Index theory, nontrivial solutions, and asymptotically linear second-order Hamiltonian systems

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Abstract

In this paper, we consider the existence and multiplicity of solutions of second-order Hamiltonian systems. We propose a generalized asymptotically linear condition on the gradient of Hamiltonian function, classify the linear Hamiltonian systems, prove the monotonicity of the index function, and obtain some new conditions on the existence and multiplicity for generalized asymptotically linear Hamiltonian systems by global analysis methods such as the Leray–Schauder degree theory, the Morse theory, the Lusternik–Schnirelman theory, etc. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

In [8] Ekeland et al. discussed the fixed endpoint problem

\[ x'' + V'(t, x) = 0, \quad (1.1) \]
\[ x(0) = x_0, \quad x(T) = x_1. \quad (1.2) \]

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where \( x_0, x_1 \in \mathbb{R}^n \) and \( T > 0 \) are fixed. Assume \( V(t, x) = V(x) \) is superquadratic at infinity and \( V(-x) = V(x) \) for any \( x \in \mathbb{R}^n \), they proved problem (1.1) and (1.2) has infinitely many solutions. When \( V(x) \) is subquadratic, one solution was found under some suitable conditions by Clarke and Ekeland [4]. On the other hand, Chang in his excellent book [2] discussed the operator equation

\[
Ax + g'(x) = 0,
\]

where \( H \) is a Hilbert space, \( A : H \to H \) is a self-adjoint operator, \( g \in C^1(H, \mathbb{R}) \) has a bounded compact differential \( g'(x) \). Assume \( g'(\theta) = \theta \) where \( \theta \in H \) is the origin, by the Morse theory he discussed nontrivial solutions of (1.3). Of course, (1.1)–(1.2) can be studied in the framework of (1.3) when \( V'(t, x) \) is asymptotically linear, i.e.,

\[
V'(t, x) = A(t)x + o(|x|)
\]

as \( |x| \to \infty \) where \( A \in L^\infty((0, 1), GL_s(\mathbb{R}^n)), \) \( GL_s(\mathbb{R}^n) \) is the group of \( n \times n \) symmetric matrices with entries in \( \mathbb{R} \) and with the norm defined by \( |A| := \sum_{i,j=1}^n |a_{ij}| \) for any \( A = (a_{ij})_{n \times n} \in GL_s(\mathbb{R}^n) \), and for any \( x \in \mathbb{R}^n \) we denote by \( |x| \) the usual norm of \( x \) in \( \mathbb{R}^n \). One can also refer to the work [18] by Wang as an example, where problem (1.1) and (1.2) was discussed with \( T = \pi, x_0 = x_1 = 0 \) and \( V''(t, x) \to B_{\infty}(t) \) as \( |x| \to \infty \). In this paper, we consider second-order Hamiltonian systems (1.1) and (1.2) with \( x_0 = x_1 = 0, T = 1, \) i.e., we will consider the following problem:

\[
x'' + V'(t, x) = 0, \quad t \in (0, 1),
\]

\[
x(0) = 0 = x(1).
\]

We assume

\[
V'(t, x) = A(t)x + o(|x|)
\]

as \( |x| \to \infty \), where \( A \in C([0, 1] \times \mathbb{R}^n, GL_s(\mathbb{R}^n)) \) and

\[
A_1(t) \leq A(t, x) \leq A_2(t)
\]

for a.e. \( t \in (0, 1) \) and \( x \in \mathbb{R}^n \) where \( A_1, A_2 \in L^\infty((0, 1), GL_s(\mathbb{R}^n)) \). Throughout this paper for any \( A_1, A_2 \in GL_s(\mathbb{R}^n) \), we denote by \( A_1 \leq A_2 \) if \( A_2 - A_1 \) is positively semi-definite, and denote by \( A_1 < A_2 \) if \( A_2 - A_1 \) is positive definite. For any \( A_1, A_2 \in L^\infty((0, 1), GL_s(\mathbb{R}^{2n})) \), we denote by \( A_1 \preceq A_2 \) if \( A_1(t) \preceq A_2(t) \) for a.e. \( t \in (0, 1) \), and denote by \( A_1 < A_2 \) if \( A_1 \preceq A_2 \) and \( A_1(t) < A_2(t) \) on a subset of \((0,1)\) with nonzero measure. Note that (1.4) is a special case of (1.6) and (1.7). As usual (1.4) is called asymptotically linear conditions, so in this paper we call (1.6) and (1.7) generalized...
asymptotically linear conditions. In order to give the existence conditions on \( A_1 \) and \( A_2 \) we need to classify the linear system

\[
x'' + A(t)x = 0, \quad x(0) = 0 = x(1)
\]

for every \( A \in L^\infty((0, 1), GL_\mathbb{R}(\mathbb{R}^n)) \). This classification gives a pair of numbers \((i(A), v(A)) \in (\mathbb{N} \cup \{0\}) \times \{0, 1, \ldots, n\}\) for any \( A \in L^\infty((0, 1); GL_\mathbb{R}(\mathbb{R}^n)) \). We call \( i(A) \) and \( v(A) \) the index and nullity of \( A \), respectively. This index is nondecreasing with respect to \( A \). By this fact, we can get new solvable conditions for (1.1) and (1.5). For example, we shall prove problem (1.1) and (1.5) has at least one solution provided \( i(A_1) = i(A_2), v(A_1) = v(A_2) = 0 \). When \( V'(t, \theta) = 0, V''(t, \theta) := B_0(t) \) and \( i(B_0) \neq i(A_1) \) we discuss the existence of multiple nontrivial solutions by the Morse theory and the Ljusternik–Schnirelman theory. These are the contents of the following sections. In Section 2 we discuss the index theory. In Section 3 we discuss applications of the index theory, and several examples will be given to show that our results could be applied to some new cases.

2. Index theory for second-order linear Hamiltonian systems

For any \( A \in L^\infty((0, 1); GL_\mathbb{R}(\mathbb{R}^n)) \), consider the following system:

\[
x'' + A(t)x = 0, \quad x(0) = 0 = x(1).
\] (2.1)

Define

\[
q_A(x, y) = \int_0^1 [x'(t) \cdot y'(t) - A(t)x(t) \cdot y(t)] \, dt \quad \forall x, y \in E,
\] (2.3)

where \( a \cdot b \) is the usual inner product for any \( a, b \in \mathbb{R}^n \), and \( E = H_0^1([0, 1]; \mathbb{R}^n) := \{ x : [0, 1] \rightarrow \mathbb{R}^n \mid x(t) \text{ is continuous on } [0, 1], \text{ satisfies (2.2), and } x' \in L^2((0, 1); \mathbb{R}^n) \} \) is a Hilbert space with the norm \( \| x \| := (\int_0^1 \| x'(t) \|^2 \, dt)^{\frac{1}{2}} \). For any \( x, y \in E \) if \( q_A(x, y) = 0 \) we say that \( x \) and \( y \) are \( q_A \)-orthogonal. As in [7,11,12] for any two subspaces \( E_1 \) and \( E_2 \) of \( E \) if \( q_A(x, y) = 0 \) for any \( x \in E_1, y \in E_2 \) we say that \( E_1 \) and \( E_2 \) are \( q_A \)-orthogonal.

**Proposition 2.1.** For any \( A \in L^\infty((0, 1); GL_\mathbb{R}(\mathbb{R}^n)) \) the following results hold:

(i) there exist \( \{ \lambda_i(A) \} \subset \mathbb{R} \) with \( \lambda_1(A) < \lambda_2(A) < \cdots \) and \( \lambda_i(A) \rightarrow +\infty \) such that

\[
x'' + (A(t) + \lambda_i(A))x = 0, \quad x(0) = 0 = x(1)
\] (2.4)
has a nontrivial solution, and if we denote the subspace of the solutions with respect to $\lambda_i = \lambda_i(A)$ by $E_i(A)$, then $\dim E_i(A) := n_i \leq n$ and $E = \bigoplus_{i=1}^{\infty} E_i(A)$; (ii) the space $E$ has a $q_A$-orthogonal decomposition

$$E = E^+(A) \oplus E^0(A) \oplus E^-(A)$$

such that $q_A$ is positive definite, null and negative definite on $E^+(A)$, $E^0(A)$ and $E^-(A)$, respectively. Moreover, $E^0(A)$ and $E^-(A)$ are finitely dimensional.

