Property \((\omega_1)\) and Weyl type theorem

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**Abstract**

In this note we define the property \((\omega_1)\), a variant of Weyl's theorem, and establish for a bounded linear operator defined on a Banach space the sufficient and necessary conditions for which property \((\omega_1)\) holds by means of the variant of the essential approximate point spectrum \(\sigma_{\omega}(T)\). In addition, the relation between property \((\omega_1)\) and hypercyclicity (or supercyclicity) is discussed.

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1. Introduction

H. Weyl [14] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has been considered by many authors. Variants have been discussed by Harte and Lee [6] and Rakočević [11,12]. In this note, we introduce a new variant of Weyl's theorem which called property \((\omega_1)\) and show how property \((\omega_1)\) follows from properties of the variant \((\omega_1)\) of the essential approximate point spectrum. Also, the relation between property \((\omega_1)\) and hypercyclicity (or supercyclicity) is discussed.

Throughout this paper, \(X\) denotes an infinite dimensional complex Banach space, \(B(X)\) the algebra of all bounded linear operators on \(X\). For an operator \(T \in B(X)\) we shall denote by \(n(T)\) the dimension of the kernel \(N(T)\), and by \(d(T)\) the codimension of the range \(R(T)\). We call \(T \in B(X)\) an upper semi-Fredholm operator if \(n(T) < \infty\) and \(R(T)\) is closed; but if \(d(T) < \infty\), \(T\) is a lower semi-Fredholm operator. An operator \(T \in B(X)\) is said to be Fredholm if both the deficiency indices \(n(T)\) and \(d(T)\) are finite. If \(T \in B(X)\) is an upper (or a lower) semi-Fredholm operator, the index of \(T\), \(\text{ind}(T)\), is defined to be \(\text{ind}(T) = n(T) - d(T)\). The ascent of \(T\), \(\text{asc}(T)\), is the least non-negative integer \(n\) such that \(N(T^n) = N(T^{n+1})\) and the descent, \(\text{dsc}(T)\), is the least non-negative integer \(n\) such that \(R(T^n) = R(T^{n+1})\). The operator \(T\) is Weyl if it is Fredholm of index zero, and \(T\) is said to be Browder if it is Fredholm "of finite ascent and descent". The upper semi-Fredholm spectrum \(\sigma_{\text{SF}}(T)\) is defined by: \(\sigma_{\text{SF}}(T) = \{\lambda \in \mathbb{C}: T - \lambda I\ \text{is not upper semi-Fredholm}\}\). Similarly, \(\sigma_{\text{SF}}(T) = \{\lambda \in \mathbb{C}: T - \lambda I\ \text{is not lower semi-Fredholm}\}\). Let \(\rho(T)\) denote the resolvent set of the operator \(T\) and \(\sigma(T) = \mathbb{C}\setminus\rho(T)\) denote the usual spectrum of \(T\). And let \(\sigma_0(T)\) denote the approximate point set of the operator \(T \in B(X)\), \(\rho_0(T) = \mathbb{C}\setminus\sigma_0(T)\). Let \(\sigma_{\omega}(T) = \{\lambda \in \mathbb{C}: T - \lambda I \notin \sigma_{\text{SF}}(T)\}\) and \(\sigma_{\text{SF}}(T) = \{T \in B(X), T\ \text{is upper semi-Fredholm and ind}(T) < 0\}\).

\(T \in B(X)\) is called a generalized upper semi-Fredholm operator [3] if there exist \(T\)-invariant subspaces \(M\) and \(N\) such that \(X = M \oplus N\) and \(T|M \in \sigma_{\text{SF}}(M), T|N\) is quasinilpotent. Clearly, if \(T\) is generalized upper semi-Fredholm, there exists \(\epsilon > 0\) such that \(T - \lambda I \in \sigma_{\text{SF}}(X)\) and \(N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]\) if \(0 < |\lambda| < \epsilon\).
Property \((\omega)\) and its stability have been studied by numerous authors, see for example [1,2], etc. The property \((\omega_1)\) which we will define has close relations with property \((\omega)\). The plan of this paper is as follows. In Section 2, we give the definition of property \((\omega_1)\) and the necessary and sufficiently conditions for \(T\) for which property \((\omega_1)\) holds are given. In Section 3, as a consequence of the main result, the relation between property \((\omega_1)\) and hypercyclicity (or supercyclicity) is discussed.

2. Property \((\omega_1)\)

**Definition 2.1.** \(T \in B(X)\) is said \([1]\) to satisfy property \((\omega)\) if
\[\sigma_a(T)\setminus\sigma_{ea}(T) = \pi_{00}(T).\]

Property \((\omega)\) implies Weyl’s theorem, Browder’s theorem and \(a\)-Browder’s theorem \([1]\). Property \((\omega_1)\) is defined as follows.

**Definition 2.2.** Property \((\omega_1)\) holds for \(T\) if
\[\sigma_a(T)\setminus\sigma_{ea}(T) \subseteq \pi_{00}(T).\]

**Remark 2.1.**
(1) Property \((\omega)\) implies property \((\omega_1)\), but the converse is not true.

For example, let \(T \in B(\ell^2)\) be defined by
\[T(x_1, x_2, x_3, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right),\]
then \(\sigma_a(T) = \sigma_{ea}(T) = \{0\}, \pi_{00}(T) = \{0\}, \sigma_a(T)\setminus\sigma_{ea}(T) = \emptyset\). This implies that property \((\omega_1)\) holds for \(T\) but property \((\omega)\) fails for \(T\).

