

where $f_0, f_\infty, f^0, f^\infty \in \{0, \infty\}$ and the above four limits are uniform in $y_1, y_2, \dots, y_{m-1}$.

We note that $f^0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f^\infty = 0$ correspond to the sublinear case. By a positive solution $y(t)$ of (1.1),(1.2) (or (1.3)), we mean that $y(t)$ is positive on $(0,1)$ and satisfies (1.1),(1.2) (or (1.3), respectively).

We remark that the upshot to the approach taken in this paper is the ability to construct Green's functions associated with (1.1),(1.2) and (1.1),(1.3) from Green's functions for conjugate problems and right focal problems, respectively, for second-order differential equations. Because of this, other problems which could be approached in this manner include equation (1.1) coupled with a generalization of Sturm-Liouville boundary conditions for the higher-order equation.

2. THE EXISTENCE OF SOLUTIONS OF (1.1),(1.2)

In this section, we apply a Guo-Krasnosel'skii fixed-point theorem [14] for operators which are of an expansion/compression type with respect to an annular region in a cone. We now present the definition of a cone in a Banach space and the Guo-Kransnosel'skii fixed-point theorem.

DEFINITION 2.1. *Let $\mathcal{B}$ be a Banach space over R. A nonempty closed set $\mathcal{P} \subset \mathcal{B}$ is said to be a cone provided the following are satisfied:*

(a) *$\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and $\alpha, \beta \geq 0$;*
(b) *if $u \in \mathcal{P}$ and $-u \in \mathcal{P}$, then $u = 0$.*

LEMMA 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{B} \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{P}$$

be a completely continuous operator such that either,

(i) *$\|Au\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, or*
(ii) *$\|Au\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.*

Then A has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Our aim is to apply Lemma 2.1 to a completely continuous operator whose kernel is the Green's function for the homogeneous problem

$$(-1)^m y^{(2m)} = 0 \tag{2.1}$$

satisfying the Lidstone boundary conditions (1.2). For the case $m = 1$, the Green's function for (2.1),(1.2) is

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

If we let $G_1(t,s) := G(t,s)$, then for $2 \leq j \leq m$, we can recursively define

$$G_j(t,s) = \int_0^1 G(t,r) G_{j-1}(r,s)\, dr.$$

Clearly,

$$G_j(t,s) \geq 0, \qquad \text{for } 2 \leq j \leq m.$$

It is noted that

$$(-1)^{m-1} \frac{\partial^{2m-2}}{\partial t^{2m-2}} G_m(t,s) = G(t,s).$$

As a result, $G_m(t,s)$ is the Green's function for (2.1),(1.2). We will make use of the following inequality:

$$G(t,s) \geq \frac{1}{4} G(s,s), \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, \quad s \in [0,1]. \tag{2.2}$$

We now establish the existence of at least one positive solution of the Lidstone boundary value problem (1.1),(1.2).

THEOREM 2.2. *Let $f : [0,1] \times I_1 \times I_2 \times \cdots \times I_m \to [0,\infty)$ where $I_i = [0,\infty)$ if i is an odd integer and $I_i = (-\infty, 0]$ if i is an even integer. Then the boundary value problem (1.1),(1.2) has at least one positive solution in the case*

(i) $f^0 = 0$ *and* $f_\infty = \infty$, *or*

(ii) $f_0 = \infty$ *and* $f^\infty = 0$.

PROOF. Let $\mathcal{B}$ denote the Banach space in $C^{2m-2}[0,1]$ as

$$\mathcal{B} = \left\{y \in C^{2m-2}[0,1] | y \text{ satisfies } (1.2)\right\}$$

with the norm endowed by

$$\|y\| = \left\|y^{(2(m-1))}\right\|_\infty.$$

We note that, for $y \in \mathcal{B}$ and $t \in [0,1]$, $|y(t)| \le \int_0^t |y'(s)|\,ds \le \|y'\|_\infty$. Hence, $\|y\|_\infty \le \|y'\|_\infty$. In addition, there exists $t_0 \in (0,1)$ such that $y'(t_0) = 0$. Then, for $t \in [0,1]$, $|y'(t)| \le |\int_{t_0}^t |y''(s)|\,ds| \le \|y''\|_\infty$. In particular, $\|y'\|_\infty \le \|y''\|_\infty$. By bootstrapping, one sees that

$$\|y\|_\infty \le \|y'\|_\infty \le \cdots \le \left\|y^{(2(m-1))}\right\|_\infty.$$