**Proof.** (i) As in the proof of Chang and Lin [3, Theorem 5.4.2], the norm $\| \cdot \|_{\lambda_0}$ defined by an inner product

$$(x, y)_{\lambda_0} := \int_0^1 [x'(t) \cdot y'(t) + (\lambda_0 - A(t))x(t) \cdot y(t)] dt \quad \forall x, y \in E$$

is equivalent to $\| \cdot \|$, where $\lambda_0$ is a positive number satisfying $\lambda_0 I_n > A$. And there is a continuously linear operator $K_{\lambda_0} : L^2 \to E$ satisfying

$$\int_0^1 x(t) \cdot y(t) dt = (x, K_{\lambda_0} y)_{\lambda_0} \quad \text{for any } x, y \in E. \quad (2.5)$$

Let $t : E \to L^2$ be the compact embedding, then $K_{\lambda_0} t : E \to E$ is self-adjoint and compact. By the spectral theory of self-adjoint compact operators, there exist $\mu_i \to 0$ and $e_{ij} \in E$, $i = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots, n_i$ such that

$$(e_{ij}, e_{lk})_{\lambda_0} = \delta_{il} \delta_{jk}, \quad K_{\lambda_0} e_{ij} = \mu_i e_{ij}. \quad (2.6)$$

From (2.5) and (2.6), we have

$$\mu_i (x, e_{ij})_{\lambda_0} = \int_0^1 x(t) \cdot e_{ij}(t) dt \quad \forall x \in E. \quad (2.7)$$

In particular, $\mu_i = \int_0^1 e_{ij}(t) \cdot e_{ij}(t) dt > 0$ for any $i \in \mathbb{N}$. Without loss of generality we assume $\mu_i$ is strictly monotonously decreasing. Denote $\lambda_i(A) = \frac{1}{\mu_i} - \lambda_0$ and $E_i(A) = \text{span}\{e_{ij}\}_{j=1}^{n_i}$, then the proof is complete except for $\dim E_i(A) = n_i \leq n$. Let $B(t) := \text{diag}\{A(t) + \lambda_i(A) I_n, I_n\}$, $x = y_1$, $-x' = y_2$, $y = (y_1, y_2)$, then (2.4) is equivalent to

$$y' = JB(t)y,$$

$$y_1(0) = 0 = y_1(1). \quad (2.8)$$
Let \( \gamma(t) \) be the matrizenant of Eq. (2.8), i.e.,

\[
\dot{\gamma}(t) = JB(t)\gamma(t), \quad \gamma(0) = I_{2n}.
\]

Then \( E_i(A) \cong \{ c \in \mathbb{R}^n | \gamma_{12}(1)c = 0 \} \subseteq \mathbb{R}^n \) where \( \gamma(1) = \begin{pmatrix} \gamma_{11}(1) & \gamma_{12}(1) \\ \gamma_{21}(1) & \gamma_{22}(1) \end{pmatrix} \). And hence \( \text{dim } E_i(A) = n_i \leq n \).

(ii) For any \( x \in E \) with \( x = \sum c_{ij}e_{ij} \), from (2.7) and that \( e_{ij} \) satisfies (2.6) we have

\[
q_A(x, x) = \sum_{i,j} \lambda_i(A)e_{ij}^2 \int_0^1 |e_{ij}(t)|^2 dt. \tag{2.9}
\]

Hence the results hold if we denote:

\[
E^+(A) = \left\{ x = \sum c_{ij}e_{ij} | c_{ij} = 0 \text{ if } \lambda_i(A) \leq 0 \right\},
\]

\[
E^0(A) = \left\{ x = \sum c_{ij}e_{ij} | c_{ij} = 0 \text{ if } \lambda_i(A) \neq 0 \right\},
\]

\[
E^-(A) = \left\{ x = \sum c_{ij}e_{ij} | c_{ij} = 0 \text{ if } \lambda_i(A) \geq 0 \right\}. \quad \square
\]

**Definition 2.2.** For any \( A \in L^\infty((0, 1); GL_s(\mathbb{R}^n)) \), we define \( i(A) = \text{dim } E^-(A) \), \( v(A) = \text{dim } E^0(A) \).

**Remark.** We call \( i(A) \) and \( v(A) \) index and nullity of \( A \), respectively. Similar definitions can be found in [7,11–15,19]. In the following we shall discuss the properties of \((i(A), v(A))\).

**Proposition 2.3.** For any \( A \in L^\infty((0, 1); GL_s(\mathbb{R}^n)) \), we have

(i) \( v(A) \) is the dimension of the solution subspace of (2.1) and (2.2) and \( v(A) \in \{0, 1, 2, \ldots, n\} \).

(ii) \( i(A) = \sum_{\lambda_i(A) < 0} n_i \), where \( \lambda_i(A) \) and \( n_i \) are defined in Proposition 2.1.

**Proof.** (i) By definition, if \( \lambda_i(A) \neq 0 \) for any \( i \in \mathbb{N} \), then \( E^0(A) = \{0\} \) and \( v(A) = 0 \); if \( \lambda_i(A) = 0 \) for some \( i \in \mathbb{N} \), then \( v(A) = \text{dim } E_i(A) \in \{0, 1, \ldots, n\} \) by the definition in Proposition 2.1.

(ii) By definition \( E^-(A) = \bigoplus_{\lambda_i(A) < 0} E_i(A) \), and \( E_i(A) \) and \( E_j(A) \) are \( q_A \)-orthogonal if \( i \neq j \). Hence the result. \( \square \)
As a result, the equality holds if and only if

\[ \phi' = \cos^2 \phi + q(t) \sin^2 \phi, \quad \phi(0) = \alpha \]

by \( \phi(t, \alpha, A) \).

**Definition 2.4** (cf. Dong [5, Definition 2], [6, Definitions 1 and 2]). For any \( A \in L^\infty(0,1) \), denote \( A \in H_k \) if \( \phi(1,0,A) = (k+1)\pi \), \( k = 0,1,2,\ldots \). Denote \( A \in F_0 \) if \( \phi(1,0,A) < \pi \); and denote \( A \in F_k \) if \( k\pi < \phi(1,0,A) < (k+1)\pi \), \( k = 1,2,3,\ldots \).

**Proposition 2.5.** When \( n = 1 \), \( A \in L^\infty(0,1) \), one has \( v(A) \in \{0,1\} \); \( i(A) = k \), \( v(A) = 0 \) if and only if \( A \in F_k \); and \( i(A) = k \), \( v(A) = 1 \) if and only if \( A \in H_k \).

**Proof.** For \( n = 1 \), from (i) of Proposition 2.3, we have \( v(A) \in \{0,1\} \). It is well-known that \( \phi(1,0,A) = 0 (\bmod \pi) \) if and only if (2.1) and (2.2) has a nontrivial solution, and that \( \phi(1,0,A + \lambda I_n) \) is strictly monotonously increasing with respect to \( \lambda \in \mathbb{R} \). As a result, \( A \in F_0 \), i.e. \( \phi(1,0,A) < \pi \) if and only if \( v(A) = 0 \) and \( \lambda_j(A) > 0 \) for \( j \in \mathbb{N} \), which is equivalent to \( (i(A), v(A)) = (0,0) \). And \( A \in F_k \) for \( k \geq 1 \) we have \( \phi(1,0,A) \in (k\pi,k\pi + \pi) \). This is equivalent to \( v(A) = 0 \) and there exist \( \lambda_1 < \lambda_2 < \cdots < \lambda_k < 0 \) such that \( \phi(1,0,A + \lambda_i(A)I_n) = i\pi \), \( i = 1,2,\ldots,k \), i.e. \( v(A) = 0 \) and \( i(A) = \# \{ j | \lambda_j(A) < 0 \} = k \), where \# denotes the total number of elements in a set \( S \). The proof is complete. \( \square \)

**Proposition 2.6.** (i) For any \( A_1, A_2 \in L^\infty((0,1); GL_s(\mathbb{R}^n)) \), if \( A_1 \preceq A_2 \), we have \( i(A_1) \leq i(A_2) \); if \( A_1 < A_2 \), we have \( i(A_1) + v(A_1) \leq i(A_2) \).

