Existence of Solutions for
Ordinary Differential Equations in Banach Spaces

Tien-Yien Li

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland,
College Park, Maryland 20742

Received December 11, 1972

1. INTRODUCTION

Let $E$ be a real Banach space with norm denoted by $\| \|$. Consider the differential equation

$$x' = f(t, x), \quad x(0) = x_0 \quad (1.1)$$

where $f: \mathbb{R}^1 \times E \to E$.

If $f$ is only assumed continuous, it has been shown that continuity of the right-hand side (or even uniform continuity on bounded sets) is insufficient for existence of solutions [8], even if $E$ is a Hilbert space, $l^p$ or $L^p$ space, $1 < p < \infty$ [9, 12]. See also the general “nonexistence” result in Ref. [10]. Nonetheless, Lasota and Yorke show that “most” (in the sense of Baire Category) differential equations have solutions when the right-hand side is continuous (see Ref. [12]).

For Eq. (1.1) there is a close relationship between conditions guaranteeing bounds on the growth of solutions and conditions guaranteeing that solutions exist. Conditions guaranteeing growth bounds thus motivate existence results. Let $x(t, x_0)$ denote the solution of Eq. (1.1) with $x(0) = x_0$, and

$$x(t, 0, X) = \{x(t, 0, x_0) : x_0 \in X\}.$$ 

If $f$ is Lipschitzian with constant $L$ then

$$\text{diam}(x(t, 0, X)) \leq e^{Lt} \text{diam}(X), \quad \text{for bounded } X, \text{ and } t \geq 0. \quad (1.2)$$

It is also well known that for $f$ Lipschitzian, Eq. (1.1) has a solution. However, there is a more general assumption guaranteeing Eq. (1.2), in particular that $f(t, x) - Lx$ is monotone for some constant $L$ (see condition (2.5)).

* Research partially supported by National Science Foundation grant GP-31386X.
† Present address: Department of Mathematics, University of Utah, Salt Lake City, Utah 84112.

Copyright © 1975 by Academic Press, Inc.
All rights of reproduction in any form reserved.
Martin showed in Ref. [4] that this condition implies existence and uniqueness of solutions when \( f \) is continuous.

Conditions can be given guaranteeing types of bounds of the growth of the "size" of \( x(t,0,X) \) which are not stated in terms of diameter of the set. Kuratowski [3] defines \( \alpha(X) \), a measure of the noncompactness of \( X \). He defines \( \alpha(X) \) to be the infimum of all \( \epsilon > 0 \), for which there exists a finite covering of \( X \) by sets of diameter \( \epsilon \). Ambrosetti [1] and Szula [2] use the condition that \( f \) is \( \alpha \)-Lipschitzian, that is that there is an \( L \geq 0 \) such that

\[
\alpha(f(X)) \leq L \alpha(X), \quad \text{for bounded } X
\]  

(1.3)

to guarantee the existence of solutions. This condition guarantees

\[
\alpha(x(t,0,X)) \leq e^{Lt} \alpha(X), \quad t \geq 0.
\]  

(1.4)

Furthermore, any Lipschitzian \( f \) is also \( \alpha \)-Lipschitzian. If \( f \) maps bounded sets into compact sets, \( f \) is again \( \alpha \)-Lipschitzian, and in fact Eq. (1.3) is satisfied with \( L = 0 \). Goebel and Rzymowski [6] extended condition Eq. (1.3) by the use of Kamke functions (see Definition 3.1).

A more general condition which also guarantees Eq. (1.4) when \( f \) is uniformly continuous is that there is an \( L \geq 0 \) such that for each set \( X \subset E \), \( f \) satisfies

\[
\alpha(\{x - hf(t,x) : x \in X\}) \geq (1 - Lh) \alpha(X) \quad \text{for all } h > 0.
\]  

(1.5)

We show in Theorem 3.1 that if \( f \) is uniformly continuous then condition (1.5) implies the existence of solution. A possibly more intuitive condition which may be substituted for (1.5) is

\[
\alpha(\{x + hf(t,x) : x \in X\}) \leq (1 + Lh) \alpha(X) \quad \text{for all } h > 0.
\]  

(1.6)

