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Matrix Subrings Having Finite Global Dimension

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We will calculate (or, in some cases, bound) the global dimension of subrings of $M_n(R)$ having certain specified forms for an arbitrary ring R with $1 \neq 0$. These subrings are used to provide examples of rings of finite global dimension with certain properties.

In the first section we consider "tiled triangular rings," i.e., rings of the form

$$\Phi = \begin{bmatrix} R & I_{12} & \cdots & I_{1n} \\ R & R & & \vdots \\ \vdots & & \ddots & I_{n-1,n} \\ R & & \cdots & R \end{bmatrix}$$

for I_{ij} two-sided ideals of R. Such rings have been considered by Fields [F1], Tarsy [T1, T2], and V. A. Jategaonkar [J-VA1, J-VA2, J-VA3] when R is a commutative discrete valuation ring or, in the case of [J-VA2], a commutative Noetherian domain of finite global dimension. Robson [R2] has been able to compute the global dimension of certain rings which have particular tiled triangular rings as a homomorphic image. Using quite different methods, we extend a result Jategaonkar proved when R is a commutative Noetherian domain of finite global dimension [J-VA2, Theorem 3.6] to show that the arbitrarily triangularly tiled ring Φ has finite right global dimension if and only if $\operatorname{rgldim} R < \infty$ and $\operatorname{rgldim}(R/I_{i,i+1}) < \infty$ for $i = 1,..., n-1$ (Theorem 1.9). We calculate exactly the right global dimension of the tiled triangular ring

$$\Gamma = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ R & R & I_2 & & \vdots \\ \vdots & & R & \ddots & I_{n-1} \\ R & R & & \cdots & R \end{bmatrix};$$

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rgldim $\Gamma = \sup_{1 \leqslant i \leqslant n-1}$ {rgldim R, rgldim$(R/I_i)+1$} (Theorem 1.6). We also obtain bounds on the lgldim Γ (Theorem 1.8). In this setting, when R is a commutative Noetherian domain of finite global dimension, Jategaonkar [J-VA2, Theorem 3.5] proved that when $I_i = I$ for all i, rgldim $\Gamma =$ rgldim R. One of the interesting features of the noncommutative case is that rgldim Γ can be greater than rgldim R. As we will illustrate, Γ provides a tool for creating prime rings of computable finite global dimension from triangular examples (where it is often easier to calculate the global dimension).

The ring Γ described above is both a right and left subidealizer in $M_n(R)$. In saying that a ring R is a right subidealizer of A in S we mean that R is a subring of S, A is a right ideal of S, A is a two-sided ideal of R and $SA = S$. Robson [R1, R2] and Goodearl [G] among others have shown that there are circumstances under which the rgldim R can be calculated (or in some cases bounded) by knowing, rgldim R/A, rgldim S, and the R-homological dimensions of simple factors of S/R or R/A. The ring Γ provides an example of a subidealizer, whose global dimension can be exactly computed, in $M_n(R)$ for any ring R. Some cases when

$$\Gamma = \begin{bmatrix} R & I \\ R & R \end{bmatrix}$$

has computable global dimension were detailed by Robson as examples of his idealizer theory [R1, Examples 7.1 and 7.8].

When a ring Γ is right hereditary, Sandomierski [S] has shown that $e\Gamma e$ is also right hereditary for any idempotent $e \in \Gamma$. There exist examples of Artinian rings Γ with rgldim $\Gamma = 2$ and rgldim $e\Gamma e = \infty$ for an idempotent $e \in \Gamma$ [ENN, S]. We conclude the first section in Example 1.13 by constructing for an arbitrary positive integer $n \geqslant 2$ a prime Noetherian PI ring Γ with Krull dimension 1, gldim $\Gamma = 2$ having an idempotent e with gldim $e\Gamma e = n$.

If the homological properties of R/I are to determine the homological properties of the ring

$$\Gamma = \begin{bmatrix} R & I \\ R & R \end{bmatrix},$$

it is necessary that R/I have global dimension at least as great as that of R. A technical device for accomplishing this involves a generalization of a result due to Eilenberg, Nagao, and Nakayama [ENN, Proposition 12]: if $R = T_n(k)$ is the ring of n by n lower triangular matrices over a field k and N is the radical of R then gldim$(R/N^2) = n-1$, while R has global dimension 1. In the second section (Lemma 2.1), we show that if $R = T_n(S)$ is the ring of lower triangular matrices over a ring S and I is the ideal of R consisting of strictly lower triangular matrices then $\operatorname{rgldim}(R/I^2) = \operatorname{rgldim}(S) + n-1$. As it is well known that $\operatorname{rgldim}(R) = \operatorname{rgldim}(S) + 1$, it follows that

$$\operatorname{rgldim}\begin{bmatrix} R & I^2 \\ R & R \end{bmatrix} = \operatorname{rgldim}(S) + n \qquad \text{for } n \geq 2.$$

Proposition 2.2 extends this result; if a given ring S of finite global dimension is the homomorphic image of a prime ring T of finite global dimension, then a prime ring $\Gamma \subset M_{2n}(T)$ is described with $\operatorname{rgldim} \Gamma = \sup\{\operatorname{rgldim}(T), \operatorname{rgldim}(S) + n\}$ (and similarly on the left). Hence by choosing n sufficiently large, $\operatorname{rgldim}(\Gamma) = \operatorname{rgldim}(S) + n$. Hence for a given ring S, when such a prime ring T exists, this construction produces a prime ring Γ of finite global dimension, with $\operatorname{rgldim}(\Gamma) = \operatorname{rgldim}(S) + n$ and $\operatorname{lgldim}(\Gamma) = \operatorname{lgldim}(S) + n$.

Fossum, Griffith, and Reiten [FGR, p. 74–75] constructed *PI* rings S so that $\operatorname{lgldim} S - \operatorname{rgldim} S = m$ for any preassigned positive integer $m \geq 2$. The second section concludes by using the methods described above to provide such examples which, in addition, are prime (Example 2.6).

The third section uses the results of the preceding sections to construct in Example 3.2 a prime *PI* affine ring of differing right and left global dimensions. By an affine ring we mean an algebra over a commutative field which is finitely generated as a ring over the field.

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I. Tiled Matrix Rings

In this section we compute and give bounds for the global dimensions of certain tiled traingular matrix rings. Throughout this section Γ will be a ring of the form

$$\Gamma = \begin{bmatrix} R & I_{12} & I_{13} & \cdots & I_{1n} \\ R & R & I_{23} & & I_{2n} \\ \vdots & & \ddots & & \vdots \\ & & & R & I_{n-1,n} \\ R & R & & \cdots & R \end{bmatrix},$$

where each I_{ij} is an ideal of the ring R.