(ii) If \( T \in O(n) \), i.e. \( T \in GL(n) \) and \( T^tT = I_n \), then \( i(T^tAT) = i(A) \), \( v(T^tAT) = v(A) \) for any \( A \in L^\infty((0,1); GL_s(\mathbb{R}^n)) \). If \( A_i \in L^\infty((0,1); GL_s(\mathbb{R}^n)) \), \( i = 1,2 \) and \( A = \text{diag}\{A_1,A_2\} \), then \( i(A) = i(A_1) + i(A_2) \), \( v(A) = v(A_1) + v(A_2) \). In particular, for any \( A \in GL_s(n) \) we have

\[
i(A) = \sum_{i=1}^n \# \{ k \in \mathbb{N} | k^2 \pi^2 < \lambda_i \}, \quad v(A) = \sum_{i=1}^n \# \{ k \in \mathbb{N} | k^2 \pi^2 = \lambda_i \},
\]

where \( \{ \lambda_i \}_{i=1}^n = \sigma(A) \), the set of eigenvalues of \( A \).

(iii) (Poincare inequality) For any \( A \in L^\infty((0,1); GL_s(\mathbb{R}^n)) \) with \( i(A) = 0 \) one has

\[
\int_0^1 |x'(t)|^2 dt \geq \int_0^1 A(t)x(t) \cdot x(t) dt \quad \forall x \in E.
\]

And the equality holds if and only if \( x \in E^0(A) \).
Proof. (i) Let $k = i(A_1)$, we assume $k \geq 1$. Let $e_1, \ldots, e_k$ be a basis of $E^-(A_1)$, and let $e_i = e_i^+ + e_i^0 + e_i^-$ with $e_i^* \in E^*(A_2)$, $* = +, 0, -$. In order to prove $i(A_2) \geq k$ we only need to show that $\{e_i^-\}_{i=1}^k$ is linear independent. In fact, if not there exist not all zero constants $c_1, \ldots, c_k$ such that $\sum_{i=1}^k c_i e_i = 0$. So $e := \sum_{i=1}^k c_i e_i = \sum_{i=1}^k c_i e_i^+ + \sum_{i=1}^k c_i e_i^0 \in E^+(A_2) \oplus E^0(A_2)$, and from (2.9) we have

$$q_{A_2}(e, e) \geq 0.$$  

But on the other hand, $E^-(A_1)$ is a linear subspace and $e_i \in E^-(A_1)$, so $e := \sum_{i=1}^k c_i e_i \in E^-(A_1) \setminus \{0\}$, and hence $q_{A_1}(e, e) < 0$. By the definition (2.3) we have

$$q_{A_2}(x, x) \leq q_{A_1}(x, x) \quad \forall x \in E.$$  

So

$$q_{A_2}(e, e) \leq q_{A_1}(e, e) < 0.$$  

This is a contradiction to (2.10). The first part is proved. Assume $v(A_1) = m$ and $\{b_i\}_{i=1}^m$ is a basis of $E^0(A_1)$. To prove $i(A_2) \geq i(A_1) + v(A_1)$ we only need to show that $e_1^-, \ldots, e_k^-, b_1^-, \ldots, b_m^-$ are linearly independent. If not there are not all zero constants $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m$ such that $\sum_{i=1}^k \alpha_i e_i^- + \sum_{i=1}^m \beta_i b_i^- = 0$. So

$$q_{A_2}(e + b, e + b) \geq 0,$$  

where $e = \sum_{i=1}^m \alpha_i e_i, b = \sum_{i=1}^k \beta_i b_i$. If $e \neq 0$ we have

$$q_{A_2}(e + b, e + b) \leq q_{A_1}(e + b, e + b) = q_{A_1}(e, e) + q_{A_1}(b, b) < 0.$$  

This is also a contradiction to (2.11). If $e = 0$, we have $x = b(t)$ is a nontrivial solution of (2.1) and (2.2) with $A(t)$ replaced by $A_1(t)$. So $b(t) \neq 0$ except for some finite possible points on $(0,1)$. By $A_1 < A_2$ we have

$$q_{A_2}(e + b, e + b) = q_{A_2}(b, b) < q_{A_1}(b, b) = 0,$$  

a contradiction to (2.11).

(ii) By definition, we have $\lambda_i(T^T A T) = \lambda_i(A)$ and $E_i(T^T A T) = T^T (E_i(A))$. So $E^-(T^T A T) = T^T (E^-(A))$, and $E^-(T^T A T) = T^T (E^-(A))$. And hence,

$$i(T^T A T) = i(A).$$  

(2.12)

Similarly,

$$E^-(\text{diag}[A_1, A_2]) = E^-(A_1) \oplus E^-(A_2)$$
and

\[ i(\text{diag}\{A_1, A_2\}) = i(A_1) + i(A_2). \quad (2.13) \]

Note that a scalar eigenvalue problem

\[ x'' + \lambda x = 0, \quad x(0) = 0 = x(1) \]

has a nontrivial solution if and only if \( \lambda = k^2 \pi^2, \ k = 1, 2, 3, \ldots \) It follows that for any \( \alpha \in \mathbb{R} \) from Definition 2.4 and Proposition 2.5 we have

\[ i(\alpha) = \#\{k \in \mathbb{N} | k^2 \pi^2 < \alpha \}. \quad (2.14) \]

Because \( \{\alpha_i\}_{i=1}^n \) are the eigenvalues of \( A \), we have \( T^\top AT = \text{diag}\{\alpha_1, \ldots, \alpha_n\} \) for some orthogonal matrix \( T \). From (2.12)–(2.14) we have

\[ i(A) = \sum_{i=1}^n i(\alpha_i) = \sum_{i=1}^n \#\{k \in \mathbb{N} | k^2 \pi^2 < \alpha_i \}. \]

(iii) For any \( x \in E \) with \( x = \sum_{i,j} c_{ij} e_{ij} \), from (2.3) and (2.9) we have

\[ \int_0^1 |x'(t)|^2 \, dt = \int_0^1 A(t)x(t) \cdot x(t) \, dt + \sum_{i,j} \lambda_i(A) c_{ij}^2 \int_0^1 |e_{ij}(t)|^2 \, dt. \]

Because \( i(A) = 0 \), by definition, \( \lambda_j(A) \geq 0 \) for any \( j \in \mathbb{N} \). So the inequality holds. And the equality is valid if and only if \( c_{ij} = 0 \) as \( \lambda_i(A) \neq 0 \). \( \square \)

By now we have proved the monotonicity of the indices (cf. (i) of Proposition 2.6). This will play an important role in the discussion of nonlinear Hamiltonian systems in the next section. But in the end of this section we will give a precise expression of the number \( i(A_2) - i(A_1) \) as \( A_2 > A_1 \) first. In order to do this, we will introduce a relative Morse index. This concept can be found in [9,13,14,19] for periodic solutions of first-order Hamiltonian systems. But here we give a completely different description, which only depends on the nullity.