See the remark in Section 3. Condition (1.5) is weaker than Martin's condition and applicable for any Banach space; however, unlike Martin, we must assume \( f \) to be uniformly continuous (at least in a neighborhood of each point). In our main existence theorem, (1.5) is actually weakened slightly by substituting a Kamke function for \( Lx \). This "Kamke" form of (1.5) is then also weaker than the conditions used by Goebel and Rzymowski [6] and by Wazewski [14]. Hence, for uniformly continuous \( f \), our existence result generalizes those in [6], [14]. Although, Wazewski requires \( f \) to be continuous rather than uniformly continuous. Proposition 2.2 and Proposition 3.1 show that condition (1.5) is more general than Cellina's \( \alpha \)-dissipative condition [7] (See Definition 2.1) which guarantees that \( \alpha(x(t,0,X)) \) is nonincreasing as a function of \( t \geq 0 \), provided \( E \) is a uniformly convex space. Therefore, our main theorem generalizes an existence result of Cellina when \( f \) is uniformly continuous.
We emphasis that it is not known if (1.5) implies (1.4) when \( f \) is continuous but not uniformly continuous. Similary our existence result says nothing for such \( f \). For existence results concerning such \( f \) along this line we refer to Refs. [2, 7], and [13].

In Section 4, we give an example which shows that (1.5) is strictly more general than Goebel–Rzymoski’s condition [6].

2. AN EXISTENCE RESULT IN UNIFORMLY CONVEX SPACES

For each \( x \) in \( E \) define the duality map \( j : E \rightarrow E^* \) by

\[
j(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^2 = \| f \|^2 \}.
\]

It is well known that \( j \) is single-valued and uniformly continuous if \( E^* \) is uniformly convex on bounded sets. We let \( B_r[A] \) be the open \( r \)-neighborhood of \( A \subset E \). For a given interval \( I \), \( C(I) \) is the Banach space of continuous functions from \( I \) into \( E \) with the usual norm \( \| x \| = \sup \{ \| x(t) \| : t \in I \} \) and \( C^1(I) \) is the subspace of \( C(I) \) consisting of \( x(\cdot) \) with continuous first derivative. Let \( R_b \) be the rectangle \( 0 < t < a, \| x - x_0 \| < b \) and \( f : R_b \rightarrow E \) be continuous and such that \( \| f \| \leq M \) on \( R_b \) (we assume that \( M \) is chosen so that \( M \geq 1 \)). Cellina [7] gave the following definition which we alter in Definition 2.2 in an obvious way.

**Definition 2.1.** The mapping \( f : R_b \rightarrow E \) is called \( \alpha \)-dissipative if for any given \( \epsilon > 0 \) there exists a finite covering \( \{0^s\} \) of \( R_b \) such that

\[
\langle f(t_1, x_1) - f(t_2, x_2), j(x_1 - x_2) \rangle \leq \epsilon
\]

whenever \((t_1, x_1)\) and \((t_2, x_2)\) belong to the same \( 0^s \).

**Definition 2.2.** The mapping \( f : R_b \rightarrow E \) is called \( \alpha \)-Lip-dissipative if there exists \( L > 0 \) such that for any given \( \epsilon > 0 \) there exists a finite covering \( \{0^s\} \) of \( R_b \) such that

\[
\langle f(t_1, x_1) - f(t_2, x_2), [j(x_1 - x_2)]\| x_1 - x_2 \| \rangle \leq L \| x_1 - x_2 \| + \epsilon \quad (2.1)
\]

whenever \((t_1, x_1)\) and \((t_2, x_2)\) belong the same \( 0^s \) and \( x_1 \neq x_2 \).

This concept is motivated by and extends the one of Cellina’s [7] and the following preliminary result generalizes Cellina’s result. The proof is quite similar to Cellina’s and is omitted. In Section 3 we give our main result which includes this one in the case of uniformly continuous \( f \).
Proposition 2.1. Let $E$ be a Banach space such that $E^*$ is uniformly convex and $R_b$ as before. Let $f : R_b \to E$ be continuous and $\alpha$-Lip-dissipative. Let $T = \min[a, b/(4M)]$ and $I = [0, T]$. Then the set of solutions of Eq. (1.1) is a nonempty and compact subset of $C(I)$.

Remark. In fact we may use, through Proposition 2.1, a more general condition by replacing Eq. (2.1) by the following condition

$$\langle f(t_1, x_1) - f(t_2, x_2), j(x_1 - x_2) \rangle \leq L \| x_1 - x_2 \| + \epsilon. \quad (2.2)$$

In Ref. [7], Cellina shows that the conclusion of Proposition 2.1 is valid in the case that $f$ is "$\alpha$-dissipative", which is the special case of Eq. (2.2) where $L = 0$.

