JOURNAL OF ALGEBRA 77, 275–285 (1982)

On Pure-Projective Modules over Artin Algebras

FRANK OKOH

*Department of Mathematics and Statistics,
Queen's University at Kingston, Kingston, Ontario, Canada K7L 3N6*

Communicated by P. M. Cohn

Received August 13, 1979

In this paper pure-projective modules over some Artin algebras are investigated. The prototype is the Artin algebra, R, consisting of 3×3 complex matrices of the form

$$\begin{bmatrix} \beta & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ 0 & 0 & \gamma \end{bmatrix}.$$

It is shown that a module over an Artin algebra is pure-projective if and only if it is a direct sum of finite-dimensional indecomposable submodules. Conditions under which an extension of a pure-projective R-module by another pure-projective R-module is pure projective are given. A homological characterization of finite-dimensional pure-projective R-modules is also obtained. Finally an $\aleph_r$-pure-projective is shown to be pure-projective if and only if $r \geqslant 1$.

INTRODUCTION

In this paper we are interested in pure-projective modules over hereditary algebras of tame type. We shall let S stand for such an algebra while T will stand for an arbitrary Artin algebra. A prototype of S is the Artin algebra, R, of 3×3 complex matrices of the form:

$$\begin{bmatrix} \beta & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ 0 & 0 & \gamma \end{bmatrix}.$$

The category of R-modules is equivalent to the category of systems. (A system is a pair of complex vector spaces (V, W) together with a $\mathbb{C}$-bilinear map from $\mathbb{C}^2 \times V$ to W; see [2] for details.) We begin in Section 1 by

0021-8693/82/080275-11$02.00/0

proving that a module over an arbitrary Artin algebra is pure-projective if and only if it is a direct sum of finite-dimensional submodules. The proof consists of putting together known results in a correct order. A pure submodule of a pure-projective module is not always pure-projective, unless it is countable generated; see [11] and Corollary 1.5 of [6]. We thank the referee for drawing our attention to [11]! Call a module $\aleph_r$-pure-projective if and only if every submodule of rank $<\aleph_r$ is pure-projective; see [12] for a definition of "rank of an S-module" and [8] for that of "rank of a system." The definition in [12] restricted to R is equivalent to the definition in [8]. The example of a subsystem of a pure-projective system which is not pure-projective given in [9] is modified to establish that a torsion-free $\aleph_r$-pure-projective system is always pure-projective if and only if $r \geqslant 1$.

In Section 2 extensions of pure-projective modules by pure-projective modules is investigated. To describe the results we recall the three classes of indecomposable finite-dimensional systems: systems of type I^m, of type II_θ^m, $\theta \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, of type III^m, respectively, m any positive integer. These correspond respectively to the indecomposable preinjective, indecomposable regular torsion and indecomposable preprojective modules of [12]. By abuse of language we shall say that an S-module is of type I if it is a direct sum of indecomposable preinjective submodules. Modules of type II or III are analogously defined. The former is an analogue of direct sums of cyclic groups, i.e., pure-projective torsion groups by Theorem 30.2 of [5]. In [3] it is shown that an extension of a pure-projective torsion group G_1 by another pure-projective torsion group G_2 is not always pure-projective except when G_2 is bounded. The results in Section 2 have a similar flavour with the exception of Proposition 2.1.

We shall assume familiarity with [9] especially Sections 0 and 1.

1. Submodules of Pure-Projective Modules

THEOREM 1.1. *A module M over an Artin algebra is pure-projective if and only if it is a direct sum of finite-dimensional submodules.*

Proof. Let $\mathscr{F}$ be the class of finite-dimensional modules over the given Artin algebra and $\{N_\alpha\}_{\alpha \in A}$ the set of finite-dimensional indecomposable submodules of M. Let $N = \bigoplus_{\alpha \in A} N_\alpha$. Purity as defined in [12] is $\mathscr{F}$-purity. Hence as in the proof of Theorem 2.3 of [8] we have a short pure-exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0.$$