**Definition 2.7.** For any \( A_1, A_2 \in L^\infty((0, 1); GL_s(\mathbb{R}^n)) \) with \( A_1 < A_2 \), we define

\[ I(A_1, A_2) = \sum_{\lambda \in [0,1)} v(\lambda A_1 + \lambda (A_2 + A_1)). \]
As $A_1 = x_1I_n$, $A_2 = x_2I_n$ we have $A_1 + \lambda(A_2 - A_1) = (x_1 + \lambda(x_2 - x_1))I_n$. From (ii) of Proposition 2.6 we have

$$I(A_1, A_2) = \sum_{\lambda \in [0,1]} n\{k \in \mathbb{N}|k^2 \pi^2 \in [x_1, x_2]\},$$

$$i(A_1) = n\{k \in \mathbb{N}|k^2 \pi^2 < x_1\},$$

$$i(A_2) = n\{k \in \mathbb{N}|k^2 \pi^2 < x_2\}.$$

And hence,

$$I(x_1I_n, x_2I_n) = i(x_2I_n) - i(x_1I_n).$$

This suggests the following

**Proposition 2.8.** For any $A_1$, $A_2 \in L^\infty((0, 1); GL_s(\mathbb{R}^{2n}))$ with $A_1 < A_2$, we have

$$I(A_1, A_2) = i(A_2) - i(A_1).$$

**Proof.** Denote $i(\lambda) := i(A_1 + \lambda(A_2 - A_1))$ for $\lambda \in [0, 1]$. Then (2.15) comes from the fact $i(\lambda + 0) = i(\lambda) + v(\lambda)$ for $\lambda \in [0, 1]$ and $i(\lambda - 0) = i(\lambda)$ for $\lambda \in (0, 1]$. We only prove the former one. By (i) of Proposition 2.6, it is enough to prove $i(\lambda + 0) \leq i(\lambda) + v(\lambda)$. For $\lambda \in [0, 1]$ fixed, there is $s \in (\lambda, 1)$ such that $i(\sigma) = i(\lambda + 0) := k$ for $\sigma \in (\lambda, s)$. By Proposition 2.1, there are $\{\lambda_i^\sigma\}_{i=1}^k \subset \mathbb{R}$, $\{e_i^\sigma\}_{i=1}^k \subset E$ such that

$$e_i^\sigma + (B_\sigma(t) + \lambda_i^\sigma)e_i^\sigma = 0,$$

$$e_i^\sigma(0) = 0 = e_i^\sigma(1),$$

where $B_\sigma(t) := A_1(t) + \sigma(A_2(t) - A_1(t))$. From the proof of Proposition 2.1 we have

$$(e_i^\sigma, e_j^\sigma)_{\lambda_0} = \delta_{ij} \mu_i^\sigma = \int_0^1 |e_i^\sigma(t)|^2 dt, \lambda_i^\sigma = \frac{1}{\mu_i^\sigma} - \lambda_0 < 0, \quad i, j = 1, 2, \ldots, k.$$

So there exist $\{\sigma_l\} \subset (\lambda, s)$ with $\sigma_l \to \lambda + 0$ as $l \to +\infty$ such that $e_i^{\sigma_l} \to e_i$ in $E$ and $e_j^{\sigma_l} \to e_j$ in $C([0,1]; \mathbb{R}^n)$. Because $\mu_i^\sigma > 0$, we have $\lambda_i^\sigma > -\lambda_0$. We may assume that $\lambda_i^{\sigma_l} \to \lambda_i$ in $\mathbb{R}$. From the Ascoli–Arzela’s theorem and (2.16) we know that $e_i^{\sigma_l} \to e_i$...
in $C^1([0, 1]; \mathbb{R}^n)$ by going to subsequences if necessary. Taking the limit as $l \to +\infty$ in (2.16)–(2.18) we have

$$\ddot{e}_i + (B_l(t) + \lambda_i)e_i = 0, \quad e_i(0) = 0 = e_i(1),$$

$$(e_i, e_j)_{\lambda_0} = \delta_{ij}, \quad \lambda_i \leq 0, \quad i, j = 1, 2, \ldots, k.$$

It follows

$$qB_k \left( \sum_{i=1}^{k} c_i e_i, \sum_{i=1}^{k} c_i e_i \right) = \sum_{i=1}^{k} \lambda_i c_i^2 \int_0^1 |e_i(t)|^2 \, dt \leq 0.$$

And hence, $k \leq i(\lambda) + v(\lambda)$. The proof is complete. □

3. Nontrivial solutions of second-order nonlinear Hamiltonian systems

Consider the following problem:

$$x'' + V'(t, x) = 0, \quad \text{(3.1)}$$

$$x(0) = 0 = x(1), \quad \text{(3.2)}$$

where $V : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$, $V' : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, $V'$ denotes the gradient of $V$ with respect to $x$.

**Theorem 3.1.** Assume

1. there exist $A \in L^\infty((0, 1) \times \mathbb{R}^n)$, and $h : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $h(t, x) = o(|x|)$ as $|x| \to \infty$ uniformly for $t \in [0, 1]$ such that

$$V'(t, x) = A(t, x)x + h(t, x), \quad \text{(3.3)}$$

2. there exist $A_1, A_2 \in L^\infty((0, 1); GL_s(\mathbb{R}^n))$ with $A_1 \leq A_2$, $i(A_1) = i(A_2)$, $v(A_2) = 0$ such that

$$A_1(t) \leq A(t, x) \leq A_2(t), \quad x \in \mathbb{R}^n, \quad \text{a.e. } t \in (0, 1), \quad \text{(3.4)}$$

or there exists $A_0 \in L^\infty((0, 1); GL_s(\mathbb{R}^n))$ with $i(A_0) + v(A_0) = 0$ such that

$$A(t, x) \leq A_0(t), \quad x \in \mathbb{R}^n, \quad \text{a.e. } t \in (0, 1). \quad \text{(3.5)}$$

Then (3.1) and (3.2) has at least one solution.
**Proof.** We first assume (3.4). By the Leray–Schauder principle we only need to prove the possible solutions of the following problem are a priori bounded with respect to the norm $\| \cdot \|$ of $E$:

$$x'' + \hat{\lambda} A_1(t)x + (1 - \hat{\lambda}) V'(t, x) = 0,$$

$$x(0) = 0 = x(1),$$

where $\hat{\lambda} \in (0, 1)$. If not, there exist $\{x_k\} \subset E$ with $\|x_k\| \to +\infty$, $\{\hat{\lambda}_k\} \subset (0, 1)$ such that

$$x_k''(t) + \hat{\lambda}_k A_1(t)x_k(t) + (1 - \hat{\lambda}_k)V'(t, x_k(t)) = 0,$$

$$x_k(0) = 0 = x_k(1).$$

Denote $y_k = x_k/\|x_k\|$, $B_k(t) = \hat{\lambda}_k A_1(t) + (1 - \hat{\lambda}_k) A(t, x_k(t))$, $e_k(t) = (1 - \hat{\lambda}_k) (V'(t, x_k(t)) - A(t, x_k(t))x_k(t))\|x_k\|^{-1}$, then (3.6) and (3.7) is equivalent to

$$y_k''(t) + B_k(t)y_k(t) + e_k(t) = 0,$$

$$y_k(0) = 0 = y_k(1).$$

From (3.3), $e_k \to 0$ in $C(0, 1)$. We may assume $y_k \to y_0$ in $E$, $y_k \to y_0$ in $C(0, 1)$, $\hat{\lambda}_k \to \hat{\lambda}_0$ and $b_{ij}^{(k)} \to b_{ij}$ in $L^2(0, 1)$ where $B_k(t) = (b_{ij}^{(k)})_{n \times n}(t)$ by going to subsequences if necessary. Denote $B_0(t) = (b_{ij}(t))_{n \times n}$, integrating (3.8), taking the limit and considering (3.9) we have

$$y_0''(t) + B_0(t)y_0(t) = 0,$$

$$y_0(0) = 0 = y_0(1).$$

By (3.4) we have $A_1 \leq B_k \leq A_2$, and hence $A_1 \leq B_0 \leq A_2$. By $i(A_2) = i(A_1)$ and $v(A_2) = 0$, from (i) of Proposition 2.6 we have $v(B_0) = 0$. This contradicts the fact that $y_0 \neq 0$ satisfies (3.10) and (3.11).