In Ref. [4], Martin proves a theorem concerning the existence of solutions of Eq. (1.1) by using one-side derivative (see Eq. (2.5) below). The following Proposition indicates that both Eqs. (2.1) and (2.2) are generalizations of Eq. (2.5).

Proposition 2.2. If $f$ is a mapping from $R_b \to E$ and $L$ is a constant, then the following conditions are equivalent:

$$\langle f(t_1, x_1) - f(t_2, x_2), [j(x_1 - x_2)]/\| x_1 - x_2 \| \rangle \leq L \| x_1 - x_2 \| \quad (2.3)$$

for all $(t_1, x_1), (t_2, x_2)$ in $R_b$ with $x_1 \neq x_2$.

$$\langle f(t_1, x_1) - f(t_2, x_2), j(x_1 - x_2) \rangle \leq L \| x_1 - x_2 \|^2 \quad (2.4)$$

for all $(t_1, x_1)$ and $(t_2, x_2)$ in $R_b$.

$$\lim_{h \to 0^-} (\| x_1 - x_2 \| + h [f(t_1, x_1) - f(t_2, x_2)]) - \| x_1 - x_2 \|/h \leq L \| x_1 - x_2 \| \quad (2.5)$$

for all $(t_1, x_1)$ and $(t_2, x_2)$ in $R_b$.

For a proof see Ref. [5, Corollary 2.2].

3. Kamke Functions and Existence Results for Arbitrary Banach Spaces

Definition 3.1. We shall say $\omega$ is a Kamke function on $[0, T]$ if

(i) $\omega : [0, T] \times [0, \infty) \to [0, \infty)$

(ii) $\omega(t, 0) = 0$

(iii) $u(t) = 0$ is the unique continuous solution of

$$u(t) \leq \int_0^t \omega(s, u(s)) \, ds$$

for which $\lim_{t \to \infty} u(t)/t$ exists and is 0.
Proposition 2.1 can be generalized by using the terminology of Kamke functions. The restatement is left to the reader. We use these functions in stating Theorem 3.1. In Theorem 3.1 we omit the assumption that $E$ is uniformly convex at the expense of having to assume $f$ is uniformly continuous by using a different dissipation condition for $f$.

**Theorem 3.1.** Let $f : \mathbb{R}^1 \times E \to E$ be uniformly continuous on $[0, T] \times B_b(x_0)$ for some $T > 0$, $b > 0$ and $x_0 \in E$. Let $\omega$ be a Kamke function on $[0, T]$. Assume $T$ is chosen so that $\|f(t, x)\| \leq b/T$, for all $(t, x) \in [0, T] \times B_b(x_0)$.

Write $g_h(t, x) := x - hf(t, x)$ for $h > 0$. Assume for any subset $X \subset B_b(x_0)$

$$\alpha(g_h(t, X)) \geq \alpha(X) - hw(t, \alpha(X)) \quad \text{for all} \quad t \in [0, T]. \quad (3.1)$$

Then Eq. (1.1) has at least one solution defined on $[0, T]$ with $u(0) = x_0$.

Remarks on special cases. In Ref. [1] Ambrosetti proves that the conclusion of Theorem 3.1 is valid under the condition

$$\alpha(f(t, X)) \leq L\alpha(X) \quad \text{for any} \quad X \subset B_b(x_0) \quad \text{and some} \quad L \geq 0. \quad (3.2)$$

In Ref. [6] K. Goebel and Rzymowski generalize (3.2) as

$$\alpha(f(t, X)) \leq \omega(t, \alpha(X)) \quad \text{for any} \quad X \subset B_b(x_0) \quad (3.3)$$

where $\omega$ is any Kamke function. Inequality (3.1) is a generalization of (3.3) since

$$\alpha(g_h(t, X)) = \alpha(x - hf(t, x) : x \in X) \leq \alpha(X) + hw(f(t, X)).$$

By letting $\omega(t, r) = Lr$, one sees that (3.3) is a generalization of (3.2) and also Martin's condition (2.5) becomes a special case of (3.1), this, of course, unlike Martin's we have to assume $f$ is uniformly continuous.

We now prove several lemmas which are used in proving Theorem 3.1.