If M is pure-projective, then N is isomorphic to $M \oplus K$. Each N_α has local endomorphism ring by Corollary 11.8 of [1]. By the Crawley–Jønsson–Warfield theorem (see Theorem 26.5 of [1]),

$$M = \bigoplus_{\beta \in B} M_\beta,$$

$$K = \bigoplus_{\gamma \in C} K_\gamma,$$

where M_β, K_γ are countably generated modules with local endomorphism rings, hence indecomposable. By Corollary 26.6 of [1] each M_β is isomorphic to some N_α. Therefore M is a direct sum of finite-dimensional indecomposable submodules. The converse follows from the definitions as in the proof of Theorem 2.3 of [8]. ∎

A straightforward modification of Example 1.5 of [9] gives a proof of

PROPOSITION 1.2. *Any unbounded pure-projective torsion-free system has a subsystem of rank* $\aleph_0$ *that is not pure-projective.*

We use Proposition 1.2 to prove

PROPOSITION 1.3. *An* $\aleph_r$*-pure-projective torsion-free system is always pure-projective if and only if* $r \geqslant 1$.

Proof. The system in Proposition 1.3 of [9] is $\aleph_0$-pure-projective by Lemma 1.3.3 of [9] but is not pure-projective by 1.3.2 by [9]. Suppose then that (V, W) is $\aleph_r$-pure-projective, $r \geqslant 1$. We shall show that (V, W) must be bounded hence pure-projective by Theorem 1.1 of [9]. Since (V, W) is $\aleph_r$-pure-projective $tc_{(V,W)}(\phi, \{w\})$ for any nonzero w in W must be of type III^m by Theorem 1.1 and the fact that the systems of type III^m are the only indecomposable pure-projective torsion-free systems of rank 1. Suppose (V, W) is not bounded then there exist $\{w_i\}_{i=1}^\infty \subset W$ such that $(V_i, W_i) = tc_{(V,W)}(\phi, \{w_i\})$ is of type III^{k_i} and $k_1 < k_2 < \cdots$. The subsystem $\sum_{i=1}^\infty (V_i, W_i)$ is of type $\bigoplus_{i=1}^\infty \mathrm{III}^{k_i}$: To see that the sum is direct one notes that $(V_1, W_1) + (V_2, W_2)$ is direct since if $(V_1, W_1) \cap (V_2, W_2) \neq (0, 0)$, then $(V_1, W_1) = (V_2, W_2)$ by Lemma 4.1 of [8], contradicting $k_1 < k_2$. We suppose that $(V_1, W_1) + \cdots + (V_n, W_n)$ is direct. Again by Lemma 4.1 of [8] if (V_{n+1}, W_{n+1}) intersects $(V_1, W_1) \dotplus \cdots \dotplus (V_n, W_n)$ nontrivially, then $(V_{n+1}, W_{n+1}) \subset (V_1, W_1) \dotplus \cdots \dotplus (V_n, W_n)$. However $k_{n+1} > \max\{k_1, k_2, \ldots, k_n\}$ and so the inclusion is impossible by the height relation in 1.4 of [9]. This proves that $\sum_{i=1}^\infty (V_i, W_i)$ is an unbounded pure-projective subsystem of (V, W). By Proposition 1.2, (V, W) would have a non-pure-projective subsystem of rank $\aleph_0$ contradicting the hypothesis that (V, W) is $\aleph_r$-pure-projective, $r \geqslant 1$. Therefore (V, W) is bounded and hence pure-projective by Theorem 1.1 of [9]. ∎

Using the table in [4] we get the following characterisation of finite-dimensional systems:

PROPOSITION 1.4. *A system* (V, W) *is finite-dimensional if and only if* $\mathrm{Ext}((V, W), \mathrm{III}^1)$ *and* $\mathrm{Ext}(\mathrm{I}^1, (V, W))$ *are finite-dimensional.*