In calculating the global dimension of Γ, modules of a particular type will play an important role. Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$ be a sequence of right R-modules such that $A_i I_{ik} \subseteq A_k$ for $i \leq k$, $k = 2,..., n$. Let $M = [A_1, A_2,..., A_n] = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as R-modules, writing elements of M as row vectors; M can be considered a Γ-module by right matrix multiplication. The symbol $[A_1,\ldots, A_n]$ will always denote this situation.

LEMMA 1.1. *If A is a right R-module, then* $\mathrm{hd}_\Gamma[A, A,\ldots, A] = \mathrm{hd}_R A$.

Proof. The proof is exactly the same as the proof that the homological dimension of $[A,\ldots, A]$ over $M_n[R]$ is equal to the homological dimension of A over R.

LEMMA 1.2. *If A is a projective R-module, then $M = [A,\ldots, A, AI_{i,i+1},\ldots, AI_{i,n}]$ is Γ-projective.*

Proof. If $A = R$, then M is a direct summand of Γ; hence the result is true if A is R-free. If A is a direct summand of a free R-module F, then M is a Γ-direct summand of the projective Γ-module $[F,\ldots, F, FI_{i,i+1},\ldots, FI_{i,n}]$.

The following lemma will be used in determining a lower bound for the global dimension of Γ. Let $e_{i,j}$ denote the usual matrix units; for $j \leqslant i$, $e_{ij} \in \Gamma$.

LEMMA 1.3. *Consider the Γ-module $M = [P, K, X_3,\ldots, X_n]$ where P is a projective R-module. Then* $\mathrm{hd}_{R/I_{12}}(P/K) \leqslant \mathrm{hd}_\Gamma M$.

Proof. The proof will be by induction on $m = \mathrm{hd}_\Gamma[P, K, X_3,\ldots, X_n]$. For $m = 0$, we will prove the conclusion for $M = [P, K, X_3,\ldots, X_n]$. Let $\{f_\alpha\} \subseteq \mathrm{Hom}_\Gamma(M, \Gamma)$ and $[p_\alpha, k_\alpha, \bar{x}_\alpha]$ where $\bar{x}_\alpha$ denotes the row vector $[x_3,\ldots, x_n]_\alpha$ be Γ-projective coordinates for M. Define R/I_{12}-homomorphisms $F_\alpha: P/K \to R/I_{12}$ by $F_\alpha(p + K) = ef_\alpha([p, 0,\ldots, 0])\, e + I_{12}$ where $e = e_{11}$ in Γ. To show that the F_α's are well-defined it suffices to show that $ef_\alpha([k, 0,\ldots, 0])\, e \in I_{12}$ for all $k \in K$. Then

$$f_\alpha([k, 0,\ldots, 0]) = f_\alpha([0, k, 0,\ldots, 0]\, e_{21}) = f_\alpha([0, k, 0,\ldots, 0])\, e_{21}.$$

Let $f_\alpha([0, k, 0,\ldots, 0]) = (a_{ij})$; note that $a_{12} \in I_{12}$. Then $ef_\alpha([k, 0,\ldots, 0])\, e = ef_\alpha([0, k, 0,\ldots, 0])\, e_{21} \cdot e = e(a_{ij})\, e_{21} e = a_{12}$. Thus F_α is well-defined.

We claim that $\{F_\alpha\}$ and $\{p_\alpha + K\}$ are R/I_{12}-projective coordinates for P/K. Identify $e\Gamma e$ with R and $[P, K, X_3,\ldots, X_n]\, e$ with P. Since $\{f_\alpha\}$ and $\{[p_\alpha, k_\alpha, \bar{x}_\alpha]\}$ are Γ-projective coordinates for $[P, K, X_3,\ldots, X_n]$, we have

$$\begin{aligned}
[p, 0,\ldots, 0] &= \sum [p_\alpha, k_\alpha, \bar{x}_\alpha]\, f_\alpha([p, 0,\ldots, 0]) \\
&= \sum [p_\alpha, k_\alpha, \bar{x}_\alpha](ef_\alpha([p, 0,\ldots, 0]) + (1 - e)\, f_\alpha([p, 0,\ldots, 0])) \\
&= \sum ([p_\alpha, 0,\ldots, 0]\, f_\alpha([p, 0,\ldots, 0] + [0, k_\alpha, \bar{x}_\alpha]\, f_\alpha([p, 0,\ldots, 0])),
\end{aligned}$$

where $([0, k_\alpha, \bar{x}_\alpha]\, f_\alpha([p, 0,\ldots, 0]))\, e \in K$ since $x_3,\ldots, x_n$ are contained in K. Hence

$$\begin{aligned}
&\textstyle\sum(p_\alpha + K)\, F_\alpha(p+K)\\
&\quad = \textstyle\sum(p_\alpha + K)(ef_\alpha([p, 0,\dots, 0])\, e + I_{12})\\
&\quad = \textstyle\sum p_\alpha ef_\alpha([p, 0,\dots, 0])\, e + K\\
&\quad = \textstyle\sum[p_\alpha, k_\alpha, \bar{x}_\alpha]\, ef_\alpha([p, 0,\dots, 0])\, e + K\\
&\quad = \textstyle\sum[p_\alpha, 0,\dots, 0]\, f_\alpha([p, 0,\dots, 0])\, e + K\\
&\quad = ([p, 0,\dots, 0] - \textstyle\sum[0, k_\alpha, \bar{x}_\alpha]\, f_\alpha([p, 0,\dots, 0]))\, e + K \quad \text{(by the above)}\\
&\quad = p + K.
\end{aligned}$$

Therefore P/K is R/I_{12}-projective.

Inductively assume that whenever $\mathrm{hd}_\Gamma[U, V, Y_3,\dots, Y_n] = s < m$ with U R-projective, then $\mathrm{hd}_{R/I_{12}}(U/V) \leqslant s$. Suppose $\mathrm{hd}_\Gamma[P, K, X_3,\dots, X_n] = m$.

Consider a Γ-projective presentation of M

$$0 \longrightarrow \ker \longrightarrow Q \xrightarrow{\sigma} M \longrightarrow 0.$$

Since M can be resolved using a direct sum of projective Γ-modules of the types given in Lemma 1.2, we may assume that Q is the form $[Q_1, Q_2,\dots, Q_n]$ with Q_1 a projective R-module. Since σ is a Γ-homomorphism, $\sigma(q)\, e_{ii} = \sigma(qe_{ii})$, and thus for each i we have an exact sequence of R-modules

$$0 \to K_i \to Q_i \to M_i \to 0,$$

where $Q_i = Qe_{ii}$, $K_i = \mathrm{Ker}\, e_{ii}$, etc. Hence we can write $\ker = [K_1, K_2,\dots, K_n]$. Since P is a projective R-module, the sequence

$$0 \to K_1 \to Q_1 \to P \to 0$$

splits, and hence K_1 is a projective R-module. Since M is not projective, $\mathrm{hd}_\Gamma \mathrm{Ker} = m - 1$, and by the induction hypothesis $\mathrm{hd}_{R/I_{12}}(K_1/K_2) \leqslant m - 1$. The module Q is Γ-projective, so Q_1/Q_2 is projective as an R/I_{12}-module. We have an exact sequence of R/I_{12}-modules

$$0 \to (Q_2 + K_1)/Q_2 \to Q_1/Q_2 \to Q_1/(Q_2 + K_1)\ (\approx P/K) \to 0.$$

Noting that $Q_2 \cap K_1 = K_2$, we have the following lattice diagram.

$$\begin{tikzcd}
 & Q_2 + K_1 \arrow[dl, dash] \arrow[dr, dash] & \\
Q_2 \arrow[dr, dash] & & K_1 \arrow[dl, dash] \\
 & K_2 &
\end{tikzcd}$$

Hence $(Q_2+K_1)/Q_2 \approx K_1/K_2$. Thus the above exact sequence yields that $\mathrm{hd}_{R/I_{12}}(P/K) \leqslant m$.