**Lemma 3.1.**

1. If $A \subset R$ then $\alpha(A) \leq \alpha(R)$,
2. $\alpha(\lambda A) = |\lambda| \alpha(A)$ for $\lambda \in \mathbb{R}$, (where $\lambda A = \{\lambda a : a \in A\}$),
3. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ where $A + B = \{x + y : x \in A \text{ and } y \in B\}$,
4. $\alpha(\bar{A}) = \alpha(A)$ where $\bar{A}$ denotes the closure of $A$,
5. $\alpha(A) = 0 \iff \bar{A}$ is compact.
Lemma 3.2. If \( H = \{ U_\beta \} \subset C(I) \) is any equicontinuous family of functions then
\[
\sup_{t \in I} \alpha(\{ U_\beta(t) : U_\beta \in H \}) = \alpha(H).
\]
For a proof see Ambrosetti [1].

Lemma 3.3. Let \( P = \{ p_n \}, Q = \{ q_n \} \) be two countable sets of points in \( E \). Then
\[
\alpha(\{ p_n \}) - \alpha(\{ q_n \}) \leq \alpha(\{ p_n - q_n \}).
\]

Proof. For any \( \varepsilon > 0 \), let \( A_i \subset E, i = 0, 1, \ldots, k \), \( B_j \subset E, j = 0, 1, \ldots, m \), be two families of open sets such that
\[
Q \subset \bigcup_{i=1}^{k} A_i \quad \text{and} \quad \bigcup_{n=1}^{\infty} \{ p_n - q_n \} \subset \bigcup_{j=1}^{m} B_j
\]
with \( \text{diam} \ A_i \leq \alpha(Q) + \varepsilon/2 \) and \( \text{diam} \ B_j \leq \alpha(\{ p_n - q_n \}) + \varepsilon/2 \) for each \( i \) and \( j \). Let \( C_{ij} = A_i + B_j \) for \( i = 0, 1, \ldots, k \) and \( j = 0, 1, \ldots, m \). Then \( \{ C_{ij} \} \) is a finite cover of \( P \) with
\[
\text{diam} \ C_{ij} \leq \text{diam} \ A_i + \text{diam} \ B_j \leq \alpha(Q) + \alpha(\{ p_n - q_n \}) + \varepsilon.
\]
It follows that \( \alpha(\{ p_n \}) \leq \alpha(\{ q_n \}) + \alpha(\{ p_n - q_n \}) + \varepsilon \) and since \( \varepsilon \) is arbitrary, the lemma is proved. \( \blacksquare \)

In the following two lemmas we assume the hypotheses of Theorem 3.1 are satisfied.

Lemma 3.4. Let \( \{ \varepsilon_n \} \) be a decreasing sequence of positive real numbers tending to zero. Then letting \( M = T/b \) there exists for each \( n \) a mapping
\[
u_n : [0, T] \rightarrow E
\]
such that
\[
u_n(0) = x_0, \tag{3.4}
\]
\[
\| u_n(t) - u_n(t') \| \leq M \| t - t' \| \quad \text{for all} \quad t, t' \in [0, T]. \tag{3.5}
\]
\[
\| u_n(t) - u_n(t') - \int_{t'}^{t} f(s, u_n(s)) \, ds \| \leq \varepsilon_n \| t - t' \|. \tag{3.6}
\]

The proof of Lemma 3.4 is standard and so is omitted. The function \( u_n \) may be choosen to be polygonal curves satisfying (3.4), (3.5), and (3.6).

Using the notation of Lemma 3.4, define \( p(t) = \alpha(X(t)) \) for each \( t \) in \( [0, T] \).
where \( X(t) = \{ u_n(t) : \text{for all } n \in \mathbb{N} \} \). We will omit "\( : \text{for all } n \in \mathbb{N} \)" from now on in specifying such sets. Hence we would write \( X(t) = \{ u_n(t) \} \).