Second we assume (3.5). We also give the proof as in the first case. Now, we have arrived at (3.8) and (3.9) with $A_1$ replaced by $A_0$ and $B_k(t) = \hat{\lambda}_k A_0(t) + (1 - \hat{\lambda}_k) A(t, x_k(t))$. By $i(B_0) + v(B_0) = 0$, from Definition 2.7 and Proposition 2.8 there exists $\varepsilon > 0$ such that $i(A_0 + \varepsilon I_n) = 0$, $v(A_0 + \varepsilon I_n) > 0$, and

$$0 = \int_0^1 (y_k''(t) + B_k(t)y_k(t)) \cdot y_k(t) \, dt + \int_0^1 y_k(t) \cdot e_k(t) \, dt$$

$$= \int_0^1 [-|y_k'(t)|^2 + B_k(t)y_k(t) \cdot y_k(t)] \, dt + \int_0^1 y_k(t) \cdot e_k(t) \, dt.$$
\begin{equation}
\leq \int_0^1 \left[-|y_k'(t)|^2 + A_0(t)y_k(t) \cdot y_k(t)\right] dt + \int_0^1 y_k(t) \cdot e_k(t) dt
\end{equation}
\begin{equation}
\leq -\varepsilon \int_0^1 |y_k(t)|^2 dt + \int_0^1 y_k(t) \cdot e_k(t) dt
\end{equation}
\begin{equation}
\rightarrow -\varepsilon \|y_0\|_{L^2}^2
\end{equation}

where we have used (iii) of Proposition 2.6 in the second inequality. This is a contradiction. The proof is complete. \(\square\)

**Example 1.** Assume \(A(t) = (a_{ij}(t))_{n \times n} = T^t \text{diag}(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))T\) with \(T \in O(n); f_i : \mathbb{R} \rightarrow [0, \alpha]\) is continuous with \(f_i(\mathbb{R}) = [0, \alpha]\) and \(\alpha > 0, i = 1, 2, \ldots, n\). Let \(F_i(x) = \int_0^t f_i(t) dt, i = 1, 2, \ldots, n\) and \(V(t, x) = \sum_{i=1}^n F_i(x_i) + \frac{1}{2} \sum_{i=1}^n a_{ii}(t)x_i^2 + \sum_{1 \leq i < j \leq n} a_{ij}(t)x_i x_j + (\sum_{i=1}^n x_i^2)^{\frac{3}{2}} \sin t\) for \(t \in [0, 1], x \in \mathbb{R}^n\). Then this \(V(t, x)\) satisfies (3.3) where \(A(t, x) = A(t) + \text{diag}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\), \(A_1(t) = A(t)\), \(A_2(t) = A(t) + \alpha n\), and \(h(t, x) = \frac{3}{2} \left(\sum_{i=1}^n x_i^2\right)^{-\frac{3}{2}} x\). If \(v(\alpha_i) = 0, i = 1, 2, \ldots, n, \alpha > 0\) is small enough, then \(v(A_1) = 0 = v(A_2), i(A_1) = i(A_2)\). Therefore, the problem (3.1) and (3.2) has a solution from Theorem 3.1. In particular, as \(f_i(x) = \alpha (\sin x)^{2i}, \alpha_i(t) = \alpha_i, \alpha_i + \alpha \cap \{k^2 \pi^2\}^+ = \emptyset\) for \(i = 1, 2, 3, \ldots, n\), then (3.1) and (3.2) has a solution.

**Theorem 3.2.** In assumption (1) if \(A(t, x) = A(t)\) with \(v(A) \neq 0\), and

\begin{align}
(x, h(t, x)) & \geq c_1 |x|^2 - b_1, \\
|h(t, x)| & \leq c_2 |x|^{\alpha - 1} + b_2
\end{align}

for some positive constants \(c_1, c_2, b_1, b_2\) and \(1 \leq \alpha < 2\), then the problem (3.1) and (3.2) has at least one solution.

**Proof.** From Definition 2.7, Proposition 2.8 and the finiteness of the index \(i(A)\), for any \(A \in L^\infty((0, 1); GL_n(\mathbb{R}))\) there exists \(\varepsilon > 0\) such that \(i(A + \varepsilon I) = i(A) + v(A)\) and \(v(A + \varepsilon I) = 0\). Denote \(A_1 = A + \varepsilon I\), we only need to prove the solutions of the following problem are a priori bounded with respect to the norm \(\| \cdot \|\) of \(E\):

\begin{align}
x'' + \lambda A_1(t)x + (1 - \lambda)A(t)x + (1 - \lambda)h(t, x) = 0, \\
x(0) = 0 = x(1).
\end{align}

If not there exist \(x_k \in E\) with \(\|x_k\| \rightarrow +\infty\) such that

\begin{align}
x_k'' + \lambda_k A_1(t)x_k + (1 - \lambda_k)A(t)x_k + (1 - \lambda_k)h(t, x_k) = 0, \\
x_k(0) = 0 = x_k(1).
\end{align}
Denote $y_k = x_k/\|x_k\|$, we may assume $y_k \rightharpoonup y_0$ in $E$ and $\lambda_k \to \lambda_0$. So $y = y_0$ is a nontrivial solution of

$$y'' + (\lambda_0 A_1(t) + (1 - \lambda_0) A(t)) y = 0, \quad y(0) = 0 = y(1).$$

It follows from (i) of Proposition 2.6 that $\lambda_0 = 0$ and

$$y_0''(t) + A(t)y_0(t) = 0, \quad y_0(0) = 0 = y_0(1).$$

From (3.13) we have

$$\int_0^1 (x_k''(t) + A(t)x_k(t)) \cdot y_0(t) \, dt + \int_0^1 \lambda_k \varepsilon x_k(t) \cdot y_0(t) \, dt$$

$$+ \int_0^1 (1 - \lambda_k) h(t, x_k(t)) \cdot y_0(t) \, dt = 0.$$ 

It follows from (3.12) and (3.15) for $k$ large enough that

$$0 \geq \int_0^1 h(t, x_k(t)) \cdot y_0(t) \, dt$$

$$= \int_0^1 h(t, x_k(t)) \cdot x_k(t) \, dt \|x_k\|^{-1} + \int_0^1 h(t, x_k(t)) \cdot [y_0(t) - y_k(t)] \, dt$$

$$\geq \|x_k\|^{-1} \int_0^1 (c_1 |x_k(t)|^2 - b_1) \, dt - \int_0^1 (c_2 |x_k(t)|^{2-1} + b_2) \, dt \cdot \|y_k - y_0\|_{C(0,1)}.$$ 

Hence,

$$0 \geq c_1 \int_0^1 |y_k(t)|^2 \, dt - b_1 \|x_k\|^{-2}$$

$$- \left( c_2 \int_0^1 |y_k(t)|^{2-1} \, dt + b_2 \|x_k\|^{-2+1} \right) \|y_k - y_0\|_{C(0,1)}$$

$$\to c_1 \int_0^1 |y_0(t)|^2 \, dt,$$

as $k \to \infty$. This is a contradiction since $\|y_0\| \neq 0$ and $c_1 > 0$. The proof is complete. □

**Remark.** We emphasize that some conditions similar to (3.12) already appeared in [10,17] in discussing nontrivial periodic solutions of first-order Hamiltonian systems with other methods.
Theorem 3.3. Assume

1. \( V \in C^2([0, 1] \times \mathbb{R}^n, \mathbb{R}) \), \( A_1(t) \leq V''(t, x) \leq A_2(t) \) for \( |x| \geq M > 0 \) with \( i(A_1) = i(A_2) \), \( v(A_2) = 0 \).

2. \( V'(t, 0) \equiv 0 \), \( A_0(t) = V''(t, 0) \) and \( i(A_1) \notin [i(A_0), i(A_0) + v(A_0)] \). Then problem (3.1) and (3.2) has at least one nontrivial solution. Moreover, if we assume (3) \( v(A_0) = 0 \), \( |i(A_1) - i(A_0)| \geq n \). Then (3.1) and (3.2) has two nontrivial solutions.