(2) Property \((\omega)\) cannot induce \(a\)-Weyl’s theorem.

Let \(A, B \in B(\ell^2)\) be defined by
\[A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),\]
\[B(x_1, x_2, x_3, \ldots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots),\]
and let \(T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\), then \(\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \{\lambda \in \mathbb{C}: |\lambda| = 1\}, \pi_{00}(T) = \emptyset\) and \(\pi_{00}^a(T) = \{0\}\). This means that property \((\omega)\) holds for \(T\) but \(a\)-Weyl’s theorem fails for \(T\).

(3) \(a\)-Weyl’s theorem cannot induce property \((\omega)\).

Let \(A \in B(\ell^2)\) be defined as in (2) and \(B \in B(\ell^2)\) be defined by
\[B(x_1, x_2, x_3, \ldots) = (x_1, 0, x_3, x_4, \ldots),\]
and let \(T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\), then \(\sigma_a(T) = \{0\} \cup \{\lambda \in \mathbb{C}: |\lambda| = 1\}, \sigma_{ea}(T) = \{\lambda \in \mathbb{C}: |\lambda| = 1\}, \pi_{00}(T) = \emptyset\) and \(\pi_{00}^a(T) = \{0\}\). This shows that \(a\)-Weyl’s theorem holds for \(T\) but property \((\omega)\) fails for \(T\).

(4) Property \((\omega_1)\) implies \(a\)-Browder’s theorem, but the converse is not true.

Let \(T \in B(\ell^2 \oplus \ell^2)\) be defined as in (3). We find that \(a\)-Browder’s theorem holds for \(T\) but property \((\omega_1)\) fails for \(T\).

The following two theorems give the relation between property \((\omega_1)\) and property \((\omega)\) as well as the necessary and sufficient conditions for \(T\) for which property \((\omega_1)\) holds.

**Theorem 2.1.** \(T \in B(X)\) satisfies property \((\omega)\) if
\[
\iff\text{property \((\omega_1)\) holds for } T \text{ and } \pi_{00}(T) = P_{00}(T), \text{ where } P_{00}(T) = \sigma(T) \setminus \sigma_b(T); \\
\iff\text{property \((\omega_1)\) holds for } T \text{ and } \sigma_{ea}(T) \cap \pi_{00}(T) = \emptyset.
\]

**Proof.** (I) Suppose that \(T\) has property \((\omega)\), then property \((\omega_1)\) holds for \(T\). Let \(\lambda \in \pi_{00}(T)\), then \(\lambda \in \sigma_a(T)\setminus\sigma_{ea}(T), \) thus \(T - \lambda I \in S_{\omega}(X)\). Since \(\lambda \in \sigma_a(T)\setminus\sigma_{ea}(T)\), we know that \(T - \lambda I\) is Browder and hence \(\lambda \notin P_{00}(T)\). Conversely, suppose \(T\) satisfies property \((\omega_1)\) and \(\pi_{00}(T) = P_{00}(T)\). Let \(\lambda \in \pi_{00}(T)\), which means that \(\lambda \in \sigma_a(T)\setminus\sigma_{ea}(T), \) thus property \((\omega)\) holds for \(T\).

(II) \(T\) has property \((\omega)\) and this implies that property \((\omega_1)\) holds for \(T\) and \(\sigma_{ea}(T) \cap \pi_{00}(T) = \emptyset\). For the converse, if \(\lambda \in \pi_{00}(T)\), \(\lambda \notin \sigma_{ea}(T)\), since \(\sigma_{ea}(T) \cap \pi_{00}(T) = \emptyset\). Then \(\lambda \in \sigma_a(T)\setminus\sigma_{ea}(T)\), hence \(\sigma_a(T)\setminus\sigma_{ea}(T) = \pi_{00}(T)\). \(\Box\)
**Remark 2.2.** Property \((\omega_1)\) holds for \(T\) if and only if \(\sigma_{eq}(T) \ni \sigma(T) \cap \pi_0(T) = \varnothing\). For example, \(T \in B(\ell^2)\) is defined by
\[
T(x_1, x_2, x_3, \ldots) = \left( 0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots \right),
\]
then \(T\) is quasinilpotent, \(\sigma_e(T) = \sigma_{eq}(T) = [0], \pi_0(T) = [0], \sigma_e(T) \setminus \sigma_{eq}(T) \subseteq \pi_0(T)\), but \(\sigma_{eq}(T) \cap \pi_0(T) = \{0\} \).

**Theorem 2.2.** The following statements are equivalent:

1. Property \((\omega_1)\) holds for \(T\);
2. \(\sigma_{eq}(T) = \sigma_e(T) \cap \sigma_0(T)\);
3. \(\sigma_{eq}(T) = \sigma_{eq}(T) \cup \pi_0(T)\);
4. \(\sigma_e(T) \setminus \sigma_{eq}(T) \subseteq \pi_0(T)\).

**Proof.** (1) \(\Leftrightarrow\) (2). Suppose \(T\) has property \((\omega_1)\). Clearly, \(\sigma_{eq}(T) \subseteq \sigma(T) \cap \sigma_0(T)\), We only need to prove that \(\sigma(T) \cap \sigma_0(T) \subseteq \sigma_{eq}(T)\). Let \(\lambda \notin \sigma_{eq}(T)\), then \(T - \lambda I \in SF^\infty(X)\), thus \(T - \lambda I\) is bounded from below or \(\lambda \notin \sigma_{eq}(T) \setminus \sigma_0(T)\). Since \(T\) has property \((\omega_1)\), we know that if \(\lambda \in \sigma_{eq}(T) \setminus \sigma_0(T)\), \(T - \lambda I\) is Browder, which means that \(\lambda \notin \sigma_{eq}(T) \setminus \sigma_0(T)\). Conversely, let \(\lambda \in \sigma_{eq}(T) \setminus \sigma_0(T)\), since \(\sigma_{eq}(T) = \sigma(T) \cap \sigma_0(T)\), it follows that \(T - \lambda I\) is Browder, hence \(\lambda \notin \pi_0(T)\), which means that property \((\omega_1)\) holds for \(T\).