**Lemma 3.5.**

\[
p(t) \leq \int_0^t \omega(s, p(s)) \, ds \quad \text{for all } t \in [0, T].
\]

**Proof.** First we prove \( p(t) \) is continuous. Let \( \delta > 0 \). By Lemma 3.3 and Ineq. (3.5)

\[
p(t + \delta) - p(t) = \omega(u_n(t + \delta)) - \omega(u_n(t)) \leq \alpha(u_n(t + \delta)) - u_n(t)) \leq 2M\delta.
\]

Therefore \( p \) is continuous and thus \( \omega(s, p(s)) \) is integrable. For proving (3.7), let \( t \in [0, T] \). Since \( f \) is uniformly continuous, for any given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |t - t'| < \delta \), \( ||x - x'|| < \delta \) implies

\[
|f(t, x) - f(t', x')| < \epsilon/4.
\]

For a positive integer \( k > Mt/\delta \), let \( h = t/k < \delta/M \) and

\[0 = t_0 < t_1 < t_2 < \cdots < t_n = t\]

where \( t_i = t_{i-1} + h, i = 1, \ldots, k \). Then,

\[
\int_0^t \omega(s, p(s)) \, ds = \lim_{k \to \infty} \left( \sum_{i=1}^k \omega(t_i, p(t_i))h \right)
\]

\[
= \lim_{k \to \infty} \left( h \cdot \sum_{i=1}^k \omega(t_i, \alpha(X(t_i))) \right)
\]

\[
\geq \lim_{k \to \infty} \sum_{i=1}^k \left[ \alpha(X(t_i)) - \alpha(g_h(t_i, X(t_i))) \right]
\]

\[
\geq \lim_{k \to \infty} \sum_{i=1}^k \left[ \alpha(X(t_i)) - \alpha(g_h(t_i, X(t_i))) \right] = \lim_{k \to \infty} S_k,
\]
where
\[
S_k \equiv \sum_{i=1}^{k} \left[ \alpha(X(t_i)) - \alpha(g_h(t_i, X(t_i))) \right]
\]
\[
= \sum_{i=1}^{k} \left( \left[ \alpha(X(t_i)) - \alpha(X(t_{i-1})) \right] - \left[ \alpha(X(t_{i-1})) - \alpha(g_h(t_i, X(t_i))) \right] \right)
\]
\[
= p(t) - \sum_{i=1}^{k} \left[ \alpha(X(t_{i-1})) - \alpha(g_h(t_i, X(t_i))) \right].
\]

By using Lemma 3.3
\[
\alpha(X(t_{i-1})) - \alpha(g_h(t_i, X(t_i))) \leq \alpha(u_n(t_{i-1}) - u_n(t_i) + hf(t_i, u_n(t_i))).
\]

But, by (3.6)
\[
\left\| u_n(t_i) - u_n(t_{i-1}) - hf(t_i, u_n(t_i)) \right\| 
\leq \left\| u_n(t_i) - u_n(t_{i-1}) - \int_{t_{i-1}}^{t_i} f(s, u_n(s)) \, ds \right\|
\]
\[
+ \left\| \int_{t_{i-1}}^{t_i} [f(s, u_n(s)) \, ds - f(t_i, u_n(t_i))] \right\|
\leq \varepsilon_n \cdot h + eh/4 < eh/2 \text{ for sufficient large } n.
\]

Therefore, \( \alpha(X(t_{i-1})) - \alpha(g_h(t_i, X(t_i))) \leq \varepsilon \cdot h \) and \( S_k \geq p(t) - \sum_{i=1}^{k} \varepsilon \cdot h = p(t) - \varepsilon t \). Hence,
\[
p(t) \leq \int_{0}^{t} \omega(s, p(s)) \, ds + \varepsilon t.
\]

Since \( \varepsilon \) is arbitrary, we have,
\[
p(t) \leq \int_{0}^{t} \omega(s, p(s)) \, ds. \]

**Proof of Theorem 3.1.** Since \( p(0) = 0 \) and \( \omega(t, 0) = 0 \) it follows from (3.5)
\[
\lim_{t \to 0^+} \frac{p(t)}{t} = 0.
\]
Hence \( p(t) = 0 \) for all \( t \in [0, T] \). Since \( \{u_n\} \) is an equi-

continuous family, by Lemma 3.2 \( \alpha(u_n) = 0 \). Therefore, there exists a

subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) converging uniformly to a function \( u \) from \( [0, T] \)

into \( E \) such that \( u(0) = x_0 \). Since \( f \) is uniformly continuous and \( u_{n_k} \to u \)

uniformly on \( [0, T] \) as \( k \to \infty \) it follows that \( f(t, u_{n_k}(t)) \to f(t, u(t)) \) uniformly

on \( [0, T] \) as \( k \to \infty \). Replacing \( n \) by \( n_k \) in (3.6) and letting \( k \to \infty \), we obtain
\[
u(t) = x_0 + \int_{0}^{t} f(s, u(s)) \, ds.
\]

It is clear from this that \( u \) is a solution of \( x' = f(t, x) \) on \([0, T]\) such that
\( \omega(0) = x_0 \) which completes the proof. \( \blacksquare \)
Remark. In the proof of Theorem 3.1 we may replace (3.1) by more intuitive condition

$$\alpha(\{x + hf(t, x) : x \in X\}) \leq \alpha(X) + h\omega(t, \alpha(X))$$ \hspace{1cm} (3.8)

and the theorem remains true.