Example 2. Let \( V(t, x) = \frac{1}{2} x^T A(t) x + f(\sum_{i=1}^n F_i(x_i)) \), where \( \lambda > 0 \) is a constant; \( f \in C^2([0, +\infty), [0, +\infty)) \), \( f(t) = 0 \) as \( t \in [0, 1] \) and \( f(t) = t \) as \( t \in [2, +\infty) \); and \( F_i(x) = \int_0^t d\theta \int_0^\theta f_i(s) ds \), \( f_i(x) = \lambda |\sin x_i|^2 + (\lambda + \varepsilon) |\cos x_i|^2 \). Then

\[
V''(t, 0) = A(t)
\]

and as \( \frac{1}{2} |x|^2 \lambda > 2 \)

\[
V''(t, x) = A(t) + \text{diag}\{f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\}
\]

and hence

\[
A(t) + \lambda I_n \leq V''(t, x) \leq A(t) + (\lambda + \varepsilon) I_n.
\]

From Theorem 3.3, if \( v(A) = 0 = v(A + \lambda I_n) \), \( i(A + \lambda I_n) > n + i(A) \), then (3.1) and (3.2) has at least two nontrivial solutions provided \( \varepsilon > 0 \) is small enough. Note that \( V(t, x) \) does not satisfy (1.4).

In order to finish the proof we need some lemmas. Let \( X \) be a Banach space and \( f \in C^1(X, \mathbb{R}) \). As in [2, Chapter 1] let \( K = \{ x \in X | f'(x) = 0 \} \), \( f_a = \{ x \in X | f(x) \leq a \} \). For an isolated critical point \( x_0 \), the critical group is defined by \( C_q(f, x_0) = H_q(f_c \cap U, (f_c \setminus \{p_0\}) \cap U; G) \) for any neighborhood \( U \) of \( p_0 \) with \( U \cap K = \{x_0\} \) and \( c = f(x_0) \). For any two regular values \( a < b \), if \( K \cap F^{-1}[a, b] = \{z_1, \ldots, z_l\} \), we denote by \( M_q = \sum_{i=1}^l \text{rank} C_q(f, z_i) \) and \( \beta_q = \text{rank} H_q(f_b, f_a; G) \).
Lemma 3.4 (cf. Chang [2, Chapter 1, Theorem 4.3]). Let \( f \in C^2(X, R^1) \) satisfy the (PS) condition, we have the following Morse inequalities:

\[
M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0,
\]

for \( q = 0, 1, 2, \ldots \).

When \( f \in C^2(X, R) \) and \( p \in K \) we have \( f''(p) \) is a self-adjoint operator. We call the dimension of the negative space corresponding to the spectral decomposing the Morse index of \( p \) and denote it by \( m_-(f''(p)) \), and denote by \( m_0(f''(p_0)) = \dim \ker f''(p_0) \).

If \( f''(p) \) has a bounded inverse we say that \( p \) is nondegenerate.

Lemma 3.5 (cf. Chang [2, Chapter 1, Theorem 5.4 and Corollary 5.1]). Assume \( f \in C^2(E, R) \) and \( p_0 \in K \) is an isolated critical point with finite index \( j := m_-(f''(p_0)) \) and nullity \( k := m_0(f''(p_0)) \). Then

1. \( C_q(f, p_0) \cong \delta_{qj}G \) or \( C_q(f, p_0) \cong \delta_{q, j+k}G \) or \( C_q(f, p_0) \cong 0 \) for \( q \leq j, q \geq j+k \).

2. If \( p_0 \) is a nondegenerate critical point, then

\[
C_q(f, p_0) \cong \delta_{qj}G, \quad q = 0, 1, 2, \ldots.
\]

3. If \( f \) has a minimal value at \( p_0 \), then

\[
C_q(f, p_0) \cong \delta_{q0}G, \quad q = 0, 1, 2, \ldots.
\]

From [2, Chapter III, Theorem 3.1, Chapter II, Theorem 5.1, 5.2, Corollary 5.2], one can prove

Lemma 3.6. Assume \( f \in C^2(X, R) \) satisfies the (PS) condition, \( f'() = 0 \), and there is a positive integer \( \gamma \) such that \( \gamma \notin [m_-(f''(\theta)), m_0(f''(\theta)) + m_-(f''(\theta))] \) and \( H_q(X, f_\alpha; G) \cong \delta_{q\gamma}G \) for some regular value \( a < f(\theta) \). Then \( f \) has a critical point \( p_0 \neq \theta \) with \( C_\gamma(f, p_0) \neq 0 \). Moreover, if \( \theta \) is a nondegenerate critical point, and \( m_0(f''(p_0)) \leq |\gamma - m_-(f''(\theta))| \), then \( f \) has another critical point \( p_1 \neq p_0, \theta \).

Proof of Theorem 3.3. Since \( \nu(A) \leq n \) for any \( A \in L^\infty((0, 1) \times R^n; GL_s(R^n)) \), by Lemma 3.6 we only need to show

\[
H_q(E, f_\alpha; G) \cong \delta_{q\gamma}G \quad (3.16)
\]

for \(-a > -f(\theta)\) is large enough, where \( \gamma = i(A_1) \). Let \( \{e_j(A)\} \) be a renumbered sequence of \( \{e_{ij}\} \) defined by (2.4) such that

\[
\ddot{e}_j(A) + (A(t) + \dot{\lambda}_j(A))e_j(A) = 0
\]
and $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. Denote by $H_1 = \text{span}\{e_j(A_1) | j \leq \gamma\}$, $H_2 = \text{span}\{e_j(A_2) | j > \gamma\}$, then

$$E = H_1 \oplus H_2.$$  \hfill (3.17)

In fact, let $\tilde{H} = \text{span}\{e_j(A_2) | j \leq \gamma\}$, then $E = \tilde{H} \oplus H_2$, i.e., for every $x \in E$ there exists uniquely pair $(\tilde{x}, x_2) \in \tilde{H} \times H_2$ such that $x = \tilde{x} + x_2$. Let $e_j(A_1) = e^+_j + e^-_j$ with $e^+_j \in H_2$, $e^-_j \in \tilde{H}$, $j = 1, 2, 3, \ldots, \gamma$, then $\{e^-_j\}$ is a base of $\tilde{H}$. There exist constants $\{x_j\}_{j=1}^\gamma$ such that $\tilde{x} = \sum_{j=1}^\gamma x_j e^-_j$, and hence $x = (\sum_{j=1}^\gamma x_j e_j(A_1)) + (x_2 - \sum_{j=1}^\gamma x_j e^+_j)$.

We have proved $E = H_1 + H_2$. Since $\int_0^1 [|x'(t)| - A_2(t)x(t) \cdot x(t)] \, dt > 0 \ \forall x \in H_2$ with $x \neq 0$, $\int_0^1 [|x'(t)| - A_1(t)x(t) \cdot x(t)] \, dt < 0 \ \forall x \in H_1$ with $x \neq 0$, and $\int_0^1 [|x'(t)| - A_1(t)x(t) \cdot x(t)] \, dt \geq \int_0^1 [|x'(t)| - A_2(t)x(t) \cdot x(t)] \, dt$ from $A_1 \leq A_2$ it follows that $H_1 \cap H_2 = \{0\}$. We have proved (3.17). In the following we will take two steps to finish the proof of (3.16).