(1) \(\Leftrightarrow\) (3). Suppose \(T\) satisfies property \((\omega_1)\). Clearly, \(\sigma_{eq}(T) \cup \pi_0(T) \subseteq \sigma(T)\). If \(\lambda \notin \sigma_{eq}(T) \cup \pi_0(T)\), then \(\lambda \notin \sigma_0(T)\). If \(\lambda \notin \sigma(T)\), then \(\lambda \notin \sigma_{eq}(T) \setminus \sigma_0(T)\), since \(T\) satisfies property \((\omega_1)\), it follows that \(\lambda \notin \pi_0(T)\). It is in contradiction to the fact that \(\lambda \notin \sigma_{eq}(T) \cup \pi_0(T)\). Thus \(\sigma_{eq}(T) = \sigma_{eq}(T) \cup \pi_0(T)\).

(1) \(\Leftrightarrow\) (4). Suppose \(T\) has property \((\omega_1)\). Let \(\lambda \in \sigma_{eq}(T) \setminus \sigma_0(T)\), then \(\lambda \in \pi_0(T)\). Since \(T - \lambda I\) is upper semi-Fredholm and \(\lambda \in \sigma(T)\), we know that \(T - \lambda I\) is Browder, hence \(\lambda \notin \pi_0(T)\). Conversely, using the fact that \(\pi_0(T) \subseteq \pi_0(T)\), if \(\sigma_0(T) \cap \sigma_{eq}(T) \subseteq \pi_0(T)\), then \(T\) has property \((\omega_1)\)  \(\Box\).

Recall that if \(T\) is generalized upper semi-Fredholm, there exists \(\epsilon > 0\) such that \(T - \lambda I \in SF^\infty(X)\) and \(N(T - \lambda I) \subseteq \bigcap_{n=1}^\infty \left( T - \lambda I \right)^2\) if \(0 < |\lambda| < \epsilon\). Now we turn to a variant of the essential approximate point spectrum which has been defined in [3]. Let
\[
\rho_1(T) = \{ \lambda \in \mathbb{C} : \text{there exists } \epsilon > 0 \text{ such that } T - \mu I \text{ is generalized upper semi-Fredholm operator if } 0 < |\mu - \lambda| < \epsilon \}
\]
and let \(\sigma_1(T) = \mathbb{C} \setminus \rho_1(T)\). Clearly, \(\sigma_1(T) \subseteq \sigma_{eq}(T) \subseteq \sigma_0(T)\) and \(\sigma_1(T) \cup \text{acc}(\sigma_0(T)) \subseteq \sigma(D)(\sigma_0(T))\), where \(\sigma(D)(T)\) is the Drazin spectrum of \(T\).

**Theorem 2.3.** The following statements are equivalent:

1. Property \((\omega_1)\) holds for \(T\) and \(\sigma_0(T) = P_0(T)\);
2. Property \((\omega_1)\) holds for \(T\) and \(\sigma_0(T) = \Pi(T)\), where \(\Pi(T) = \sigma(T) \setminus \sigma(D)(T)\);
3. \(T\) is f-polaroid and satisfies property \((\rho_1)\);
4. \(\sigma_q(T) = \sigma_D(T) = \sigma_1(T) \cup \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)]\).

**Proof.** (1) \(\Rightarrow\) (4). \(\sigma_1(T) \cup \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)] \subseteq \sigma_0(T)\) and \(\sigma_1(T) \cup \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)] \subseteq \sigma(D)(T)\) is clear. We only need to prove that \(\sigma_0(T) \subseteq \sigma_1(T) \cup \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)]\). Let \(\lambda_0 \notin \sigma_1(T) \cup \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)]\). Then there exists \(\epsilon > 0\) such that \(T - \lambda I \in SF^\infty(X)\) and \(N(T - \lambda I) \subseteq \bigcap_{n=1}^\infty \left( T - \lambda I \right)^2\), if \(0 < |\lambda' - \lambda| < \epsilon\). If \(n(T - \lambda I) > 0\), then \(\lambda' \notin \sigma_0(T)\). Since property \((\omega_1)\) holds for \(T\), it follows that \(T - \lambda' I\) is Browder. Then \(\text{acc}(T - \lambda' I) < +\infty\) and \(N(T - \lambda I) = N(T - \lambda' I) \cap \bigcap_{n=1}^\infty \left( T - \lambda' I \right)^2 = \{0\}\), a contradiction. Hence \(T - \lambda I\) is bounded from below if \(0 < |\lambda' - \lambda| < \epsilon\).

Then \(\lambda \in \sigma_0(T)\) or \(\rho_0(T)\). Since \(\sigma_0(T) = P_0(T)\), \(T - \lambda I\) is Browder if \(\lambda \in \sigma_0(T)\). It implies that \(\lambda \in \sigma_0(T)\). Hence \(\lambda \in \text{acc}(\sigma_0(T)) \cup [\sigma(T) \cap \rho_0(T)]\).