Denote

$$\mathcal{P} = \left\{ y \mid y \in \mathcal{B},\ y \ge 0, \text{ and } \min_{1/4 \le t \le 3/4} (-1)^{m-1} y^{(2(m-1))}(t) \ge \frac{\left\|y^{(2(m-1))}\right\|_\infty}{4} \right\}.$$

It is obvious that $\mathcal{P}$ is a cone in $\mathcal{B}$. Let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be the operator

$$(\mathcal{A}y)(t) = \int_0^1 G_m(t,s) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\,ds. \tag{2.3}$$

Then $y \in \mathcal{B}$ is a solution of (1.1),(1.2) if and only if $\mathcal{A}y = y$. By the assumption on f and nonnegativity of G_m, the solution is positive. In order to apply Lemma 2.1, we need to verify $\mathcal{AP} \subseteq \mathcal{P}$. Now let $y \in \mathcal{P}$,

$$\begin{aligned}
\min_{1/4 \le t \le 3/4} (-1)^{m-1} (\mathcal{A}y)^{(2m-2)}(t) &= \min_{1/4 \le t \le 3/4} \int_0^1 G(t,s) f\left(s, y(s), y''(s), \dots, y^{(2m-1)}(s)\right)\,ds \\
&\ge \frac{1}{4} \int_0^1 G(s,s) f\left(s, y(s), y''(s)), \dots, y^{(2(m-1))}(s)\right)\,ds \\
&\ge \frac{1}{4} \left\|(\mathcal{A}y)^{(2(m-1))}\right\|_\infty.
\end{aligned}$$

We conclude that $\mathcal{A}y \in \mathcal{P}$. It is routine to see that $\mathcal{A}$ is completely continuous. We now first deal with the superlinear case, that is, $f^0 = 0$ and $f_\infty = \infty$. Since $f^0 = 0$, we may choose $\mathcal{H}_1 > 0$ so that $f(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p) \le \eta p$, $\forall\, t, y_1, \dots, y_{m-1}$, $0 < p \le \mathcal{H}_1$, where η satisfies

$$\eta \int_0^1 G(s,s)\,ds \le 1. \tag{2.4}$$

Thus, if $y \in \mathcal{P}$ and $\|y\| = \mathcal{H}_1$, then we have

$$\begin{aligned}
(-1)^{m-1} (\mathcal{A}y)^{(2(m-1))}(t) &= \min_{1/4 \le t \le 3/4} \int_0^1 G(t,s) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\,ds \\
&\le \int_0^1 G(s,s) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\,ds \\
&\le \int_0^1 G(s,s) \eta \left((-1)^{m-1} y^{(2(m-1))}(s)\right)\,ds \\
&\le \|y\| \eta \int_0^1 G(s,s)\,ds \\
&\le \|y\|.
\end{aligned}$$

Now, if we let
$$\Omega_1 = \{y \in \mathcal{B} \mid \|y\| < \mathcal{H}_1\},$$
then,
$$\|\mathcal{A}y\| \le \|y\|, \qquad y \in \mathcal{P} \cap \partial\Omega_1.$$
Further, since $f_\infty = \infty$, there exists $\hat{\mathcal{H}}_2$ such that
$$f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right) \ge \rho p, \text{ for } p \ge \hat{\mathcal{H}}_2, \qquad \forall\, t, y_1, \dots, y_{m-1},$$
where ρ is chosen so that
$$\frac{\rho}{4}\int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)\, ds \ge 1. \tag{2.5}$$
Let $\mathcal{H}_2 := \max\{2\mathcal{H}_1, 4\hat{\mathcal{H}}_2\}$ and $\Omega_2 = \{y \in \mathcal{B} \mid \|y\| < \mathcal{H}_2\}$. Then $y \in \mathcal{P}$ and $\|y\| = \mathcal{H}_2$ imply
$$\min_{1/4 \le t \le 3/4} (-1)^{m-1} y^{(2(m-1))}(t) \ge \frac{\left\|y^{(2(m-1))}\right\|_\infty}{4} \ge \hat{\mathcal{H}}_2,$$
and so
$$\begin{aligned}
(-1)^{m-1}(\mathcal{A}y)^{(2(m-1))}\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\, ds \\
&\ge \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\, ds \\
&\ge \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) \rho\left((-1)^{m-1} y^{(2(m-1))}(s)\right)\, ds \\
&\ge \rho\frac{1}{4}\left\|y^{(2(m-1))}\right\|_\infty \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)\, ds \\
&\ge \left\|y^{(2(m-1))}\right\|_\infty.
\end{aligned}$$
Hence, $\|\mathcal{A}y\| \ge \|y\|$, for $y \in \mathcal{P} \cap \partial\Omega_2$.