Proof. If (V, W) is finite-dimensional, then $\mathrm{Ext}((V, W), \mathrm{III}^1)$ and $\mathrm{Ext}(I^1, (V, W))$ are finite-dimensional by [4]. Suppose then that $\mathrm{Ext}((V, W), \mathrm{III}^1)$ and $\mathrm{Ext}(\mathrm{I}^1, (V, W))$ are finite-dimensional: If the divisible part of (V, W), $\mathrm{Div}(V, W)$, were infinite-dimensional, then it would either contain a system of type $\mathrm{II}_\theta^\infty$ or $\mathscr{R}$ or a system of type $\bigoplus_{j \in J} \mathrm{I}^{k_j}$, $\mathrm{Card}(J)$ infinite. The hypothesis and the table in [4] rule out these possibilities. Therefore $(V, W) = \mathrm{Div}(V, W) \oplus (V_1, W_1)$, (V_1, W_1) reduced and $\mathrm{Div}(V, W)$ finite-dimensional. Suppose $t(V_1, W_1)$ is infinite-dimensional, then again the given Ext's would be infinite-dimensional because, for any n, $t(V_1, W_1)$ would have a direct summand of type $\mathrm{II}_{\theta_1}^{k_1} \oplus \cdots \oplus \mathrm{II}_{\theta_n}^{k_n}$, θ_i's not necessarily different, by Corollary 9.16(b) of [1]. So $t(V_1, W_1)$ is finite-dimensional. It is a direct summand of (V_1, W_1) by Theorems 5.5 and 9.12 of [1]. So $(V_1, W_1) = t(V_1, W_1) \oplus (V_2, W_2)$, where (V_2, W_2) is torsion-free. Because $\mathrm{Ext}((V, W), \mathrm{III}^1)$ is finite-dimensional, (V_2, W_2) does not have subsystems of type III^m for arbitrarily large m, i.e., (V_2, W_2) is bounded. Therefore by Theorem 1.1 of [9], (V_2, W_2) is a direct sum of subsystems of type III^{m_j}. (V_2, W_2) is of type $\bigoplus_{j \in J} \mathrm{III}^{m_j}$. If $\mathrm{Card}(J)$ were infinite, the hypothesis that $\mathrm{Ext}(I^1, (V, W))$ is finite-dimensional would be contradicted. Thus (V, W) is finite-dimensional. ∎

In a similar vein one proves

PROPOSITION 1.5. *A system* (V, W) *is a direct sum of a projective system and a finite-dimensional system if and only if* $\mathrm{Ext}(V, W)$, $\mathrm{III}^1 \oplus \mathrm{III}^2)$ *is finite-dimensional.*

2. EXTENSIONS OF PURE-PROJECTIVE MODULES BY PURE-PROJECTIVE MODULES

The first two propositions dispose of cases that are already treated in the literature or readily deduced therefrom.

PROPOSITION 2.1. *Extensions of pure-projective S-modules by pure-projective S-modules are pure-projective in the following cases:*

(i) *Extensions of modules of type* I *by modules of type* I.

(ii) *Extensions of modules of type* I *by modules of type* II.

(iii) *Extensions of type* I *by modules of type* III.

(iv) *Extensions of modules of type* II *by modules of type* III.

Proof. (i) See Proposition 3.4 of [12].

(ii) A module of type I is divisible, hence $\mathrm{Ext}(\mathrm{II}, \mathrm{I}) = 0$ by Corollary 3.5 of [12] and the fact that $\mathrm{Ext}(\bigoplus_{j\in J} A_j, B)$ is isomorphic to $\prod_{j\in J} \mathrm{Ext}(A_j, B)$.

(iii) Similar to (ii).

(iv) Follows from the facts that the torsion part of a module is a pure submodule of the module, Theorem 4.1 of [12] and a module of type III is pure-projective. ∎

PROPOSITION 2.2. *If* $0 \to G_1 \to G \to G_2 \to 0$ *is an exact sequence with* G_1, G_2 *of type* II. *Then* G *is always pure-projective if and only if* G_2 *is bounded.*

Proof. See [3]. ∎

Let M be a torsion S-module with no preinjective direct summand. Then by 4.5 of [12], $M = \sum_{t\in T} \cdot M_t$, where each M_t may be considered as a module over a principal ideal domain. We shall call M a bounded module if $M_t = 0$ for all but a finite number of t in T, and each nonzero M_t is bounded as a module over a PID. For future use we note that the results on bounded R-modules, i.e., systems, proved in [10] generalize to torsion S-modules with only formal changes. In particular Theorems 2.9 and 3.5 are valid for S-modules. Hence bounded modules are pure-projective and pure injective.