The following proposition establishes a lower bound on the global dimension of Γ. We define the global dimension of the zero-ring to be -1. We allow the possibility that any of the dimensions in the inequality of the following proposition might be infinite.

PROPOSITION 1.4. *Let*

$$\Gamma = \begin{bmatrix} R & I_{12} & I_{13} & \cdots & I_{1n} \\ R & R & I_{23} & & I_{2n} \\ \vdots & & \ddots & & \vdots \\ & & & R & I_{n-1,n} \\ R & R & & \cdots & R \end{bmatrix}.$$

Then $\operatorname{rgldim} \Gamma \geqslant \sup\{\operatorname{rgldim} R, \operatorname{rgldim}(R/I_{i,i+1})+1\}$.

Proof. The proof will be by induction on n. The case $n=1$ is trivial. Inductively assume that the result is true for all such tiled matrix rings of size less than n. Let $e=e_{22}+\cdots+e_{nn}$. Then $e\Gamma e$ is isomorphic to the $(n-1)\times(n-1)$ tiled matrix ring

$$\begin{bmatrix} R & I_{23} & & \cdots & I_{2n} \\ R & R & I_{34} & & I_{3n} \\ \vdots & & \ddots & & \vdots \\ & & & R & I_{n-1,n} \\ R & R & & \cdots & R \end{bmatrix}$$

and hence by induction $\operatorname{rgldim} e\Gamma e \geqslant \sup\{\operatorname{rgldim} R, \operatorname{rgldim}(R/I_{i,i+1})+1\}$. By Harada [H] $\operatorname{rgldim} \Gamma \geqslant \operatorname{rgldim} e\Gamma e$.

Let $R \supset K \supseteq I_{12}$ be such that $\mathrm{hd}_{R/I_{12}}(R/K) = \operatorname{rgldim} R/I_{12}$. If $M=[R, K, KI_{23}+I_{13},..., KI_{2n}+I_{1n}]$, then $M \subseteq e_{nn}\Gamma$ is a right ideal of Γ which by Lemma 1.3 has $\mathrm{hd}_\Gamma M \geqslant \mathrm{hd}_{R/I_{12}}(R/K) = \operatorname{rgldim} R/I_{12}$. Hence $\operatorname{rgldim} \Gamma \geqslant \operatorname{rgldim}(R/I_{12})+1$.

We next consider a tiled triangular matrix ring of a special form. Let

$$\Gamma = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ R & R & I_2 & & I_{n-1} \\ \vdots & & \ddots & & \vdots \\ & & & R & I_{n-1} \\ R & R & & \cdots & R \end{bmatrix}$$

for ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{n-1}$ of R.

As before we consider right Γ-modules of the type $[A_1, \ldots, A_n]$. Of course, not every right Γ-module is isomorphic to a module of this type, but if M is an arbitrary Γ-module, M does have a homomorphic image which is of this type. Let $e_i = e_{ii}$ and let $h_i = e_{i1}$. We claim the map $\sigma: M \to [Mh_1, Mh_2, \ldots, Mh_n]$ is a Γ-module homomorphism onto a Γ-module of the type described above. Since $mh_{i+1} = m(e_{i+1,i})\, h_i \in Mh_i$, we have $Mh_{i+1} \subseteq Mh_i$. Next note that for $a \in I_{k-1}$ and $mh_i \in Mh_i$, $mh_i a = mh_i e_{11} a = mh_i(ae_{1k})\, h_k \in Mh_k$. Hence $[Mh_1, \ldots, Mh_n]$ is of the desired type. It is not hard to check that σ is a Γ-homomorphism. Let

$$J = \sum h_i \Gamma = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ R & I_1 & I_2 & & I_{n-1} \\ \vdots & & & \ddots & \vdots \\ R & I_1 & I_2 & \cdots & I_{n-1} \end{bmatrix};$$

the right ideal J is a two-sided ideal of Γ, J_Γ is projective, and $(\ker \sigma)\, J = 0$. For an arbitrary Γ-module M the exact sequence $0 \to \ker \sigma \to M \to^{\sigma} [Mh_1, \ldots, Mh_n] \to 0$ and the fact that $\ker \sigma$ is a module over Γ/J will be useful in computing $\operatorname{hd}_\Gamma M$.

LEMMA 1.5. $\operatorname{hd}_\Gamma[A_1, A_2, \ldots, A_n] \leqslant \sup_i\{\text{rgldim } R, \text{rgldim}(R/I_i) + 1\}$.

Proof. Consider the exact sequence

$$0 \to [A_1, A_2, \ldots, A_n] \to [A_1, \ldots, A_1] \to \text{Coker} \to 0.$$

Let

$$K = \begin{bmatrix} R & I_1 & I_2 & I_3 & \cdots & I_{n-1} \\ R & I_1 & I_2 & I_3 & & I_{n-1} \\ R & R & I_2 & I_3 & & I_{n-1} \\ \vdots & & R & I_3 & & \vdots \\ & & & \ddots & & \\ R & & \cdots & & R & I_{n-1} \end{bmatrix}$$

be the ideal of Γ obtained from Γ by replacing the entry in the i, ith position by I_{i-1}; K is a two-sided ideal of Γ which is projective as a right ideal. Since $(\text{Coker})\, K = 0$, Coker is a module over Γ/K. The ring Γ/K is isomorphic to $(R/I_1) \times (R/I_2) \times \cdots \times (R/I_{n-1})$. Hence $\operatorname{hd}_\Gamma \text{Coker} \leqslant \sup\{\text{rgldim}(R/I_i)\} + 1$. By Lemma 1.1, $\operatorname{hd}_\Gamma[A_1, \ldots, A_1] = \operatorname{hd}_R A_1$, hence $\operatorname{hd}_\Gamma[A_1, A_2, \ldots, A_n] \leqslant \sup\{\text{rgldim } R, \text{rgldim}(R/I_i) + 1\}$.