Therefore, by the first part of Lemma 2.1, it follows that $\mathcal{A}$ has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$, such that $\mathcal{H}_1 \le \|y\| \le \mathcal{H}_2$. This completes the superlinear part of the theorem.

We now establish the existence of a positive solution of (1.1),(1.2) for the sublinear case, that is, $f_0 = \infty$ and $f^\infty = 0$. We first choose $\mathcal{H}_1 > 0$ such that $f(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p) \ge \hat{\eta} p$ for $0 < p \le \mathcal{H}_1$, $\forall\, t, y_1, \dots, y_{m-1}$ where
$$\frac{\hat{\eta}}{4}\int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)\, ds \ge 1. \tag{2.6}$$
Then for $y \in \mathcal{P}$ and $\|y\| = \mathcal{H}_1$, we have
$$\begin{aligned}
(-1)^{m-1}(\mathcal{A}y)^{(2(m-1))}\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\, ds \\
&\ge \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right)\, ds \\
&\ge \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) \hat{\eta}\left((-1)^{m-1} y^{(2(m-1))}(s)\right)\, ds \\
&\ge \hat{\eta}\frac{1}{4}\left\|y^{(2(m-1))}\right\|_\infty \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)\, ds \\
&\ge \left\|y^{(2(m-1))}\right\|_\infty.
\end{aligned}$$

Thus, we may let

$$\Omega_1 = \{y \in \mathcal{B} \mid \|y\| < \mathcal{H}_1\},$$

so that

$$\|\mathcal{A}y\| \geq \|y\|, \qquad y \in \mathcal{P} \cap \partial\Omega_1.$$

Now, since $f^\infty = 0$, there exists $\hat{\mathcal{H}}_2 > 0$ so that $f(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p) \leq \lambda p$, for $0 < p \leq \hat{\mathcal{H}}_2$, where $\lambda > 0$ satisfies

$$\lambda \int_0^1 G(s,s)\,ds \leq 1. \tag{2.7}$$

We consider two subcases.

CASE I. Suppose f is bounded, say $f(t, y_1, y_2, \dots, y_m) \leq N$ for all $(t, y_1, y_2, \dots, y_m) \in [0,1] \times I_1 \times I_2 \times \cdots \times I_m$. In this case, we may choose

$$\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, N\int_0^1 G(s,s)\,ds\right\},$$

so that for $y \in \mathcal{P}$ with $\|y\| = \mathcal{H}_2$, we have

$$\begin{aligned}(-1)^{m-1}(\mathcal{A}y)^{(2(m-1))}(t) &\leq \int_0^1 G(s,s) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right) ds\\ &\leq N\int_0^1 G(s,s)\,ds\\ &\leq \mathcal{H}_2,\end{aligned}$$

and therefore, $\|\mathcal{A}y\| \leq \|y\|$.

CASE II. If f is unbounded, then we define a function $f^* : [0,\infty) \to [0,\infty)$ by

$$\begin{aligned}f^*(r) := \max\Big\{f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right) \mid t \in [0,1], 0 \leq (-1)^{i-1}y_i \leq r,&\\ i = 1,2,\dots m-1,\ 0 < p \leq r\Big\}.&\end{aligned}$$

It is easy to see that f^* is nondecreasing and $\lim_{r\to\infty}(f^*(r)/r) = 0$, and

$$f^*(r) \leq \lambda r, \qquad \text{for } r > \hat{\mathcal{H}}_2. \tag{2.8}$$

Choose $\mathcal{H}_2 > \max\{2\mathcal{H}_1, \hat{\mathcal{H}}_2\}$, then we have

$$f\left(t, y_1, y_2, \dots, y_{m-1}, (-1)^{m-1}p\right) \leq f^*(\mathcal{H}_2), \tag{2.9}$$

for $t \in [0,1]$, $0 \leq (-1)^{i-1}y_i \leq \mathcal{H}_2$, $i = 1, 2, \dots m-1$, $0 < p \leq \mathcal{H}_2$.