PROPOSITION 2.3. *An extension of a bounded* S*-module* M_2 *by an* S*-module* M_1 *of type* I *is pure-projective.*

Proof. Let $0 \to M_2 \to^{\kappa} M \to^{\rho} M_1 \to 0$ be an exact sequence. For any torsion-free S-module, N, $\mathrm{Ext}(N, M_1) = 0 = \mathrm{Ext}(N, M_2)$ by Proposition 4.7 of [12] and Theorem 3.5 of [10]. Therefore $\mathrm{Ext}(N, M) = 0$ for all torsion-free S-modules N. Hence, by Theorem 3.5 of [10], M is a direct sum of a divisible module and a bounded module. By 4.2 of [10], M is torsion and a torsion divisible module is a direct sum of a pure-projective module and Prüfer-type divisible modules, by Corollary 2 of 4.7 in [10]. Hence to show that M is pure-projective it is enough to show that it has no Prüfer module as a direct summand. Suppose $S^{\omega} = \bigcup_n S^n$ is such a summand of M. Let $M = S^{\omega} \dot{+} M'$. Since M_2 is bounded, $\mathrm{proj}_{S^{\omega}} | \kappa(M_2)$ is contained in S^{n_0} for some n_0. We have the exact sequence

$$0 \longrightarrow (S^{n_0} + M_2)/S^{n_0} \xrightarrow{\bar{\kappa}} S^{\omega}/S^{n_0} \oplus M' \xrightarrow{\bar{\rho}} M_1/\rho(S^{n_0}) \longrightarrow 0$$

where $\bar{\kappa}$ and $\bar{\rho}$ are induced by κ and ρ. The image of $\bar{\kappa}$ has zero component in S^ω/S^{n_0} and so S^ω/S^{n_0} is a direct summand of $M_1/\rho(S^{n_0})$. This is impossible because the latter module is of type I by 3.4 of [12] and so cannot have the regular module S^ω/S^{n_0} as a direct summand. Therefore M is pure-projective.

For our counterexamples we shall recall the description of indecomposable finite-dimensional systems by chains: Let (V, W) be a system, $v_i \in V$, $w_i \in W$, and (a, b) a fixed basis of $\mathbb{C}^2$.

(a) A chain $((v_1, v_2,..., v_m), (w_1, w_2,..., w_{m-1}))$ is said to be of type I^m if $av_1 = 0 = bv_m$, $av_{i+1} = bv_i = w_i$, $i = 1,..., m-1$.

(b) A chain $((v_1, v_2,..., v_m), (w_1, w_2,..., w_m))$ is said to be of type II^n_∞ if $av_1 = 0$, $bv_m = w_m$, $av_{i+1} = bv_i = w_i$, $i = 1,..., m-1$. Let $b_\theta = b - \theta a$ for $\theta \in \mathbb{C}$. If $b_\theta v_1 = 0$, $av_m = w_m$, $b_\theta v_{i+1} = av_i = w_i$, the chain is said to be of type II^n_θ.

(c) A chain $((v_1, v_2,..., v_{m-1}), (w_1, w_2,..., w_m))$ is said to be of type III^m if $av_1 = w_1$, $bv_{m-1} = w_m$, $av_i = w_i = bv_{i-1}$, $i = 2,..., m-1$. If $m = 1$, the chain is $(\varnothing, w_1)$.

Let V^1 and W^1 be the respective spans of the v_i's and w_i's. The subsystem (V^1, W^1) of (V, W) is called the subsystem spanned by $((v_i), (w_i))$. In case the v_i's and w_j's form bases of V^1 and W^1, respectively, (V^1, W^1) is itself called a subsystem of type I^m, II^m_θ, or III^m depending on the type of chain which spans it.