THEOREM 1.6. *Let*

$$\Gamma = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ R & R & I_2 & & \vdots \\ \vdots & & \ddots & & \\ & & & R & I_{n-1} \\ R & R & \cdots & R & R \end{bmatrix}.$$

Then rgldim $\Gamma = \sup_{1 \leqslant i \leqslant n-1}\{\text{rgldim } R, \text{rgldim}(R/I_i) + 1\}$.

Proof. By Proposition 1.4 rgldim $\Gamma \geqslant \sup_{1 \leqslant i \leqslant n-1}\{\text{rgldim } R, \text{rgldim}$ $(R/I_i) + 1\}$. To show the other inequality consider the sequence

$$0 \longrightarrow \ker \sigma \longrightarrow M \xrightarrow{\sigma} [Mh_1, Mh_2,..., Mh_n] \longrightarrow 0.$$

As noted earlier, $(\ker \sigma) J = 0$. Since $J = K^{n-1}$ (K as defined in the previous lemma), the factors of the filtration

$$0 \subseteq (\ker \sigma) K^{n-2} \subseteq (\ker \sigma) K^{n-3} \subseteq \cdots \subseteq (\ker \sigma) K \subseteq \ker \sigma$$

are all modules over Γ/K. Hence the homological dimension of each factor is less than or equal to $\sup\{\text{rgldim}(R/I_i)\} + 1$. Consequently $\text{hd}_\Gamma \ker \sigma \leqslant \sup\{\text{rgldim}(R/I_i) + 1\}$. By Lemma 1.5 $\text{hd}_\Gamma[Mh_1,..., Mh_n] \leqslant \sup\{\text{rgldim } R, \text{rgldim}(R/I_i) + 1\}$; hence the same bound holds for $\text{hd}_\Gamma M$.

COROLLARY 1.7. *Let*

$$\Gamma = \begin{bmatrix} R & I & I & \cdots & I \\ R & R & I & & I \\ \vdots & & & \ddots & \vdots \\ R & R & & R & I \\ R & R & \cdots & R & R \end{bmatrix}.$$

Then

$$\text{rgldim } \Gamma = \sup\{\text{rgldim } R, \text{rgldim}(R/I) + 1\}$$

and

$$\text{lgldim } \Gamma = \sup\{\text{lgldim } R, \text{lgldim}(R/I) + 1\}.$$

We next obtain bounds on the left global dimension of

$$\Gamma = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ R & R & I_2 & & \vdots \\ \vdots & & R & & \\ & & & \ddots & I_{n-1} \\ R & & & \cdots & R \end{bmatrix}.$$

THEOREM 1.8.

$$\sup\{\operatorname{lgldim} R, \operatorname{lgldim}(R/I_i)+1\} \leqslant \operatorname{lgldim} \Gamma$$
$$\leqslant \sup_{1 \leqslant k \leqslant n-1} \{\operatorname{lgldim} R, \operatorname{lgldim}(R/I_k)+n-k\}.$$

Proof. We begin by noting that Γ is both a right and a left subidealizer in $S = M_n(R)$; it is a left subidealizer at the left ideal

$$A = \begin{bmatrix} R & I_1 & I_2 & \cdots & I_{n-1} \\ \vdots & & & \ddots & \vdots \\ R & I_1 & I_2 & \cdots & I_{n-1} \end{bmatrix}$$

and a right subidealizer at the right ideal

$$B = \begin{bmatrix} I_{n-1} & \cdots & I_{n-1} \\ \vdots & \ddots & \vdots \\ I_{n-1} & & I_{n-1} \\ R & \cdots & R \end{bmatrix}.$$

Thus by [R2] for any left S-module, $\operatorname{hd}({}_S M) = \operatorname{hd}({}_\Gamma M)$. Note that S is a projective Γ-module.

We note that the lower bound holds by Proposition 1.4. The proof of the upper bound is by induction on n, the case $n=1$ being trivial and case $n=2$ by above.

Consider the exact sequence $0 \to \Gamma \to S \to S/\Gamma \to 0$ and let M be an arbitrary left Γ-module. The long exact sequence of Tor gives $0 = \operatorname{Tor}^1_\Gamma(S, M) \to \operatorname{Tor}^1_\Gamma(S/\Gamma, M) \to \Gamma \otimes_\Gamma M \to S \otimes_\Gamma M \to (S/\Gamma) \otimes_\Gamma M \to 0$. Let $\bar{M}$ be the image of $\Gamma \otimes_\Gamma M$ in $S \otimes_\Gamma M$. We obtain two exact sequences of left Γ-modules

$$0 \to \bar{M} \to S \otimes_\Gamma M \to (S/\Gamma) \otimes_\Gamma M \to 0 \tag{1}$$

and

$$0 \to \operatorname{Tor}^1_\Gamma((S/\Gamma), M) \to M \to \bar{M} \to 0. \tag{2}$$

In sequence (1), $\operatorname{hd}_\Gamma(S \otimes_\Gamma M) = \operatorname{hd}_S(S \otimes_\Gamma M) \leqslant \operatorname{gldim} R$. Since $(S/R) \otimes_\Gamma M$ is a left module over Γ/B and B is a projective left Γ-module we have

$$\operatorname{hd}_\Gamma((S/\Gamma) \otimes_\Gamma M) \leqslant \operatorname{lgldim}(\Gamma/B) + 1.$$

Therefore $\operatorname{hd}(\bar{M}) \leqslant \sup\{\operatorname{gldim} R, \operatorname{gldim}(\Gamma/B)+1\}$. In sequence (2),

$\mathrm{Tor}_\Gamma^1((S/\Gamma), M)$ is a left module over Γ/B; hence $\mathrm{hd}_\Gamma(\mathrm{Tor}_\Gamma^1((S/\Gamma), M)) \leqslant \mathrm{lgldim}(\Gamma/B)+1$. Therefore since M is arbitrary,

$$\mathrm{lgldim}\ \Gamma \leqslant \sup\{\mathrm{lgldim}\ R,\ \mathrm{lgldim}(\Gamma/B)+1\}.$$

Furthermore

$$\Gamma/B \approx \begin{bmatrix} R/I_{n-1} & I_1/I_{n-1} & I_2/I_{n-1} & \cdots & I_{n-2}/I_{n-1} \\ R/I_{n-1} & R/I_{n-1} & I_2/I_{n-1} & & \vdots \\ \vdots & & \ddots & & I_{n-2}/I_{n-1} \\ R/I_{n-1} & & \cdots & & R/I_{n-1} \end{bmatrix},$$

a ring to which the induction hypothesis applies. Hence $\mathrm{lgldim}\ \Gamma \leqslant \sup\{\mathrm{lgldim}\ R,\ \sup\{\mathrm{lgldim}(R/I_{n-1}),\ \mathrm{lgldim}(R/I_k)+n-1-k\}+1\}$ giving the desired result.