Let $y \in \mathcal{P}$ and $\|y\| = \mathcal{H}_2$. We also know

$$\|y\|_\infty \leq \|y'\|_\infty \leq \cdots \leq \left\|y^{(2(m-1))}\right\|_\infty = \mathcal{H}_2.$$

This together with (2.8) implies that

$$\begin{aligned}(-1)^{m-1}(\mathcal{A}y)^{2(m-1)}(t) &= \int_0^1 G(t,s) f\left(s, y(s), y''(s), \dots, y^{(2(m-1))}(s)\right) ds\\ &\leq \int_0^1 G(s,s) f^*(\mathcal{H}_2)\,ds\\ &\leq \lambda\mathcal{H}_2 \int_0^1 G(s,s)\,ds\\ &\leq \mathcal{H}_2\\ &= \left\|y^{(2(m-1))}\right\|_\infty.\end{aligned}$$

Therefore, in either case, we may put

$$\Omega_2 = \{y \in \mathcal{B} \mid \|y\| < \mathcal{H}_2\},$$

and for $y \in \mathcal{P} \cap \partial\Omega_2$, we have $\|\mathcal{A}y\| \leq \|y\|$. By the second part of Lemma 2.1, it follows that (1.1),(1.2) has a positive solution.

EXAMPLE. Consider the boundary value problem

$$y^{(6)} = \frac{720}{[(360t^2 - 360) + 1]^r} \left(y^{(4)} + 1\right)^r,$$

$$y(0) = y''(0) = y^{(4)}(0) = 0, \qquad y(1) = y''(1) = y^{(4)}(1) = 0,$$

Let $f(t, y_1, y_2, y_3) = (720/[(360t^2 - 360) + 1]^r)(y_3 + 1)^r$. We see that it is superlinear if $r > 1$ and sublinear if $0 < r < 1$. One such solution is given by $y(t) = t^6 - 3t^5 + 5t^3 + 3t$.

3. THE EXISTENCE OF SOLUTIONS OF (1.1),(1.3)

We now establish the existence of at least one positive solution of the boundary value problem (1.1),(1.3).

THEOREM 3.1. *Let $f : [0,1] \times I_1 \times I_2 \times \cdots \times I_m \to [0,\infty)$ where $I_i = [0,\infty)$ if i is an odd integer and $I_i = (-\infty, 0]$ if i is an even integer. Then the boundary value problem (1.1),(1.3) has at least one positive solution in the case*

(i) *$f^0 = 0$ and $f_\infty = \infty$, or*

(ii) *$f_0 = \infty$ and $f^\infty = 0$.*

PROOF. In order to apply Lemma 2.1, we need to assign a suitable Banach space $\mathcal{B}$, cone $\mathcal{P}$, and operator $\mathcal{A}$. Let $\mathcal{B}$ denote the Banach space in $C^{2m-2}[0,1]$ as

$$\mathcal{B} = \left\{y \in C^{2m-2}[0,1] \mid y \text{ satisfies } (1.3)\right\},$$

with the norm endowed by

$$\|y\| = \left\|y^{(2(m-1))}\right\|_\infty.$$

Denote

$$\mathcal{P} = \left\{ y \mid y \in \mathcal{B},\ y \geq 0, \text{ and } \min_{1/2 \leq t \leq 1} (-1)^{m-1} y^{(2(m-1))}(t) \geq \frac{\left\|y^{(2(m-1))}\right\|_\infty}{2} \right\}.$$

It is obvious that $\mathcal{P}$ is a cone in $\mathcal{B}$. Let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be the operator

$$(Ay)(t) = \int_0^1 K(t,s) f\left(s, y(s), y''(s), \ldots, y^{(2(m-1))}(s)\right) ds,$$

where $K(t,s)$ is the Green's function for (2.1) with boundary condition (1.3). The rest of the proof is essentially the same as that in Theorem 2.2, where one uses

$$(-1)^{m-1} \frac{\partial^{2m-2}}{\partial t^{2m-2}} K(t,s) = H(t,s)$$

and

$$H(t,s) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1. \end{cases}$$

This certainly establishes that (1.1),(1.3) has at least one positive solution if f is either superlinear or sublinear.

EXAMPLE. Consider the boundary value problem

$$y^{(6)} = \frac{720}{[(360t^2 - 720t) + 1]^r} \left(y^{(4)} + 1\right)^r,$$

$$y(0) = y''(0) = y^{(4)}(0) = 0, \qquad y'(1) = y'''(1) = y^{(5)}(1) = 0.$$

Let $f(t, y_1, y_2, y_3) = (720/[(360t^2 - 720t) + 1]^r)(y_3 + 1)^r$. We see that it is superlinear if $r > 1$ and sublinear if $0 < r < 1$. One such solution is given by $y(t) = t^6 - 6t^5 + 40t^3 - 96t$.

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