**Step 1:** For $-a > -f(\emptyset)$ large enough we have

$$H_q(H, f_a; G) = H_q(M, M \cap f_a; G), \quad q = 0, 1, 2, \ldots,$$  \hfill (3.18)

where $M \subset H$ will be defined later. Since $e''_j + (A_2(t) + \lambda_j)e_j = 0$, we have

$$\lambda_j \int_0^1 e_j \cdot e_i \, dt = \int_0^1 (e'_j \cdot e'_i - A_2(t)e_j \cdot e_i) \, dt$$

$$= - \int_0^1 (e''_j \cdot e_i + A_2(t)e_j \cdot e_i) \, dt = \lambda_j \int_0^1 e_j \cdot e_i \, dt$$

where we denote by $e_j = e_j(A_2)$. Let $\int_0^1 e_j \cdot e_i \, dt = 0$ if $i \neq j$ and $\lambda_i = \lambda_j$, and $\lambda_j \int_0^1 |e_j|^2 \, dt = 1$. Then $\forall x \in H_2$ with $x = \sum_{j>\gamma} c_j e_j$, $\|x\|_2 := (\int_0^1 [|x'(t)|^2 - A_2(t)x(t) \cdot x(t)] \, dt)^{\frac{1}{2}} = \sum_{j>\gamma} c_j^2$. So $H_2$ is a Banach space under the norm $\| \cdot \|_2$. Similarly $H_2$ is also a Banach space under the norm $\| \cdot \|$. Moreover, from (iii) of Proposition 2.6 we have $C_1 \|x\|_2 \leq \|x\| \leq C_2 \|x\| \ \forall x \in H_2,$  \hfill (3.19)

where $C_2 > 0$ is a constant. Let $\|x\|_1 = (-\int_0^1 [|x'(t)| - A_1(t)x(t) \cdot x(t)] \, dt)^{\frac{1}{2}} \ \forall x \in H_1$, then $\| \cdot \|_1$ is a norm of $H_1$, and there are constants $C_3, C_4 > 0$ such that

$$C_3 \|x\| \leq \|x\|_1 \leq C_4 \|x\| \ \forall x \in H_1.$$  \hfill (3.20)
Since $\forall x = x_1 + x_2$ with $x_1 \in H_1, x_2 \in H_2$, we have from (3.19) and (3.20) and assumption (1) that

$$
(df(x), x_2 - x_1) = \int_0^1 [(x'(t), x_2'(t) - x_1'(t)) - (V'(t, x(t)), x_2(t) - x_1(t))] dt
$$

$$
= \int_0^1 [(|x_2'(t)| - |x_1'(t)|^2 \left( \int_0^1 V''(t, \theta x(t)) d\theta x_2(t), x_2(t) \right) + \left( \int_0^1 V''(t, \theta x(t)) d\theta x_1(t), x_1(t) \right)] dt
$$

$$
\geq \|x_2\|^2_2 + \|x_1\|^2_1 - C_5(\|x_2\| + \|x_1\|)
$$

$$
\geq C_2^2\|x_2\|^2 - C_5\|x_2\|^2 + C_3^2\|x_1\|^2 - C_5\|x_1\|,
$$

where $C_5 > 0$ is a constant. And hence, there exists $R_0 > 0$ such that

$$(df(x), x_2 - x_1) > 1 \quad \forall x \in E \text{ with } \|x_2\| > R_0 \text{ or } \|x_1\| > R_0.$$
In fact, because \( V(t, 0) = 0 \), \( V'(t, 0) = 0 \), we have

\[
V(t, x) = \int_0^1 V'(t, \theta x) \cdot x \, d\theta \\
= \int_0^1 \theta \, d\theta \int_0^1 V''(t, \theta_1 \theta x) \, d\theta_1 x \cdot x.
\]

By assumption (1) we have

\[
V(t, x) \leq \int_0^1 \theta \, d\theta \int_0^1 A_2(t) x \cdot x \, d\theta_1 + C_6 = \frac{1}{2} A_2(t) x \cdot x + C_6
\]

and

\[
V(t, x) \geq \frac{1}{2} A_1(t) x \cdot x - C_7.
\]

Therefore, for every \( x \in \mathcal{M} \) we have

\[
f(x) = \int_0^1 \left[ \frac{1}{2} |x'(t)|^2 - V(t, x(t)) \right] \, dt \\
\leq \frac{1}{2} \int_0^1 [\|x'(t)\|^2 - A_1(t) x(t) \cdot x(t)] \, dt + C_7 \\
= \frac{1}{2} \int_0^1 [\|x'_1(t)\|^2 - A_1(t) x_1(t) \cdot x_1(t)] \, dt \\
+ \frac{1}{2} \int_0^1 [\|x'_2(t)\|^2 - A_1(t) x_2(t) \cdot x_2(t)] \, dt \\
+ \int_0^1 [x'_1(t) \cdot x'_2(t) - A_1(t) x_1(t) \cdot x_2(t)] \, dt + C_7 \\
\leq -C_8 \|x_1\|^2 + C_9 \quad (3.24)
\]

and in a similar way we have

\[
f(x) \geq \frac{1}{2} \int_0^1 [\|x'(t)\|^2 - A_2(t) x(t) \cdot x(t)] \, dt - C_6 \\
\geq -C_9 \|x_1\|^2 - C_11, \quad (3.25)
\]

where all the above \( C_i \)s are positive constants and \( \lambda > 0 \) such that \( i (A_2 - \lambda I_n) = 0 \). Since \( H_1 \) is finite and any norm is equivalent to \( \| \cdot \| \). We obtain from (3.24) and
(3.25) that

\[ f(x) \to -\infty \iff \|x_1\| \to +\infty \text{ uniformly in } x_2 \in H_2 \cap B_{R_0}. \]

Thus, there exist \( T > 0, a_1 < a_2 < -T, R_1 > R_2 > R_0 \) such that

\[(H_2 \cap B_{R_0}) \oplus (H_1 \setminus B_{R_1}) \subset f_{a_1} \cap \mathcal{M} \subset (H_2 \cap B_{R_0}) \oplus (H_1 \setminus B_{R_0}) \subset f_{a_2} \cap \mathcal{M}.\]

We now begin to define a deformation from \( \mathcal{M} \cap f_{a_2} \) to \( \mathcal{M} \cap f_{a_1} \). Consider the flow defined by (3.21) and (3.22). For every \( x \in \mathcal{M} \cap (f_{a_2} \setminus f_{a_1}) \), since \( \sigma(t,x) = e^{-t}x_2 + e^t x_1, f(\sigma(t,x)) \) is continuous with respect to \( t, f(\sigma(0,x)) = f(x) > a_1 \) and \( f(\sigma(t,x)) \to -\infty \) as \( t \to +\infty \), so the time \( t = T_1(x) \) arriving at \( f_{a_1} \cap \mathcal{M} \) exists uniquely and is defined by \( f(\sigma(t,x)) = a_1 \). Since

\[ \frac{d}{dt} f(\sigma(t,x)) = \langle df(\sigma(t,x)), \sigma'(t,x) \rangle \]

\[ = \langle df(e^{-t}x_2 + e^t x_1), -e^{-t}x_2 + e^t x_1 \rangle \leq -1 \]

as \( t > 0 \). The continuity of \( t = T_1(x) \) comes from the implicit function theorem. Define

\[ \eta_1(t,x) = x, \quad x \in f_{a_1} \cap \mathcal{M} \]

\[ = \sigma(T_1(x),x), \quad x \in \mathcal{M} \cap (f_{a_2} \setminus f_{a_1}) \]

then \( \eta_1 : [0,1] \times f_{a_2} \cap \mathcal{M} \to f_{a_2} \cap \mathcal{M} \) is continuous, and is a deformation from \( f_{a_2} \cap \mathcal{M} \) to \( f_{a_1} \cap \mathcal{M} \) and \( \tau_1 = \eta_1(1,(\cdot)) : \mathcal{M} \cap f_{a_2} \to \mathcal{M} \cap f_{a_1} \) is a strong deformation retract. Define

\[ \tau_2(x) = x, \quad \|x_1\| \geq R_1 \]

\[ = x_2 + \frac{x_1}{\|x_1\|} R_1, \quad \|x_1\| < R_1 \]

then

\[ \tau_2 : (H_2 \cap B_{R_0}) \oplus (H_1 \setminus \text{int}(B_{R_2})) \to (H_2 \cap B_{R_0}) \oplus (H_1 \setminus \text{int}(B_{R_1})) \]

is a strong deformation retract. Let \( \tau = \tau_2 \circ \tau_1 \), we obtain a strong deformation retract:

\[ \tau : \mathcal{M} \cap f_{a_2} \to (H_2 \cap (B_{R_0})) \oplus (H_2 \setminus \text{int}(B_{R_1})). \]
Hence,

\[ \hat{H}_q(\mathcal{M}, \mathcal{M} \cap f_{a_2}; G) \cong \hat{H}_q((H_2 \cap B_{R_0}) \oplus H_1, (H_2 \cap B_{R_0}) \oplus (H_1 \setminus \text{int}(B_{R_1})); G) \cong \hat{H}_q(H_1 \cap B_{R_1}, \hat{\hat{\partial}}(H_1 \cap B_{R_1}); G) \cong \delta_{q_0}G. \]

This is (3.23). And (3.16) comes from (3.18) and (3.23). The proof is complete. □

**Theorem 3.7.** Assume that \( V \in C^2([0, 1] \times \mathbb{R}^n, \mathbb{R}^n) \), \( V''(t, x) \leq B_0(t) \) for every \( t \in [0, 1], x \in \mathbb{R}^n, i(B_0) + v(B_0) = 0, V'(t, \theta) = \theta, i(V''(t, \theta)) \geq 1 \), then (3.1) and (3.2) has at least a nontrivial solution. Moreover, if \( v(V''(\cdot, \theta)) = 0 \), then (3.1) and (3.2) has at least two nontrivial solutions.