We now obtain an upper bound on a general tiled triangular ring.

THEOREM 1.9. *Let*

$$\Gamma = \begin{bmatrix} R & I_{12} & \cdots & I_{1n} \\ R & R & & \vdots \\ \vdots & & \ddots & I_{n-1,n} \\ R & R & \cdots & R \end{bmatrix};$$

$\sup\{\mathrm{rgldim}(R),\ \mathrm{rgldim}(R/I_{i,i+1})+1\} \leqslant \mathrm{rgldim}\ \Gamma \leqslant \sup\{\mathrm{rgldim}(R),\ \mathrm{rgldim}(R/I_{n-1,n})+1\}+n-2+\sum_{i=1}^{n-2} \mathrm{rgldim}(R/I_{i,i+1})$. *Hence* $\mathrm{rgldim}\ \Gamma < \infty$ *if and only if* $\mathrm{rgldim}(R/I_{i,i+1}) < \infty$ *for* $i=1,\dots,n-1$ *and* $\mathrm{rgldim}\ R < \infty$.

Proof. The lower bound follows from Proposition 1.4. The proof of the upper bound is by induction on n; the case $n=2$ follows by Theorem 1.6. Inductively assume the upper bound for matrix rings of smaller size.

Let U be an arbitrary right ideal of Γ. We can write $U \approx [U_1,\dots,U_n]$, where $U_i \approx Ue_{ii}$. Let $e = e_{22} + \cdots + e_{nn}$. The induction hypothesis applies to

$$e\Gamma e = \begin{bmatrix} R & I_{23} & \cdots & I_{2n} \\ R & R & & \vdots \\ \vdots & & R & \\ & & \ddots & I_{n-1,n} \\ R & & \cdots & R \end{bmatrix}$$

and hence $\operatorname{rgldim}(e\Gamma e) \leqslant \sup\{\operatorname{rgldim}(R), \operatorname{rgldim}(R/I_{n-1,n})+1\}+n-3+\sum_{i=2}^{n-2} \operatorname{rgldim}(R/I_{i,i+1})$. Consider the exact sequence of right Γ modules, $0 \to [U_2, U_2, U_3,..., U_n] \to [U_1, U_2,..., U_n] \to \text{coker} \to 0$.
Let

$$J = \begin{bmatrix} I_{12} & I_{12} & I_{13} & \cdots & I_{1n} \\ R & R & I_{23} & & \vdots \\ \vdots & & R & & \\ & & & \ddots & I_{n-1,n} \\ R & & \cdots & & R \end{bmatrix}$$

be the two-sided ideal of Γ obtained from Γ by replacing the R in the 1, 1 position by the ideal I_{12}; note that $\Gamma/J \approx R/I_{12}$. Since coker is a module over Γ/J, $\operatorname{hd}_\Gamma(\text{coker}) < \operatorname{hd}_\Gamma(J) + 1 + \operatorname{rgldim}(R/I_{12})$. Clearly $\operatorname{hd}_\Gamma(J) = \operatorname{hd}_\Gamma[I_{12}, I_{12}, I_{13},..., I_{1n}]$. Note that $[I_{12}, I_{12}, I_{13},..., I_{1n}] \approx [I_{12}, I_{13},..., I_{1n}] \otimes_{e\Gamma e} e\Gamma$. Hence by [H], $\operatorname{hd}_\Gamma[I_{12}, I_{12}, I_{13},..., I_{1n}] = \operatorname{hd}_{e\Gamma e}[I_{12}, I_{13},..., I_{1n}]$. Therefore $\operatorname{hd}_\Gamma(\text{coker}) \leqslant (\operatorname{rgldim}(e\Gamma e)-1)+1+\operatorname{rgldim}(R/I_{12})$. Similarly $\operatorname{hd}_\Gamma[U_2, U_2,..., U_n] = \operatorname{hd}_{e\Gamma e}[U_2, U_3,..., U_n]$. Since $[U_2, U_3,..., U_n]$ is a submodule of a projective $e\Gamma e$-module $M_n(R)\,e$, $\operatorname{hd}_\Gamma[U_2, U_2,..., U_n] \leqslant \operatorname{rgldim}(e\Gamma e)-1$. Therefore $\operatorname{hd}_\Gamma(U) \leqslant \operatorname{rgldim}(e\Gamma e)+\operatorname{rgldim}(R/I_{12})$ and hence $\operatorname{rgldim}\Gamma \leqslant \operatorname{rgldim}(e\Gamma e)+\operatorname{rgldim}(R/I_{12})+1$, giving the result.

In the case that R is a commutative Noetherian domain of finite global dimension, Theorem 1.9 reduces to the following result of Jategaonkar.

COROLLARY 1.10. [J-VA2, Theorem 3.6]. *Let R be a commutative Noetherian domain of global dimension $d < \infty$. Then a triangular tiled order Γ in $M_n(R)$ has finite global dimension if and only if* $\operatorname{gldim}\Gamma \leqslant (n-1)\,d$.

Proof. When R is a commutative Noetherian domain of finite global dimension $d < \infty$, if $\operatorname{gldim}(R/I) < \infty$ then $\operatorname{gldim}(R/I) \leqslant d-1$, and hence the inequality in the preceding proposition reduces to $\operatorname{gldim}\Gamma \leqslant d+(n-2)+(n-2)(d-1)=(n-1)\,d$.

We note that in [J-VA2, Proposition 3.8] an example is given of ideals I_{ij} in any regular local ring R of global dimension 2 so that a particular triangular tiled order in $M_n(R)$ has $\operatorname{gldim}\Gamma = 2(n-1)$, and hence the upper bound above can be attained.

EXAMPLE 1.11. *To illustrate these constructions, consider the subring of $M_n(Z)$*

$$R=\begin{bmatrix} Z & 2Z & \cdots & & 2Z \\ Z & Z & & & \cdot \\ \cdot & & \cdot & & \\ \cdot & & & Z & 2Z \\ Z & & \cdots & & Z \end{bmatrix}$$

and the ideals

$$I=\begin{bmatrix} 2Z & 2Z & \cdots & 2Z \\ Z & 2Z & \cdots & 2Z \\ \cdot & & \cdot & \cdot \\ \cdot & & 2Z & \cdot \\ Z & \cdots & Z & 2Z \end{bmatrix} \quad \textit{and} \quad J=\begin{bmatrix} 2Z & & \cdots & & 2Z \\ 2Z & 2Z & & & \cdot \\ Z & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & 2Z & \\ Z & \cdots & Z & 2Z & 2Z \end{bmatrix};$$

we have gldim $R=1$, $R/J \approx T_n(Z_2)/N^2$, *so that* gldim$(R/J)=n-1$. *It follows that* $\Gamma=\left[\begin{smallmatrix} R & J \\ R & R \end{smallmatrix}\right]$ *is a Noetherian prime PI ring of Krull dimension* 1 *and global dimension n.*

The following result follows from Theorem 1.9.