We can now show that it is necessary in Proposition 2.3 that M_2 be bounded. Using the chain representations above one sees that a system of type $\mathrm{I}^{m+1} \oplus \mathrm{II}^m_\infty$ contains a "diagonal" subsystem of type II^m_∞. We illustrate this in the case $m = 1$: let $(u_1, u_2), (z_1))$ and $((x_1), (y_1))$ span chains of type I^2, II^1_∞, respectively. The diagonal subsystem of type II^1_∞ is spanned by $((u_1 + x_1), (z_1 + y_1))$. Let (V, W) be a system of type $\oplus_{k=1}^\infty \mathrm{I}^k \oplus \mathrm{II}^\infty_\infty$. A system of type $\mathrm{II}^\infty_\infty$ is an ascending union of systems of type II^n_∞. It is indecomposable. Therefore, by Theorem 1.1, (V, W) is not pure-projective. $(V_1, W_1) = \oplus_{k=1}^\infty \cdot (V^k, W^k)$, where (V^k, W^k) is the diagonal subsystem of $\mathrm{I}^{k+1} \oplus \mathrm{II}^k_\infty$ of type II^k_∞. The quotient $(V, W)/(V_1, W_1)$ is of type $\oplus_{k=1}^\infty \mathrm{I}^k$. Therefore (V, W) is a non-pure-projective extension of a system of type II by one of type I.

PROPOSITION 2.4. *An extension of a system of type* III *by one of type* I *is always pure-projective if and only if the system of type* III *is finite-dimensional.*

Proof. Suppose we have an exact sequence

$$0 \longrightarrow (V_1, W_1) \longrightarrow (V, W) \longrightarrow (V_2, W_2) \longrightarrow 0$$

with (V_1, W_1) finite-dimensional of type III and (V_2, W_2) of type I. Let $(V, W) = (V^1, W^1) \dotplus (V^2, W^2)$, where (V^1, W^1) is the maximal pure-projective divisible submodule of (V, W). Since (V_1, W_1) is finite-dimensional its image under the projection of (V, W) onto (V^1, W^1) is finite-dimensional. So we may suppose that it is contained in $(V^1_1, W^1_1) \dotplus (V^2, W^2)$, where (V^1_1, W^1_1) is a finite-dimensional direct summand of (V^1, W^1). So $(V, W) = (V^1_1, W^1_1) \dotplus (V^1_2, W^1_2) \dotplus (V^2, W^2)$ with $(V^1, W^1) = (V^1_1, W^1_1) \dotplus (V^1_2, W^1_2)$. Since (V^1_2, W^1_2) is pure-projective we may now suppose that $(V, W) = (V^1, W^1) \dotplus (V^2, W^2)$ with (V^1, W^1) finite-dimensional. In that case we shall show that (V_2, W_2) is finite-dimensional, hence (V, W) would be pure-projective. Let $t(V^2, W^2) = \sum_{\theta \in \tilde{\mathbb{C}}} \cdot t(V^2, W^2)_\theta$ be the decomposition of the torsion part of (V^2, W^2) into its primary parts. Suppose $t(V^2, W^2)_\theta \neq 0$ for infinitely many θ in $\mathbb{C}$. As (V_1, W_1) is finite dimensional, (V, W) would have a direct summand of type II^n_ν (say) such that (V_1, W_1) is contained in a direct complement. This would imply that (V_2, W_2) which is of type I has a direct summand of type II^n_ν, a contradiction. Therefore $t(V^2, W^2)$ has only finitely many eigenvalues. If (V_2, W_2) is infinite dimensional, then there are infinitely many linearly independent elements which have any $\theta \in \tilde{\mathbb{C}}$ as an eigenvalue. Choose $\nu \in \tilde{\mathbb{C}}$ not an eigenvalue of (V^2, W^2) and v_n's linearly independent elements in V^2 such that $b_\nu v_n = w_n \in W_1$. We note that $w_n \neq 0$ since ν is not an eigenvalue of (V^2, W^2). Since W_1 is finite-dimensional, there exist a positive integer k and $\alpha_1, \alpha_2, \dots, \alpha_k$ nonzero complex numbers such that

$$\sum_{i=1}^{k} \alpha_i w_i = 0.$$

Thus

$$b_\nu \left(\sum_{i=1}^{k} \alpha_i v_i \right) = 0.$$

But $\sum_{i=1}^{k} \alpha_i v_i \neq 0$, contradicting the assumption that ν is not an eigenvalue of (V^2, W^2). Thus (V_2, W_2) is finite-dimensional as required.