**Proof.** As proved before \( f \) satisfies the (PS) condition and is bounded from below. So \( f \) has a minimal critical point \( p_0 \), and \( C_0(f, p_0) \cong G \). By (2) of Lemma 3.5 we also have \( C_0(f, \theta) = 0 \). Then \( p_0 \neq \theta \) is a nontrivial solution. We also get

\[ \hat{H}_q(H, f_{a_2}; G) \cong \hat{H}_q(B_R, f_{a_2} \cap B_R; G) \cong \hat{H}_q(B_R, \phi; G) \cong \delta_{q_0}G \]

for \( q = 0, 1, 2, \ldots, -a \) is large enough such that for every \( x \in f_{a_2} \), we have \( x \notin K \cap B_R \) as proved in the former theorem. If \( v(V''(\cdot, \theta)) = 0 \), then \( C_q(f, \theta) = \delta_{q_0}G, q = 0, 1, 2, \ldots, \) and \( \gamma := i(V''(\cdot, \theta)) \). Assume apart from \( p_0 \), \( f \) has not any other nontrivial solutions, then from Lemma 3.4 the \( \gamma + 1 \)th Morse inequality gives \(-1 \geq 0\), a contradiction. The proof is complete. □

**Remark.** This result can be obtained as a corollary of Chang [1, Theorem 5.3.1].

When the potential \( V \) is symmetric, i.e., \( V(t, -x) = V(t, x) \) for every \( t \in [0, 1], x \in \mathbb{R}^n \), we can find more than two nontrivial solutions.

**Theorem 3.8.** Assume \( V(t, 0) = 0 \) and

1. there are \( B_1, B_2 \in L^\infty((0, 1); GL_s(\mathbb{R}^n)) \) with \( i(B_1) = i(B_2), v(B_1) = v(B_2) = 0 \) such that

\[ V'(t, x) = B(t, x)x + o(|x|) \quad \text{as} \quad |x| \to \infty, \]

\[ B_1(t) \leq B(t, x) \leq B_2(t) \quad \forall t \in [0, 1], x \in \mathbb{R}^n. \]
(2) there are $A_1, A_2 \in L^\infty((0, 1) \times \mathbb{R}^n; GL_s(\mathbb{R}^n))$ with $i(A_1) = i(A_2), v(A_1) = v(A_2) = 0$ such that

$$V'(t, x) = A(t, x)x + o(|x|) \quad \text{as } |x| \to 0,$$

$$A_1(t) \leq A(t, x) \leq A_2(t) \quad \forall t \in [0, 1], x \in \mathbb{R}^n.$$

(3) For every $t \in [0, 1], x \in \mathbb{R}^n$, one has $V(t, -x) = V(t, x)$.

Then the problem (3.1) and (3.2) has at least $|i(B_1) - i(A_1)|$ pairs of solutions.

This result can be obtained from [1, Theorems 4.3.4 and 4.3.6]. We only prove it in the case $i(A_1) > i(B_1)$. This will be finished by a Ljusternik–Schnirelman-type theorem. For a Banach space $X$, let $\Sigma$ be the set of all symmetric closed subsets of $X$. Let $\gamma$ be the genus on $\Sigma$. Define

$$c_n = \inf\{a|\gamma(f_a) \geq n\}.$$

It was proved that $c_n$ is a critical value if $f$ satisfies the (PS) condition and $c_n$ is finite. Moreover, we have

**Lemma 3.9 (cf. Chang [1, Theorem 4.3.4]).** Assume $f \in C^1(X, \mathbb{R}^1)$ satisfies the (PS) condition, $f(\theta) = 0, f(-x) = f(x)$, and

(1) there are a $m$-dimensional subspace $X_1$ and a constant $\rho > 0$ such that

$$\sup_{x \in X_1 \cap S_\rho} f(x) < 0,$$

(2) there is a $j$-dimensional subspace $\tilde{E}$ such that

$$\inf_{x \in X_2^\perp} f(x) > -\infty,$$

where $X_2^\perp \subset X$ is a subset of $X$ such that $X_2^\perp \oplus X_2 = X$.

Then $f$ has at least $m - j$ pairs of critical points if $m - j > 0$.

**Proof of Theorem 3.8.** Let $X_1 := \text{span}\{e_j(A_1)\mid \lambda_j(A_1) < 0\}$, then $i(A_1) = \dim X_1$. By assumption (2), we have

$$V(t, x) = \int_0^1 V'(t, \theta x) d\theta \cdot x$$

$$\geq \frac{1}{2} A_1(t)x \cdot x + o(|x|^2) \quad \text{as } x \to 0.$$
And hence the assumption (1) of Lemma 3.9 is valid, where \( f \) is defined as in (3.23). Let \( X_2 = \text{span}\{e_j(B_2) | \lambda_j(B_2) < 0\} \), then \( i(B_2) = \dim X_2 \), and the assumption (2) of Lemma 3.9 is also satisfied. Therefore, the proof is complete. □

**Example 3.** Let \( V(t,x) = \frac{1}{2} x^t A(t)x + f(\sum_{i=1}^{n} F_i(x_i)) + g(\sum_{i=1}^{n} G_i(x_i)) \), where \( f \in C^1((0, +\infty), [0, +\infty)) \), \( f(t) = 0 \) as \( t \in [0, 1] \) and \( f(t) = t \) as \( t \in [2, +\infty) \); and \( F_i(x) = \int_0^x s f_i(s) \, ds, \) \( f_i(x) = \lambda_i \sin x_i^t + (\lambda_2 \varepsilon_2) \cos x_i^t \); \( g(x) = x \) as \( x \in [0, 1] \), \( g(x) = 0 \) as \( x \in [2, +\infty) \), \( F_i(x) = \int_0^x s \sin(s^{-1}) \, ds \). Assume \( A \in L^\infty((0,1) \times \mathbb{R}^n; GL_\mathbb{R}(\mathbb{R}^n)) \) with \( v(A) = 0 = v(A + \varepsilon_1 I_n) = v(A + \lambda I_n) = v(A + (\lambda + \varepsilon_2) I_n) \), \( i(A + \varepsilon_1 I_n) = i(A) \), \( i(A + \lambda I_n) = i(A + (\lambda + \varepsilon_2) I_n) > i(A) \). Then as \( |x| \) is small enough we have

\[
V'(t,x) = (A(t) + \varepsilon \text{diag}(|\sin(x_1^{-1})|, |\sin(x_2^{-1})|, \ldots, |\sin(x_n^{-1})|)) x
\]

and

\[
A(t) \leq A(t,x) \leq A(t) + \varepsilon_1 I_n.
\]

And as \( |x| \) is large enough we have

\[
V'(t,x) = (A(t) + \text{diag}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))) x,
\]

\[
A(t) + \lambda I_n \leq B(t,x) \leq A(t) + (\lambda + \varepsilon_2) I_n.
\]

From Theorem 3.8, (3.1) and (3.2) has \( |i(A + \lambda I_n) - i(A)| \) pairs of solutions. Note that \( V(t,x) \) is not twice differentiable, so it does not satisfy the known results.

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**References**