PROPOSITION 1.12. *Let*

$$\Gamma=\begin{bmatrix} R & I & J \\ R & R & I \\ R & R & R \end{bmatrix}$$

for I, J *two-sided ideals of* R, $I^2 \subseteq J \subseteq I$. *Then* rgldim $\Gamma \leqslant \sup\{$rgldim R, rgldim$(R/I)+1\}+1+$rgldim(R/I).

We note that when R is a commutative discrete valuation ring, I is the maximal ideal of R and $J=I^2$, Tarsy [T1, Theorem 10 or 11] showed that rgldim $\Gamma=2$; hence the upper bound above can be assumed.

EXAMPLE 1.13. *Applying the preceding proposition to the ring R and the ideal I of Example* 1.11, *we have*

$$\Gamma=\begin{bmatrix} R & I & J \\ R & R & I \\ R & R & R \end{bmatrix} \qquad \text{with gldim}(R/I)=0, \text{ so gldim } \Gamma \leqslant 2.$$

Letting

$$e=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad e\Gamma e=\begin{bmatrix} R & J \\ R & R \end{bmatrix} \qquad \text{and gldim}(e\Gamma e)=n.$$

Thus, even for a Noetherian prime PI ring of Krull dimension 1 *and global dimension* 2, *there is no bound on* gldim$(e\Gamma e)$, *even when these are finite.*

II. Building Prime Rings

In this section we will generalize a result of Eilenberg *et al.* [ENN, Proposition 12] which we will then use to reduce the problem of finding a prime *PI* affine ring of differing right and left global dimensions to finding an affine *PI* ring of differing dimensions, which is a homomorphic image of a prime *PI* affine ring of finite global dimension, where the kernel of the homomorphism is finitely generated as a two-sided ideal.

Throughout this section let S be an arbitrary ring, T_m be the ring of lower triangular $m \times m$ matrices over S, L be the ideal of strictly lower triangular $m \times m$ matrices over S, and $W_m = T_m/L^2$. If $a \in T_m$, we will denote the coset $a + L^2$ by a^*; every coset of W_m has a unique bidiagonal representation of the form

$$\begin{bmatrix} * & 0 & & \cdots & & 0 \\ * & * & 0 & & & 0 \\ 0 & * & * & & & \vdots \\ \cdot & & & * & * & 0 \\ 0 & \cdots & & 0 & * & * \end{bmatrix}.$$

Theorem 2.1.

$$\text{rgldim } W_m = \text{rgldim } S + m - 1$$

and

$$\text{lgldim } W_m = \text{lgldim } S + m - 1.$$

Proof. We will prove rgldim $W_m \leqslant$ rgldim $S + m - 1$ by induction on m. The case $m = 1$ being trivial, inductively assume that the result is true for size $m-1$ by $m-1$ (or smaller). Let $n =$ rgldim S.

The ring $W_m \approx \left[\begin{smallmatrix} S & 0 \\ M & W_{m-1} \end{smallmatrix}\right]$. By Field's results [F2, Corollary 5] rgldim $W_m \leqslant \sup\{\text{rgldim } W_{m-1} + \text{rhd}_S S + 1, \text{ rgldim } S\} \leqslant n + m - 1$.

Let I be a right ideal of S with $\text{rhd}_S I = n - 1$. Then

$$\text{hd}_{W_m} \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \\ 0 & & \cdots & 0 \end{bmatrix}^* = n - 1.$$

For a right ideal I of S let $N_k(I)$ be the right ideal of T_m, $N_k(I)=[S,\dots,S, I, 0,\dots,0]$ obtained by taking the kth row of T_m and restricting the diagonal elements to elements of I. For a right S-module U let $M_k(U)=[U, U,\dots,U, 0,\dots,0]=\{(u_1,\dots,u_m): u_i\in U, u_j=0 \text{ for } j>k\}$. Note that for $k\geqslant 3$ $M_k(U)L^2=M_{k-2}(U)$; by $M_0(V)$ we mean 0. It is not hard to see that $\mathrm{hd}_{W_m} N_2(I)^*=n$ since $\mathrm{hd}_{\left[\begin{smallmatrix} S & 0\\ S & S\end{smallmatrix}\right]} \left[\begin{smallmatrix} 0 & 0\\ S & I\end{smallmatrix}\right]=n$.

We will next prove that $\mathrm{hd}_{W_m} N_k(I)^*=n+k-2$. The proof is by induction on k; the cases $k=1, 2$ having been done, take $k\geqslant 3$.

Let $0\to K\to Q\to^{\sigma} I\to 0$ be a short exact sequence of S-modules with Q S-projective. Consider the exact sequence of W_m-modules

$$0\longrightarrow \ker\hat{\sigma}\longrightarrow M_{k-1}(S)/M_{k-3}(S)\oplus M_k(Q)/M_{k-2}(Q)\xrightarrow{\hat{\sigma}} N_k(I)^*\longrightarrow 0 \tag{$*$}$$

given by

$$\begin{aligned}\hat{\sigma}(a^*, b^*)&=\hat{\sigma}((s_1,\dots,s_{k-1},0,\dots,0)+M_{k-3}(S),\\ &\qquad (q_1,\dots,q_k,0,\dots,0)+M_{k-2}(Q))\\ &=(\sigma(q_1)-s_1,\dots,\sigma(q_{k-1})-s_{k-1},\sigma(q_k),0,\dots,0)^*.\end{aligned}$$

Note that $\ker\hat{\sigma}=\{((s_1,\dots,s_{k-1},0,\dots,0)+M_{k-3}(S),\ (q_1,\dots,q_k,0,\dots,0)+M_{k-2}(Q)): q_k\in K \text{ and } \sigma(q_{k-1})=s_{k-1}\}$. Consider the following W_m-submodule of $\ker\hat{\sigma}$:

$$A=\{(a^*, b^*)\in\ker\hat{\sigma}: q_{k-1}\in K \text{ (this implies } s_{k-1}=0)\}.$$

We have the following exact sequence of W_m-modules

$$0\to M_{k-2}(S)/M_{k-3}(S)\to N_{k-1}(S)/M_{k-3}(S)\to N_{k-1}(I)/M_{k-2}(S)\to 0. \tag{$**$}$$

Define a map α: $\ker\hat{\sigma}\to N_{k-1}(I)/M_{k-2}(S)$ by

$$\alpha(a^*, b^*)=(\sigma(q_1),\dots,\sigma(q_{k-1}),0,\dots,0)+N_{k-1}(I)/M_{k-2}(S).$$