(4) \(\Rightarrow\) (1). Let \(\lambda_0 \in \sigma_0(T) \setminus \sigma_{eq}(T)\), then \(T - \lambda_0 I \in SF^\infty(X)\) and \(n(T - \lambda_0 I) > 0\). Using the punctured neighborhood theorem, there exists \(\epsilon > 0\) such that \(T - \lambda I \in SF^\infty(X)\) and \(N(T - \lambda I) \subseteq \bigcap_{n=1}^\infty \left( T - \lambda I \right)^2\), if \(0 < |\lambda - \lambda_0| < \epsilon\). Thus \(\lambda_0 \notin \sigma_1(T)\). Also \(\lambda_0 \notin \text{acc}(\sigma_0(T))\). If \(\lambda_0 \in \text{acc}(\sigma_0(T))\), then there exists \(\lambda_1\) such that \(0 < |\lambda_1 - \lambda_0| < \epsilon\) and \(\lambda_1 \in \sigma_0(T)\). Since \(T - \lambda_1 I \in SF^\infty(X)\) and \(\lambda_1 \in \sigma_0(T)\), we know that \(T - \lambda_1 I\) is Browder [5, Corollary 4.9]. Hence \(\text{acc}(T - \lambda_1 I) < +\infty\) and \(N(T - \lambda_1 I) = \infty\).
$N(T - λ_1l) \cap \bigcap_{i=1}^{n} R[(T - λ_1l)^{i}] = \{0\}$. Thus $T - λ_1l$ is invertible. It is in contradiction to the fact that $λ_1 \in σ(T)$. Therefore $λ_0 \notin σ_1(T) \cup acc(isoσ(T)) \cup [σ(T) \cap ρ_0(T)]$, which means that $λ_0 \notin σ_0(T)$. Thus $λ_0 \in π_{00}(T)$ and property $(ω_1)$ holds for $T$. Suppose $λ_0 \in isoσ_0(T)$. Then $λ_0 \notin σ_1(T) \cup acc(isoσ(T)) \cup [σ(T) \cap ρ_0(T)]$. Hence $λ_0 \notin σ_0(T)$ and $λ_0 \in π_{00}(T)$.

Similarly, we can prove $(2) \Rightarrow (4)$.

$(3) \Rightarrow (4)$. We only need to prove that $σ_0(T) \subseteq σ_1(T) \cup acc(isoσ(T)) \cup [σ(T) \cap ρ_0(T)]$. Let $λ_0 \notin σ_1(T) \cup acc(isoσ(T)) \cup [σ(T) \cap ρ_0(T)]$. Using the similar way of $(1) \Rightarrow (4)$, we know that $λ_0 \in isoσ_0(T) \cup ρ_0(T)$. If $λ_0 \in isoσ_0(T)$, since $T$ is f-a-polaroid, it follows that $λ_0 \in σ_0(T) \setminus ρ_0(T)$. Since property $(ω_1)$ holds for $T$, we know that $T - λ_0l$ is Browder; if $λ_0 \in ρ_0(T)$, the fact that $λ_0 \notin (T \cap ρ_0(T)$ tells us that $T - λ_0l$ invertible. Thus $λ_0 \notin σ_0(T)$ and $λ_0 \notin σ_0(T)$.

$(4) \Rightarrow (3)$. We need only to prove $T$ is f-a-polaroid. Let $λ_0 \in isoσ_0(T)$, then $λ_0 \notin σ_1(T) \cup acc(isoσ(T)) \cup [σ(T) \cap ρ_0(T)]$. Hence $λ_0 \notin σ_0(T)$, which means that $T - λ_0l \in SF_∞(X)$. $\square$

Example 2.1. Let $T \in B(l^2)$ be defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$

then $σ(T) = \{λ \in \mathbb{C}: 0 \leq |λ| \leq 1\}$. $σ_0(T) = \{λ \in \mathbb{C}: 0 \leq |λ| \leq 1\}$. $σ_0(T) = \{λ \in \mathbb{C}: |λ| = 1\}$. $σ_0(T) = \{λ \in \mathbb{C}: |λ| = 1\}$.

Thus $σ(T) = \{λ \in \mathbb{C}: |λ| = 1\}$. $acc(σ(T)) = \emptyset$, $σ(T) \cap ρ_0(T) = \{λ \in \mathbb{C}: 0 \leq |λ| < 1\}$. Therefore $σ_0(T) = σ_0(T) = σ_1(T) \cup acc(σ(T)) \cup [σ(T) \cap ρ_0(T)]$. Using Theorem 2.3, $T$ satisfies property $(ω_1)$, $T$ is f-a-polaroid and $isoσ_0(T) = P_{00}(T)$.

In the following, let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $σ(T)$ and are not constant on any component of $σ(T)$.

Theorem 2.4. Suppose property $(ω_1)$ holds for $T$, then the following statements are equivalent:

$(1)$ For any $f \in H(T)$, property $(ω_1)$ holds for $T$;

$(2)$ For any $f \in H(T)$, $σ_{ea}(f(T)) = f(σ_{ea}(T))$, and if $σ_0(T) \neq σ_{ea}(T)$, then $σ(T) = σ_{ea}(T)$;

$(3)$ For each pair $λ, μ \in \mathbb{C} \setminus σ_{fa}(T)$, $ind(T - λl)ind(T - μl) \geq 0$, and if $σ_0(T) \neq σ_{ea}(T)$, then $σ(T) = σ_{ea}(T)$.