Condition (3.1) seems quite similar in some ways to the condition that $f$ is $\alpha$-Lip-dissipative. The following proposition shows any $\alpha$-Lip-dissipative $f$ automatically satisfies (3.1).

**Proposition 3.1.** Assume $f$ is $\alpha$-Lip-dissipative in $[0, T] \times B_b(x_0)$. For all $(t, x)$ in $[0, T] \times B_b(x_0)$ and $h \in \mathbb{R}^+$ write $g_h(t, x) = x - hf(t, x)$. Then for any subset $X \subset B_b(x_0)$ and $h \geq 0$, we have

$$\alpha(g_h(t, x)) \geq (1 - Lh) \alpha(X) \text{ for all } t \text{ in } [0, T].$$ \hspace{1cm} (3.1)

**Proof.** Let $h$ be arbitrary, $t$ in $[0, T]$, and $X \subset B_b(x_0)$. Then for any $\epsilon > 0$ there exists finite covering $\{B_{i_t}\}_{t=1}^n$ of $g_h(t, X)$ with

$$\text{diam } B_i \leq \alpha(g_h(t, X)) + \epsilon/3.$$

Let $A_i = \{x : x - hg(t, x) \in B_i\}$ then

$$X \subset \bigcup_{i=1}^n A_i.$$

By assumption there exists finite covering $\{O^i\}$ of $[0, T] \times B_b(x_0)$ such that

$$h < f(t_1, x_1) - f(t_2, x_2), j(x_1 - x_2) /\| x_1 - x_2 \| \leq Lh \| x_1 - x_2 \| + \epsilon/3$$

for all $(t_1, x_1), (t_2, x_2)$ belonging to the same $O^i$. Set $A_i' = \{x : (t, x) \in O^i\}$ and $A_{ij} = A_i \cap A_i'$. We have $X \subset \bigcup A_{ij} \text{ i = 1, 2, ..., n; j = 1, 2, ..., m}$. For $(t_1, x_1), (t_2, x_2) \in A_{ij}$,

$$\| x_1 - x_2 \| - \epsilon/3$$

$$\leq \langle x_1 - x_2, \frac{j(x_1 - x_2)}{\| x_1 - x_2 \|} \rangle - h \langle f(t_1, x_1) - f(t_2, x_2), j(x_1 - x_2) /\| x_1 - x_2 \| \rangle$$

$$= \langle (x_1 - h(t_1, x_1)) - (x_2 - h(t_2, x_2)), j(x_1 - x_2) /\| x_1 - x_2 \| \rangle$$

$$\leq \| (x_1 - h(t_1, x_1)) - (x_2 - h(t_2, x_2)) \| \leq \text{diam } B_i$$

$$\leq \alpha(g_h(t, X)) + \epsilon/3.$$
Therefore,
\[
\text{diam } A_{ij} \leq \alpha(g_h(t, X)) + \epsilon/3 + \epsilon/3 < \alpha(g_h(t, X)) + \epsilon
\]
and hence
\[
(1 - Lh) \alpha(X) \leq \alpha(g_h(t, X)) + \epsilon.
\]

Since \( \epsilon \) is arbitrary, we have
\[
(1 - Lh) \alpha(X) \leq \alpha(g_h(t, X)). \ 
\]

As a result of this proposition and Theorem 3.1, we see that Proposition 2.1 can be extended to any Banach space, provided \( f \) is uniformly continuous.

**Corollary 3.1.** If \( f \) is \( \alpha \)-Lip-dissipative and uniformly continuous then Eq. (1.1) has a solution.

### 4. An Example

Condition (3.1) is automatically satisfied for any \( f \) in finite dimensional space since \( \alpha(X) = 0 \) for every bounded set \( X \), so it is strictly more general than (2.5). In this section we give an example showing that condition (3.1) is strictly more general than Goebel-Rzymoski’s condition (3.3) which is weaker than Ambrosetti’s \( \alpha \)-Lipschitzian condition. It will be shown that there exists a continuous function \( f \) from \( l^\infty \) to \( l^\infty \) satisfying (2.5) but \( f \) does not satisfy \( \alpha(f(S)) \leq \omega(\alpha(S)) \) for any Kamke function \( \omega \), where \( S \) is a bounded set in \( E \).