Suppose (V_1, W_1) is infinite-dimensional. We want to construct an extension of (V_1, W_1) by a system of type I, which is not pure-projective.

We first do the case where $\operatorname{rank}(V_1, W_1) = \aleph_0$. So let $(V_1, W_1) = \bigoplus_{n=1}^{\infty} (V_{k_n}, W_{k_n})$ where (V_{k_n}, W_{k_n}) is of type III^{k_n}. Let P denote the torsion-free system $(\mathbb{C}[\xi], \mathbb{C}[\xi])$. We map (V_{k_1}, W_{k_1}) to the subsystem of P spanned by $((1, \xi, \dots, \xi^{k_1 - 1}), (1, \xi, \dots, \xi^{k_1}))$, if $k_1 \geqslant 2$. If $k_1 = 1$, map (V_{k_1}, W_{k_1}) to $(0, \mathbb{C} \cdot 1)$, (V_{k_2}, W_{k_2}) is mapped to the subsystem spanned by $((\xi^{k_1 - 1}, \dots, \xi^{k_1 + k_2 - 1}), (\xi^{k_1 + 1}, \dots, \xi^{k_1 + k_2}))$. Other components are similarly mapped into P making sure at each stage that there is no overlap. One readily sees that $P/(V_1, W_1)$ is of type I. By taking an appropiate number of direct sums of P one obtains an extension of an arbitrary system of type III by a system of type I with the extension not pure-projective. Systems of type $\mathrm{II}_\theta^\infty$ can also be used. ∎

The torsion-free rank 1 system $\wedge_J$ given by

$$V = \left[\left\{ \frac{1}{\xi - \theta} : \theta \in J \subset \mathbb{C} \right\} \right],$$

$$W = [V + 1],$$

is an extension of $(0, \mathbb{C} \cdot 1)$—a system of type III^1—by a system of type $\oplus_{\theta \in J} \mathrm{II}_\theta^1$. $\wedge_J$ is infinite-dimensional if J is infinite and so an extension of a finite-dimensional system of type III by an unbounded system of type II is not necessarily pure-projective. If the system of type II is bounded and the system of type III is not finite-dimensional the extension need not be pure-projective as the following example shows: The purely simple system of rank two in Theorem 3.1 of [8] is an extension of a system of type $\oplus_{\aleph_0} \mathrm{III}^1$ by a system of type $\oplus_{\aleph_0} \mathrm{II}_\infty^1$: The system of type $\oplus_{\aleph_0} \mathrm{III}^1$ is $(0, [\{\xi^k + \alpha_{2k} w_2 : k = 0, 1, 2, \dots\}])$.

However we have

LEMMA 2.5. *Let* $0 \to (V_1, W_1) \to (V, W) \to (V_2, W_2) \to 0$ *be an exact sequence in which* (V_1, W_1) *is torsion-free and finite-dimensional and* (V_2, W_2) *is bounded and of type* II. *Then* (V, W) *is pure-projective.*

Proof. From the facts that (V_1, W_1) is finite-dimensional and (V_2, W_2) contains $(t(V, W) + (V_1, W_1))/(V_1, W_1)$ which is isomorphic to $t(V, W)/(V_1, W_1) \cap t(V, W)$ we conclude that $t(V, W)$ is also bounded. So $(V, W) = t(V, W) \dotplus (X, Y)$ for some torsion-free subsystem, (X, Y), of (V, W) by Theorem 3.3 of [10]. As in the proof of Proposition 2.4 we may suppose that $t(V, W)$ is finite-dimensional and under that assumption prove that (V_2, W_2) is finite-dimensional. Suppose $\dim(V_2, W_2) = r$ (say). The exact sequence

$$0 \longrightarrow t(V, W) \longrightarrow (V, W) \longrightarrow (V, W)/t(V, W) \longrightarrow 0$$

gives the exact sequence

$$\mathrm{Ext}((V, W)/t(V, W), (V_2, W_2)) \longrightarrow \mathrm{Ext}((V, W), (V_2, W_2)) \longrightarrow \mathrm{Ext}(t(V, W), (V_2, W_2)) \longrightarrow 0.$$

The first entry is 0 because (V_2, W_2) is bounded; hence pure-projective (Theorem 3.3 of [10]). If r is an infinite cardinal, then $\dim \mathrm{Ext}((V, W),$