Proof. $(1) \Rightarrow (2)$. Let $σ_{ea}(f(T)) \subseteq σ_{ea}(f(T))$ hold. We need to prove $f(σ_{ea}(T)) \subseteq σ_{ea}(f(T))$. Let $μ_0 \notin σ_{ea}(f(T))$, then $f(T - μ_0l) \notin SF_∞(X)$. Let

$$f(T - μ_0l) = (T - λ_1l)^{μ_1}(T - λ_2l)^{μ_2} \cdots (T - λ_kl)^{μ_k}g(T),$$

where $λ_1 \neq λ_2$ and $g(T)$ is invertible. Thus $T - λ_1l$ is upper semi-Fredholm operator and $μ_0 \in ρ_0(f(T))$ or $μ_0 \in σ_0(f(T)) \setminus σ_{ea}(f(T))$. If $μ_0 \in ρ_0(f(T))$, then $T - μ_0l$ is bounded from below, which means that each $T - λ_1l$ is bounded from below. Then $μ_0 \notin f(σ_{ea}(T))$. If $μ_0 \in σ_0(f(T)) \setminus σ_{ea}(f(T))$, since property $(ω_1)$ holds for $T$, we know that $T - μ_0l$ is Browder. Hence $T - λ_1l$ is Browder and $λ_1 \notin σ_{ea}(T)$.

Next we will prove if $σ_0(T) \neq σ_{ea}(T)$, then $σ(T) = σ_{ea}(T)$. Let $λ_0 \in σ_0(T) \setminus σ_{ea}(T)$. Then $T - λ_0l$ is Browder because property $(ω_1)$ holds for $T$. For any $μ_0 \notin σ_0(T)$, $n(T - μ_0l) = 0$. Let $f(T) = (T - μ_0l)(T - λ_0l)$, then $0 \in σ_0(f(T)) \setminus σ_{ea}(f(T))$. Since $f(T)$ has property $(ω_1)$, we know that $T - μ_0l$ is Browder. The fact $n(T - μ_0l) = 0$ tells us that $T - μ_0l$ is invertible, which means that $μ_0 \notin σ(T)$, hence $σ(T) = σ_{ea}(T)$.

$(2) \Rightarrow (1)$. Let $μ_0 \in σ_{ea}(f(T)) \setminus σ_{ea}(f(T))$, then $f(T - μ_0l) \in SF_∞(X)$ and $n(f(T - μ_0l) > 0$. Let

$$f(T - μ_0l) = (T - λ_1l)^{μ_1}(T - λ_2l)^{μ_2} \cdots (T - λ_kl)^{μ_k}g(T),$$

where $λ_1 \neq λ_2$ and $g(T)$ is invertible. Since $σ_{ea}(f(T)) = f(σ_{ea}(T))$ and $μ_0 \notin σ_{ea}(f(T))$, it follows that $λ_1 \neq σ_{ea}(T)$. Then $T - λ_1l \notin SF_∞(X)$. Let $n(T - λ_1l) = 0$ if $1 \leq i \leq j$ and $n(T - λ_1l) > 0$ if $j < i \leq k$. Then $T - λ_1l$ is bounded from below if $1 \leq i < j$. Using the fact $σ(T) = σ_{ea}(T)$ we know that $T - λ_1l$ is invertible. If $j < i \leq k$, then $λ_1 \notin σ_{ea}(T) \setminus σ_{ea}(T)$, since $T$ has property $(ω_1)$, $T - λ_1l$ is Browder. Thus $f(T) - μ_0l$ is Browder and $μ_0 \in π_{00}(f(T))$. Hence property $(ω_1)$ holds for $T$.

$(1) \Rightarrow (3)$. Suppose that there exist $λ, μ \in \mathbb{C} \setminus σ_{fa}(T)$ such that $ind(T - λl)ind(T - μl) < 0$. Let $ind(T - λl) = k > 0$, then $T - λl$ is Fredholm. If $ind(T - μl) = t < 0$, $t \neq ∞$, then let $f(T) = (T - λl)^{k}(T - μl)^{t}$; if $ind(T - μl) = ∞$, then let $f(T) = (T - λl)^{t}(T - μl)^{∞}$.

Thus $0 \in σ_0(f(T)) \setminus σ_{ea}(f(T))$, since $f(T)$ has property $(ω_1)$, we know that $f(T)$ is Browder. Thus $T - λl$ and $T - μl$ are Browder. It is in contradiction to the fact that $ind(T - λl) > 0$. Hence for each pair $λ, μ \in \mathbb{C} \setminus σ_{fa}(T)$, $ind(T - λl)ind(T - μl) \geq 0$.

$(3) \Rightarrow (1)$. Let $μ_0 \notin σ_{ea}(f(T)) \setminus σ_{ea}(f(T))$, then $f(T - μ_0l) \in SF_∞(X)$. Let

$$f(T - μ_0l) = (T - λ_1l)^{μ_1}(T - λ_2l)^{μ_2} \cdots (T - λ_kl)^{μ_k}g(T),$$

where $λ_1 \neq λ_2$ and $g(T)$ is invertible. Then for any $λ_1, λ_2, T - λ_1l$ is upper semi-Fredholm and $\sum_{i=1}^{k} ind(T - λ_i,l)^{μ_i} \leq 0$. By condition $ind(T - λ_1l) \leq 0$, we get that $T - λ_1l \in SF_∞(X)$. Using the same way of $(2) \Rightarrow (1)$, we can prove property $(ω_1)$ holds for $f(T)$. $\square$