Consider the function \( g(u) = \frac{v}{u^{1/3}} \). It is easily verified that the function \( g \) satisfies the following inequalities:

\[
m | u - v | \leq | g(u) - g(v) | \tag{4.1}
\]

for all \( u, v \in [-(3m)^{-3/2}, (3m)^{-3/2}] \) and \( m > 0 \).

\[
| u - v + h(g(u) - g(v)) | \geq | u - v | \tag{4.2}
\]

for all \( h \leq 0 \) and \( u, v \in (-\infty, \infty) \).

Let \( E = l^\infty \), i.e., the set of all bounded sequences with the supremum norm. For \( x = (x_1, x_2, \ldots, x_n, \ldots) \in E \), define \( f(x) = (g(x_1), g(x_2), \ldots, g(x_n), \ldots) \). Let
Let $f_i(x)$ denote the $i$th components of $f(x)$. It is easily verified that $f$ is continuous. Also,

$$D_-(x, y, f) = \lim_{h \to 0^-} h^{-1}(\| x - y + h(f(x) - f(y))\| - \| x - y \|)$$

$$= \lim_{h \to 0^-} h^{-1}(\sup_i |x_i - y_i + h(f_i(x) - f_i(y))| - \sup_i |x_i - y_i|)$$

$$= \lim_{h \to 0^-} h^{-1}(\sup_i |x_i - y_i + h[g(x_i) - g(y_i)]| - \sup_i |x_i - y_i|).$$

For $h < 0$,

$$\sup_i |x_i - y_i + h[g(x_i) - g(y_i)]| - \sup_i |x_i - y_i| \geq \sup_i |x_i - y_i + h[g(x_i) - g(y_i)]| - \sup_i |x_i - y_i|.$$ 

By (4.2), $\sup_i |x_i - y_i + h[g(x_i) - g(y_i)]| > \sup_i |x_i - y_i|$, hence,

$$D_-(x, y, f) \leq 0.$$ 

Therefore, (2.5) is satisfied with $L = 0$.

On the other hand, for any $m > 0$, we define a set

$$A_m = \{x \in E : \| x \| < (3m)^{-3/2}\}.$$ 

Then $\alpha(A_m) = 2 \times (3m)^{-3/2}$.

By applying (4.1), we have,

$$\| f(x) - f(y)\| = \sup_i |f_i(x) - f_i(y)| = \sup_i |g(x_i) - g(y_i)|$$

$$\geq \| g(x_i) - g(y_i)\| = m \| x - y \|.$$ 

Hence, $\| f(x) - f(y)\| \geq m \| x - y \|$. It follows that

$$\alpha(f(A_m)) \geq \max(A_m).$$ 

The proof of (4.3) is standard. See, for example, Szulfla [2]. Suppose that there exists a Kamke function $\omega$ from $[0, \infty) \to [0, \infty)$ such that

$$\alpha(f(S)) \leq \omega(\alpha(S)) \quad \text{for all bounded } S \subset E.$$ 

Then, by (4.3), we have,

$$\max(A_m) \leq \alpha(f(A_m)) \leq \omega(\alpha(A_m)).$$ 

Let $r = \alpha(A_m) = 2 \times (3m)^{-3/2}$.

$$k_1 = (1/2)^{-3/2} \times (1/3), \quad \text{and} \quad k_1 \equiv k_1^r \leq \omega(r).$$
for all \( r > 0 \). For \( u(t) = (2k_1t/3)^{3/2} \), we have,

\[
\begin{align*}
u(t) &= \int_0^t k_1(2k_1t/3)^{1/2} dt \\
&= \int_0^t k_1[u(t)]^{3/2} dt \\
&\leq \int_0^t \omega(u(t)) dt
\end{align*}
\]

Also, both \( u(t)/t = (2k_1/3)^{3/2} t^{1/2} \) and \( u(t) \) go to zero as \( t \to 0^+ \). This contradicts to the fact that \( \omega \) is a Kamke function.

**References**