It can be checked that α is well defined, onto, and that $\ker\alpha=A$. But $A\approx M_{k-2}(S)/M_{k-3}(S)\oplus M_k(K)/M_{k-2}(K)$ under the W_m-map that takes (a^*, b^*) to $(a^*, (0,\dots,0,q_{k-1},q_k,0,\dots,0)+M_{k-2}(K))$. One can check that $\mathrm{hd}_{W_m} M_k(S)/M_{k-1}(S)=k-1$ and $\mathrm{hd}_{W_m}(M_k(U)/M_{k-2}(U))=\mathrm{hd}_S(U)$. Hence $\mathrm{hd}_{W_m} A=\sup\{k-3, n-2\}$. By induction $\mathrm{hd}_{W_m} N_{k-1}(I)/M_{k-3}(S)=n+k-3$ and $\mathrm{hd}_{W_m}(M_{k-2}(S)/M_{k-3}(S))=k-3$; therefore $(**)$ shows that

$\operatorname{hd}_{W_m}(N_{k-1}(I)/M_{k-2}(S)) = n+k-3$ $(n \geqslant 1)$. Considering the exact sequence of W_m-modules

$$0 \to A \to \ker \hat{\sigma} \to N_{k-1}(I)/M_{k-2}(S) \to 0$$

and the long exact sequence of Ext yields $\operatorname{hd}_{W_m} \ker \hat{\sigma} = \operatorname{hd}_{W_m} N_{k-1}(I)/M_{k-2}(S) = n+k-3$ provided $n \geqslant 2$. Since $M_{k-1}(S)/M_{k-3}(S) \oplus M_k(Q)/M_{k-2}(Q)$ is W_m-projective, sequence $(*)$ shows that $\operatorname{hd}_{W_m} N_k(I)^* = n+k-2$ for $n \geqslant 2$. By considering the right ideal $N_m(I)^*$, we have that rgldim $W \geqslant n+m-1$, and hence we have equality $(n \geqslant 2)$.

Since $W_m(S[t]) \approx W_m(S)[t]$, the restriction that n be greater than or equal to 2 can be removed.

PROPOSITION 2.2. *Let T be a ring and I be an ideal of T. Let Γ be the subring of $M_n(T)$*

$$\Gamma = \begin{bmatrix} T & I & \cdots & I \\ T & T & I & I \\ \vdots & & \ddots & \vdots \\ & & & I \\ T & & \cdots & T \end{bmatrix}$$

and let L be the ideal of Γ

$$L = \begin{bmatrix} I & I & \cdots & I \\ I & I & I & I \\ T & I & \ddots & \vdots \\ \vdots & & & I \\ T & \cdots & T\ I & I \end{bmatrix}$$

having I on the diagonal and subdiagonal, and let Φ be the ring $\Phi = \left[\begin{smallmatrix} \Gamma & L \\ \Gamma & \Gamma \end{smallmatrix}\right]$. Then

(1) rgldim Φ = sup{rgldim T, rgldim$(T/I) + n$} *and* lgldim Φ = sup{lgldim T, lgldim$(T/I) + n$}.

(2) *If T is prime and I is nonzero, then Φ is prime.*

(3) *If T is PI so is Φ.*

(4) *If T is right Noetherian, then so is Φ.*

(5) *If T is affine and I is finitely generated as a two-sided ideal, then Φ is affine.*

(6) *If* rgldim$(T/I) \neq$ lgldim(T/I), *then for n sufficiently large,* rgldim $\Phi \neq$ lgldim Φ.

Proof. The results follow easily from Corollary 1.7 and Proposition 2.1 since $\Gamma/L \approx T_n/L^2$ for $S = T/I$.

This result will be used in the next section to produce a prime, affine *PI* ring of differing right and left global dimensions. As a simple illustration of this proposition, we next use it to construct prime *PI* (but not affine) rings of differing right and left global dimensions. First we need the following result of Small [Sm3].

LEMMA 2.3 *If R is a left subidealizer in a ring T of a left ideal M of T then*

$$\operatorname{lgldim}(R) \leqslant \sup\{\operatorname{lgldim}(T), \operatorname{lgldim}(R/M) + \operatorname{hd}_R(R/M) + 1\}.$$

In particular, if T and R/M both have finite left global dimension, so does R.

The arguments of [G, Theorem 2.10] can be used to prove the following result.

LEMMA 2.4. *If R is a left subidealizer in T of a left ideal M of T then* $\operatorname{rgldim} R \leqslant \operatorname{rgldim} T + \operatorname{rgldim}(R/M) + 1$. *In particular, if T and R/M both have finite right global dimension, so does R.*

EXAMPLE 2.5. *Let R be a commutative discrete valuation ring with quotient field K, let $T = \left[\begin{smallmatrix} R + xK[x] & xK[x] \\ K[x] & K[x] \end{smallmatrix}\right]$ and let I be the two-sided ideal of T $I = \left[\begin{smallmatrix} xK[x] & xK[x] \\ xK[x] & xK[x] \end{smallmatrix}\right]$. Then $T/I \approx \left[\begin{smallmatrix} R & 0 \\ K & K \end{smallmatrix}\right]$ which has* $\operatorname{rgldim}(T/I) = 2$ *and* $\operatorname{lgldim}(T/I) = 1$. *Since*

$$\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}^{-1} T \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R + xK[x] & K[x] \\ xK[x] & K[x] \end{bmatrix}.$$

T has the same right and left global dimensions. As T is a right subidealizer in $M_2(K[x])$, it follows from Lemma 2.3 that $\operatorname{rgldim} T \leqslant 3$. *Hence by choosing $n = 3$ in Proposition 2.2, we can construct a prime PI ring Φ with* $\operatorname{rgldim} \Phi = 5$ *and* $\operatorname{lgldim} \Phi = 4$; *the ring Φ is a subring of $M_{12}(K[x])$ and has the same PI degree as $M_{12}(K[x])$.*

The results of this section can be used to easily produce prime *PI* rings Γ with $\operatorname{lgldim} \Gamma - \operatorname{rgldim} \Gamma = m$ for any preassigned positive integer $m \geqslant 2$.