If $σ_{fa}(T) = σ_1(T)$, then for any $λ \in ρ_{fa}(T)$, $ind(T - λl) \leq 0$ [3], where $ρ_{fa}(T) = \{λ \in \mathbb{C}: T - λl$ is semi-Fredholm operator]. In addition, if $σ_{fa}(T) = σ_1(T)$, then $T$ is f-a-polaroid. In fact, let $λ_0 \in isoσ_0(T)$, then $λ_0 \notin σ_1(T)$. Thus $λ_0 \notin σ_{fa}(T)$.
and $T - \lambda_0 I$ is lower semi-Fredholm. Since $\lambda_0 \in \sigma_{\text{reg}}(T)$, it follows that $T$ has single valued extension property in $\lambda_0$. Then $\text{asc}(T - \lambda_0 I) < +\infty$ and $n(T - \lambda_0 I) < \infty$. This implies that $T - \lambda_0 I$ is Fredholm operator. By condition $\text{ind}(T - \lambda I) \leq 0$, we get that $T - \lambda_0 I \in \mathcal{SF}_{\text{w}}(X)$. Hence $T$ is a-polaroid.

**Corollary 2.1.** Suppose $\sigma_{\text{SF}_{\text{w}}}(T) = \sigma_1(T)$ and property $(\omega_1)$ holds for $T$, then for any $f \in H(T)$, $f(\sigma_1(T)) = \sigma_1(f(T))$.

**Proof.** Let $\mu_0 \in f(\sigma_1(T))$ and $\mu_0 = f(\lambda_0)$, where $\lambda_0 \in \sigma_1(T)$. If $\mu_0 \notin f(\sigma_1(T))$, then there exists $\delta > 0$ such that $f(T) - \mu I$ is generalized upper semi-Fredholm if $0 < |\mu - \mu_0| \leq \delta$. For any $\mu$, there exists $\delta' > 0$ such that $f(T) - \mu' I \in \mathcal{SF}_{\text{w}}(X)$ and $N(f(T) - \mu' I) \leq 1 = \cap_{i=0}^{\infty} R(f(T) - \mu' I)^{i+1}$. Let

$$f(T) - \mu I = (T - \lambda_1 I)^{m_1} (T - \lambda_2 I)^{m_2} \cdots (T - \lambda_k I)^{m_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then for any $\lambda_i' (1 \leq i \leq k)$, $T - \lambda_i' I$ is upper semi-Fredholm. Since $\sigma_{\text{SF}_{\text{w}}}(T) = \sigma_1(T)$, it follows that $\text{ind}(T - \lambda_i' I) \leq 0$. If $n(T - \lambda_i' I) = 0$, then $\text{asc}(T - \lambda_i' I) < +\infty$; if $n(T - \lambda_i' I) > 0$, then $\lambda_i' \in \sigma_{\text{reg}}(T) \setminus \sigma_0(T)$. Since property $(\omega_1)$ holds for $T$, we know that $T - \lambda_i' I$ is Browder and $\text{asc}(T - \lambda_i' I) < +\infty$. Hence $\text{asc}(T - \mu I) < +\infty$ and $N(f(T) - \mu I') = N(f(T) - \mu I) \cap \cap_{i=0}^{\infty} R(f(T) - \mu' I)^{i+1} = \{0\}$. Thus $\mu \notin \sigma_{\text{reg}}(f(T)) \cup \sigma_{\text{reg}}(f(T))$. For $\lambda_0$, there exists $\epsilon > 0$ such that $f(\lambda) \in \sigma_{\text{reg}}(f(T)) \cup \sigma_{\text{reg}}(f(T))$ if $0 < |\lambda - \lambda_0| < \epsilon$. Then $\lambda \in \sigma_{\text{reg}}(T) \cup \sigma_0(T)$. Thus $\lambda \notin \sigma_1(T)$. Since $\sigma_{\text{SF}_{\text{w}}}(T) = \sigma_1(T)$, $T - \lambda_0 I$ is Fredholm with the index $\text{ind}(T - \lambda_1 I) \leq 0$. Now we have proved that for $\lambda_0 > 0$ such that $T - \lambda_0 I \in \mathcal{SF}_{\text{w}}(X)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Thus $\lambda_0 \notin \sigma_1(T)$. It is in contradiction to the fact that $\lambda_0 \in \sigma_1(T)$. Hence $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$ for any $f \in H(T)$.

Conversely, let $\mu_0 \notin f(\sigma_1(T))$ and $f(T) - \mu_0 I = (T - \lambda_1 I)^{m_1} (T - \lambda_2 I)^{m_2} \cdots (T - \lambda_k I)^{m_k} g(T)$, where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $\lambda_i \notin \sigma_1(T)$. Since $\lambda_i \notin \sigma_{\text{SF}_{\text{w}}}(T)$, it follows that $T - \lambda_i I$ is Fredholm with the index $\text{ind}(T - \lambda_i I) \leq 0$. Then $f(T) - \mu_0 I \in \mathcal{SF}_{\text{w}}(X)$ and $\mu_0 \notin f(\sigma_1(T))$. □

### 3. Property $(\omega_1)$ and hypercyclic (supercyclic) operators

In the following, $H$ denotes an infinite dimensional complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on $H$. For $x \in H$, the orbit of $x$ under $T$ is the set of images of $x$ under successive iterates of $T$: $\text{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$. A vector $x \in H$ is hypercyclic if the set of scalar multiples of $\text{Orb}(T, x)$ is dense in $H$, and $x$ is hypercyclic if $\text{Orb}(T, x)$ is dense. A hypercyclic operator is one that has a hypercyclic vector. We similarly define the notion of supercyclic operator. We denote by $HCH(H)$ ($SC(H)$) the set of all hypercyclic (supercyclic) operators in $B(H)$ and $HCH(H) \cap SC(H)$ the norm-closure of the class $HCH(H)$ ($SC(H)$). Supercyclic operators were introduced by Hilden and Wallen in 1974 [8]. Many fundamental results of the theory of hypercyclic and supercyclic operators were established by C. Kitai in her thesis [9]. The essential facts for hypercyclic operators and supercyclic operators were described by Herrero in 1991 [7].