$(V_2, W_2)) = \dim \operatorname{Ext}(t(V, W), (V_2, W_2)) \leqslant r$, because $t(V, W)$ is finite-dimensional. On the other hand, the exact sequence

$$0 \longrightarrow (V_1, W_1) \longrightarrow (V, W) \longrightarrow (V_2, W_2) \longrightarrow 0$$

leads to the exact sequence

$$\operatorname{Hom}((V_1, W_1), (V_2, W_2)) \longrightarrow \operatorname{Ext}((V_2, W_2), (V_2, W_2)) \\ \longrightarrow \operatorname{Ext}((V, W), (V_2, W_2)) \longrightarrow \operatorname{Ext}((V_1, W_1), (V_2, W_2)).$$

The last entry is 0 because (V_2, W_2) is pure-injective while the first entry is r-dimensional, if r is an infinite cardinal, and the second entry is 2^r-dimensional; see table in [4]. Therefore $\dim \operatorname{Ext}((V, W), (V_2, W_2))$ is 2^r, a contradiction. Therefore r must be finite. As in Proposition 2.4 (V, W) is pure-projective. ∎

We note that it is easy to show that any extension of a pure-projective system by a finite-dimensional system is pure-projective.

We summarise everything after Proposition 2.4 in

THEOREM 2.6. *An extension of a system of type* III *by a system of type* II *is always pure-projective only in the following cases*:

(i) *The system of type* III *is finite-dimensional while that of type* II *is bounded.*

(ii) *The system of type* II *is finite-dimensional.*

LEMMA 2.7. *Let (V_1, W_1) and (V_2, W_2) be torsion-free systems of bounded height not exceeding m. Then any extension (V, W) of (V_1, W_1) by (V_2, W_2) is also of bounded height not exceeding m.*

Proof. Suppose (V, W) has a subsystem (X, Y) of type III^{m+2}. Then by Lemma 4.1 of [8] and the height relation in the proof of Lemma 1.4 of [9] we conclude that $(X, Y) \cap (V_1, W_1) = 0$. Thus, $((X, Y) + (V_1, W_1))/(V_1, W_1)$ is a subsystem of (V_2, W_2) of type III^{m+2} contradicting the hypothesis that (V_2, W_2) is of bounded height not exceeding m. Therefore, (V, W) is of bounded height not exceeding m.

PROPOSITION 2.8. *An extension of a system of type* III *by a bounded system of type* III *is pure-projective.*

Proof. Let (V, W) be an extension of (V_1, W_1) by (V_2, W_2) where (V_1, W_1), (V_2, W_2) are of type III and the latter is of bounded height not exceeding m (say). Put $(V_1, W_1) = (V_3, W_3) \dot{+} (V_4, W_4)$ with (V_3, W_3) a direct sum of systems of type III^k, $k \leqslant m$, and (V_4, W_4) a direct sum of systems of type III^l, $l > m$. The projection of (V_1, W_1) onto (V_3, W_3) and the inclusion of (V_1, W_1) in (V, W) yield by pushout the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (V_3, W_3) \dotplus (V_4, W_4) & \xrightarrow{(\kappa,\lambda)} & (V, W) & \longrightarrow & (V_2, W_2) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (\pi,\rho)} & & \Big\downarrow{\scriptstyle (\Phi,\Psi)} & & \Big\| & & \\
0 & \longrightarrow & (V_3, W_3) & \longrightarrow & (U, Z) & \longrightarrow & (V_2, W_2) & \longrightarrow & 0
\end{array}$$

By Lemma 2.7, (U, Z) is of bounded height not exceeding m. Hence by Lemma 2.8 of [10] it is direct sum of systems of type III^k, $k \leqslant m$. Since (π, ρ) is onto, so is (Φ, Ψ). Moreover, $\mathrm{Ker}(\Phi, \Psi)$ is isomorphic to $\mathrm{Ker}(\pi, \rho) = (V_4, W_4)$. The last two statements follow from the definition of the pushout and the fact that (κ, λ) is monic. We now have an exact sequence

$$0 \longrightarrow (V_4, W_4) \longrightarrow (V, W) \longrightarrow (U, Z) \longrightarrow 0.$$

By Proposition 1E(b) and (c) of [12] and the table in [4] this sequence splits. Therefore, (V, W) is pure-projective.