EXAMPLE 2.6. *Following the Fossum, Griffith, and Reiten construction, let S be a commutative Noetherian integral domain of sufficient cardinality such that its field of quotients Q has* $2 \leqslant \operatorname{hd}_S Q = \operatorname{gldim} S = n < \infty$ *(see*

Osofsky [0, *Corollary* 6.8]). *Let* $R=Q[x_1,..., x_{m-1}]$ *be the polynomial ring in* $m-1$ *commuting indeterminates with* $1 \leqslant m-1 \leqslant n$. *The ring* $\Gamma=\begin{bmatrix} R & 0 \\ Q & S \end{bmatrix}$ *has* $\operatorname{lgldim} \Gamma - \operatorname{rgldim} \Gamma = m$ [FGR, p. 74]. *Let* $T=\begin{bmatrix} R[t] & tR[t] \\ R[t] & S+tR[t] \end{bmatrix}$ *where* $R[t]$ *is the polynomial ring over R in the commuting indeterminate t. Then T is a left subidealizer in* $M_2(T[t])$ *of the left ideal* $\begin{bmatrix} R[t] & tR[t] \\ R[t] & tR[t] \end{bmatrix}$; *hence by Lemmas* 2.3 *and* 2.4 *T has finite right and left global dimensions. Let I be the two-sided ideal of T,* $I=\begin{bmatrix} tR[t] & tR[t] \\ M & tR[t] \end{bmatrix}$, *where* $M=\langle t, x_1,..., x_{m-1}\rangle$ *is the maximal ideal of* $R[t]$ *generated by* $\{t, x_1,..., x_{m-1}\}$. *Then* $T/I \approx \begin{bmatrix} R & 0 \\ Q & S \end{bmatrix}$. *Thus by choosing sufficiently large matrices, Proposition* 2.2 *guarantees the existence of a prime PI ring of left global dimension* $k+n+m$ *and right global dimension* $k+n$.

III. An Affine Example

In this section we will use the results of the previous sections to construct an affine prime *PI* ring with differing right and left global dimensions. Auslander [A] showed that for right and left Noetherian rings the right and left global dimensions are equal; elsewhere we have shown [KK] that affine *PI* right hereditary rings are left hereditary. Kaplansky [K] gave the first example of a ring with differing global dimensions; other examples were given by Small [Sm1, Sm2]. Jategaonkar [J-AV] has shown for all m, n, $1 \leqslant m \leqslant n \leqslant \infty$ there is a left Noetherian ring R with $\operatorname{lgldim} R=m$, and $\operatorname{rgldim} R=n$; as mentioned in the preceding section, Fossum, Griffith, and Reiten [FGR] produced *PI* rings whose right and left global dimension differ by any preassigned integer greater than or equal to 2. We first note that since affine rings are countable dimension vector spaces over a field, the right and left dimensions of an affine ring can differ by at most 1 [Je].

The constructions [Sm1, Sm2, FGR] are based on triangular matrix rings; our present construction is as well. By the results of the previous sections, it suffices to produce an affine *PI* ring A of differing right and left global dimensions, an affine prime *PI* ring Γ of finite right and left global dimensions with two-sided ideal L, with L finitely generated as a two-sided ideal of Γ, and $\Gamma/L \approx A$. To construct A we use the following lemma, which is an immediate consequence of Corollaire to Theorème 1 [PR].

Lemma 3.1. *The ring* $A=\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *with* ${}_RM_S$ *a bimodule and* $\operatorname{gldim} S=0$ *has* $\operatorname{lgldim} A=\sup\{\operatorname{lgldim} R,\ \operatorname{hd}_R M+1\}$ *and* $\operatorname{rgldim} A=\sup\{\operatorname{rgldim} R, \operatorname{wd}_R M+1\}$.

The ring A is now easily constructed. Let $R=\begin{bmatrix} k[x] & k[x, x^{-1}] \\ 0 & k[x, x^{-1}] \end{bmatrix}$ and $N=\begin{bmatrix} 0 & k[x, x^{-1}] \\ 0 & 0 \end{bmatrix}$ for any field k; it is not difficult to show that R is an affine ring of rgldim $R = $ lgldim $R = 2$, N is a two-sided ideal of R with $\mathrm{hd}_R N = 1$, and $\mathrm{wd}_R N = 0$. Let

$$A = \begin{bmatrix} R & R/N \\ 0 & \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \end{bmatrix};$$

then A is an affine PI ring with rgldim $A = 2$ and lgldim $A = 3$.

EXAMPLE 3.2. To construct an affine prime PI ring Γ of finite global dimension having A as a homomorphic image, we begin by defining such a ring T mapping onto R. Let $S = k[x, x^{-1}, t]$ and $T = \left[\begin{smallmatrix} k[x]+tS & S \\ tS & S \end{smallmatrix}\right]$. Clearly T is an affine prime PI ring, and by Lemma 2.3, lgldim $T \leqslant 3$ (it can be shown by other methods that lgldim $T = 2$). Since in $M_2(Q(S))$ $\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]^{-1} T \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} k[x]+tS & tS \\ S & S \end{smallmatrix}\right]$, lgldim $T = $ rgldim T. Now $I = \left[\begin{smallmatrix} tS & tS \\ tS & tS \end{smallmatrix}\right]$ is a two-sided ideal of T with $T/I \approx R$; it is easy to check that I is finitely generated as a two-sided ideal of T. Furthermore $J = \left[\begin{smallmatrix} tS & S \\ tS & tS \end{smallmatrix}\right]$ is a finitely generated two-sided ideal of T with $T/J \approx R/N$; note that $tT \subseteq I \subseteq J$. Next let

$$\sum = \begin{bmatrix} T & T \\ tT & \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + tT \end{bmatrix};$$

one can check that $\sum$ is an affine, prime (it contains $M_4(t^2S)$, an essential ideal of $M_4(S)$) PI ring. Lemmas 2.3 and 2.4 show that $\sum$ has finite right and left global dimensions, since it is a right subidealizer of the right ideal $U = \left[\begin{smallmatrix} T & T \\ tT & tT \end{smallmatrix}\right]$ in $M_2(T)$ and $\sum/U$ and $M_2(T)$ both have finite right and left global dimensions. Let $K = \left[\begin{smallmatrix} I & J \\ tT & tT \end{smallmatrix}\right]$; it is easily checked that K is a two-sided ideal of $\sum$ and $\sum/K \approx A$. We would be done if K is finitely generated as a two-sided ideal of $\sum$, but it is not. However, the only problem in generating K is producing all of S in the first row, fourth column entry of K; the matrix unit e_{14} generates only $k[x] + tS$. Hence we modify $\sum$ as follows. Consider the ring $\Omega = \left[\begin{smallmatrix} T & T \\ I & T \end{smallmatrix}\right] \supseteq \sum$; it is not hard to check that Ω is an affine prime PI ring; since both T and $T/I \approx A$ have finite global dimensions, so does Ω by Corollary 1.7. Furthermore, K is a right ideal of Ω and ${}_{\sum}K_{\Omega}$ is finitely generated as a bimodule. Finally, consider the ring $\Gamma = \left[\begin{smallmatrix} \sum & K \\ \Omega & \Omega \end{smallmatrix}\right]$; it is easily checked that Γ is an affine prime PI ring which has finite global dimensions because it is a right subidealizer of $L = \left[\begin{smallmatrix} K & K \\ \Omega & \Omega \end{smallmatrix}\right]$ in $M_2(\Omega)$, a ring of finite global dimensions, and $\Gamma/L \approx \sum/K \approx A$. The ring Γ, a subring of $M_8(S)$, may itself have differing right and left global dimensions, but Proposition 2.2 guarantees the existence of an affine prime PI ring of differing right and left global dimensions.

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