**Theorem 3.1.** Suppose $T \in B(H)$ has property $(\omega_1)$, then

1. $T \in HCH(H) \Leftrightarrow \sigma_0(T) = \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T) \setminus \partial D$ is connected;\)
2. $T \in SC(H) \Leftrightarrow \sigma_0(T) = \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T) \setminus \partial D$ is connected for some $r \geq 0$.

**Proof.** Suppose $T \in HCH(H)$. Since $\sigma(T) = \sigma_0(T)$, it follows that $\sigma_1(T) \subseteq \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$. Let $\lambda_0 \notin \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$. There exists $\epsilon > 0$ such that $T - \lambda I$ is generalized upper semi-Fredholm if $0 < |\lambda - \lambda_0| < \epsilon$. Using the fact that property $(\omega_1)$ holds for $T$, we can prove that $\lambda \in \sigma_0(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Therefore $\lambda \in \sigma_0(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$. Thus $\lambda_0 \in \sigma_0(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$, it follows that $\lambda_0 \notin \partial D$. Thus $\lambda_0 \notin \sigma_0(T)$. Hence $\sigma_0(T) = \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T)) \cup \sigma(T)$. Since $T \in HCH(H)$ and property $(\omega_1)$ holds for $T$, $\sigma_0(T) \cup \partial D = \sigma_1(T) \cup \partial D = \sigma(T) \cup \partial D$ is connected.

Conversely, we prove $\sigma_0(T) = \sigma(T)$ first. Let $\lambda \notin \sigma_0(T)$, then $T - \lambda I$ is Weyl. Since $T$ has property $(\omega_1)$, this implies that Browder’s theorem holds for $T$, we know that $T - \lambda I$ is Browder and $\lambda \notin \sigma_0(T)$. Since $\sigma_0(T) = \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T))$, it follows that $\lambda \notin \sigma_0(T)$. Then $T - \lambda I$ is invertible. Therefore $\sigma(T) = \sigma_0(T) = \sigma(T)$ and $\sigma(T) = \sigma(T) \setminus \partial D$. Since $\sigma(T) \cup \partial D$ is connected, we know that $\sigma_0(T) \cup \partial D$ is connected. If there exists $\lambda \in \rho_0(T)$ such that $\text{ind}(T - \lambda I) > 0$, then $\lambda \notin \sigma_0(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T))$. If $\lambda \in \sigma_0(T)$, then $T - \lambda I$ is Browder. It is in contradiction to the fact that $\text{ind}(T - \lambda I) < 0$. Thus $\lambda \notin \sigma_1(T) \cup \text{acc}([\sigma(T)] \cup \sigma(T))$. Thus $T - \lambda I$ is Browder, it is a contradiction again. Hence for any $\lambda \in \rho_0(T)$, $\text{ind}(T - \lambda I) \geq 0$. Then $T \in HCH(H)$ [7].

Similarly, we can prove $(2)$. □

**Remark 3.1.** If $T \in HCH(H)$, then $\sigma_0(T) = \sigma(T)$, $\sigma_0(T) = \sigma_0(T)$. $T \in HCH(H)$ implies that:

- $T$ has property $(\omega) \Leftrightarrow$ Weyl’s theorem holds for $T \Leftrightarrow T$ satisfies a-Weyl’s theorem;
- property $(\omega_1)$ holds for $T \Leftrightarrow T$ satisfies Browder’s theorem $\Leftrightarrow$ Browder’s theorem holds for $T$. 
Thus by [3, Theorems 2.1, 2.2 and Corollary 3.5], we have the following results.

**Theorem 3.2.** Suppose $T \in \overline{HC(H)}$, then

1. property $(\omega_1)$ holds for $T \iff acc\sigma_b(T) = \sigma_1(T) \cup acc[iso\sigma_a(T)];$
2. $T$ is f-isoloid and property $(\omega)$ holds for $T \iff acc\rho(T) = \sigma_1(T) \cup acc[iso\sigma_a(T)];$
3. $\sigma(T) = s_1(T) \iff T$ is f-isoloid and property $(\omega)$ holds for $T.$

**Corollary 3.1.** If $T \in B(H)$ is f-a-polaroid and property $(\omega_1)$ holds for $T,$ then

1. $T \in \overline{HC(H)} \iff \sigma(T) = \sigma_1(T)$ and $\sigma(T) \cup \partial D$ is connected;
2. $T \in \overline{SC(H)} \iff \sigma(T) = \sigma_1(T)$ or $\sigma(T) = \sigma_1(T) \cup \{\lambda\},$ where $\lambda \neq 0$ and $T - \lambda I$ is Browder, and $\sigma(T) \cup \partial\sigma(D)$ is connected for some $r \geq 0.$