We now give an example to show that "boundedness of (V_2, W_2)" is necessary in the last proposition. Let $(V_1, W_1) = (0, \mathbb{C}w)$, $w \neq 0$ and $(V_2, W_2) = \oplus_{k=1}^{\infty} (V_{2k}, W_{2k})$ where (V_{2k}, W_{2k}) is of type III^k spanned by the chain $((v_{21}, v_{22}, \ldots, v_{2,k-1}), (w_{21}, w_{22}, \ldots, w_{2k}))$. Let

$$V = V_1 \oplus V_2,$$
$$W = W_1 \oplus W_2.$$

We make (V, W) a system as follows: Let (a, b) be a basis of $\mathbb{C}^2$. For k even put

$$av_{2i} = w_{2i}, \qquad i = 1, \ldots, k-1,$$
$$bv_{2i} = w_{2,i+1}, \qquad i = 1, \ldots, k-1.$$

For $3 \leqslant k$ odd put

$$av_{2i} = w_{2i} \quad \text{if} \quad i \neq \frac{k+1}{2},$$
$$av_{2,(k+1)/2} = w_{2,(k+1)/2} + w,$$
$$bv_{2i} = w_{2,i-1}, \qquad i = 1, \ldots, k-1.$$

This makes (V, W) an extension of (V_1, W_1) by (V_2, W_2). For any odd $k \geqslant 3$, w is contained in the range space of a subsystem of (V, W) that is a direct sum of two subsystems of type $\mathrm{III}^{(k+1)/2}$ and $\mathrm{III}^{(k-1)/2}$, respectively. This implies that (V, W) cannot be pure-projective for if it were, then w would be contained in the range space of a finite-dimensional direct summand (X, Y) of (V, W). We now show that this is not possible. Let $m_0 = \max\{m: (X, Y)$ has a direct summand of type $\mathrm{III}^m\}$.

Choose k odd such that $(k-1)/2 > m_0$. By the height relation in the proof of 1.4 of [8] w must be contained in the range space of a direct complement of (X, Y) contradicting $w \in Y$. Therefore, (V, W) is not pure-projective. ∎

Remark 2.9. By using pullback and/or pushout and the results of this section one can obtain conditions under which an extension of an arbitrary pure-projective system by another pure-projective system is pure-projective.

References

1. F. Anderson and K. R. Ruller, "Rings and Categories of Modules, Springer Verlag, New York/Heidelberg/Berlin, 1973.
2. N. Aronszajn and U. Fixman, Algebraic spectral problems, *Studia Math.* **30** (1968), 273–338.
3. J. Dieudonne, Sur les p-groupes abeliens infinis, *Portugal. Math.* **11** (1952), 1–5.
4. U. Fixman and F. Okoh, Extensions of pairs of linear transformations between infinite-dimensional vector spaces, *Linear Algebra Appl.* **19** (1978), 275–291.
5. L. Fuchs, "Infinite Abelian Groups," Vol. I, Academic Press, New York/London, 1970.
6. R. Kietpynski and D. Simson, On pure homological dimension, *Bull. Acad. Polon. Sci.* **23** (1975), 1–6.
7. S. MacLane, "Homology," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
8. F. Okoh, A bound on the rank of purely simple systems, *Trans. Amer. Math. Soc.* **232** (1977), 169–186.
9. F. Okoh, Direct sums and direct products of canonical pencils of matrices, *Linear Algebra Appl.* **25** (1979), 1–26.
10. F. Okoh, A splitting criterion for pairs of linear transformations, *Illinois J. Math.* **22** (1978), 379–388.
11. F. Okoh, Hereditary algebras that are not pure hereditary, "Proceedings of the International Congress on Representations of Algebras II," Lecture Notes in Mathematics, No. 832, pp. 432–437, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
12. C. M. Ringel, Infinite-dimensional representations of finite-dimensional hereditary algebras, *Symposia Math. Ins. Alta Mat.* **23** (1979), 321–412.