**Proof.** Suppose $T \in \overline{HC(H)}.$ We need to prove $\sigma(T) \subseteq \sigma_1(T).$ Let $\lambda_0 \notin \sigma_1(T).$ Then there exists $\epsilon > 0$ such that $T - \lambda I$ is generalized upper semi-Fredholm if $0 < |\lambda - \lambda_0| < \epsilon.$ For this $\lambda,$ there exists $\epsilon' > 0$ such that $T - \lambda' I \in SF_+^\infty(X)$ and $N(T - \lambda' I) \subseteq \bigcap_{n=0}^{\infty} R(T - \lambda' I)^n$ if $0 < |\lambda' - \lambda| < \epsilon'.$ Since $\sigma_{sa}(T) = \sigma_{sf}(T),$ we know that $T - \lambda' I$ is Weyl. $T$ has property $(\omega_1),$ this implies that Browder's theorem holds for $T.$ Then $T - \lambda' I$ is Browder. Thus $N(T - \lambda' I) = N(T - \lambda' I) \cap \bigcap_{n=0}^{\infty} R(T - \lambda' I)^n = \{0\},$ which means that $T - \lambda' I$ is invertible. Therefore $\lambda \in iso\sigma(T) \cup \rho(T).$ If $\lambda \in iso\sigma(T),$ since $T$ is f-a-polaroid, it follows that $T - \lambda I$ is Weyl. Then $T - \lambda I$ is Browder. It is in contradiction to the fact that $\sigma(T) \setminus \sigma_{sf}(T) = \emptyset$ [11]. Thus $\lambda \in \rho(T).$ It induces that $\lambda_0 \notin iso\sigma(T) \cup \rho(T).$ Using the same way, we prove that $T - \lambda I$ is invertible, which means that $\lambda \notin \sigma(T).$ Since $\sigma(T) \subseteq iso\sigma(T) \subseteq \sigma(T)$ and $\sigma(T) = \sigma_1(T),$ it follows that $\sigma_{sf}(T) = \sigma(T).$ The condition $T \in \overline{HC(H)}$ tells us $\sigma_{sf}(T) \cup \partial D$ is connected, then $\sigma(T) \cup \partial D$ is connected.

For the converse, suppose $\sigma(T) = \sigma_1(T)$ and $\sigma(T) \cup \partial D$ is connected. Since $\sigma_{sf}(T) = \sigma(T),$ it follows that $\sigma_{sf}(T) \cup \partial D$ is connected. Using the fact that $iso\sigma(T) \cap \sigma_1(T) = \emptyset$ and $\sigma(T) = \sigma_1(T),$ we know that $iso\sigma(T) = \emptyset.$ Thus $\sigma(T) \setminus \sigma_{sf}(T) = \emptyset.$ If there exists $\lambda \in \rho_{sf}(T)$ such that $ind(T - \lambda I) < 0,$ then $\lambda \notin \sigma_1(T)$ hence $\lambda \notin \sigma(T),$ which means that $T - \lambda I$ is invertible.

It is in contradiction to the fact that $\sigma(T) \setminus \sigma_1(T) = \emptyset.$ Hence for any $\lambda \in \rho_{sf}(T),$ $ind(T - \lambda I) \geq 0.$ Using Lemma 3.1 in [3], $T \in \overline{HC(H)}$.

Similarly, we can prove (2). \( \square \)

The Weyl's theorem for $T$ is not sufficient for the Weyl's theorem for $T + F$ with finite rank [10]. So does a-Weyl's theorem [4]. But if $T \in \overline{HC(H)}$ or $T \in \overline{SC(H)},$ we have:

**Theorem 3.3.** Suppose that $T \in \overline{HC(H)}$ or $T \in \overline{SC(H)}.$ If $T$ is f-a-polaroid and property $(\omega_1)$ holds for $T,$ then for every finite rank operator $F$ commuting with $T,$ $T + F$ is f-a-polaroid and satisfies property $(\omega_1).$

**Proof.** Suppose $T \in \overline{HC(H)}.$ From Theorem 2.3, we need to prove that $\sigma_0(T + F) \subseteq \sigma_1(T + F) \cup acc\sigma_{iso}(T + F) \cup [\sigma(T + F) \cap \rho_0(T + F)].$ Let $\lambda_0 \notin \sigma_1(T + F) \cup acc\sigma_{iso}(T + F) \cup [\sigma(T + F) \cap \rho_0(T + F)].$ Then there exists $\epsilon > 0$ such that $T + F - \lambda I$ is generalized upper semi-operator if $0 < |\lambda - \lambda_0| < \epsilon.$ For this $\lambda,$ there exists $\epsilon' > 0$ such that $T + F - \lambda'I \in SF_+^\infty(X)$ if $0 < |\lambda' - \lambda| < \epsilon'.$ Then $T - \lambda' I \in SF_+^\infty(X).$ Since $T \in \overline{HC(H)},$ $T - \lambda' I$ is Weyl. $T$ satisfies property $(\omega_1),$ this induces that $T - \lambda' I$ is Browder. Then $T - \lambda' I$ is invertible. Therefore $\lambda \in iso\sigma(T) \cup \rho(T).$ We claim that $\lambda \in \rho(T).$ If $\lambda \in iso\sigma(T),$ the fact that $T$ is f-a-polaroid and property $(\omega_1)$ imply that $T - \lambda I$ is Browder. Since $\sigma(T) \setminus \sigma_{sf}(T) = \emptyset,$ we know that $T - \lambda I$ is invertible, a contradiction. Once again, we get that $\lambda_0 \in \rho(T).$ Thus $T - \lambda_0 I$ is invertible. Then $T + F - \lambda_0 I$ is Browder. \( \square \)

This proves that $T + F$ is f-a-polaroid and satisfies $(\omega_1).$ \( \square \)

**References**