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On two-path convexity in multipartite tournaments

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Abstract

In the context of two-path convexity, we study the *rank, Helly number, Radon number, Caratheodory number*, and *hull number* for multipartite tournaments. We show the maximum Caratheodory number of a multipartite tournament is 3. We then derive tight upper bounds for rank in both general multipartite tournaments and clone-free multipartite tournaments. We show that these same tight upper bounds hold for the Helly number, Radon number, and hull number. We classify all clone-free multipartite tournaments of maximum Helly number, Radon number, hull number, and rank.

1. Introduction

Convexity has been studied in many contexts. In graphs and digraphs, convex sets are usually subsets of the vertex set determined by paths within the graph. For a (directed) graph $T = (V, E)$ and a set $\mathcal{P}$ of (directed) paths in T, a subset $A \subseteq V$ is $\mathcal{P}$*-convex* if, whenever $v, w \in A$, any (directed) path in $\mathcal{P}$ that originates at v and ends at w can involve only vertices in A. For $S \subseteq V$, the *convex hull of* S, denoted $C(S)$, is defined to be the smallest convex subset containing S. We denote the set of convex subsets of T by $\mathcal{C}(T)$.

In the case $\mathcal{P}$ is the set of geodesics in T, we get *geodesic convexity*, which was introduced by F. Harary and J. Nieminen in [6] (see also [2] and [1]). When $\mathcal{P}$ is the set of all chordless paths, we get *induced path convexity* (see [4]). Other types of convexity include *path convexity* (see [8, 7]) and *triangle path convexity* (see [3]). This paper considers *two-path convexity*, in which the path set $\mathcal{P}$ is the set of all 2-paths (see also [10,5,9]).

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The most studied convex invariants are the Helly, Radon, and Caratheodory numbers (see [11, Ch. 2]). Let $T = (V, E)$ be a (directed) graph, and let $F \subseteq V$. Then F is *H-independent* if $\bigcap_{p \in F} C(F - \{p\}) = \emptyset$. The *Helly number* $h(T)$ is the size of a largest H-independent set. The set F is *C-independent* if $C(F) \not\subseteq \bigcup_{a \in F} C(F - \{a\})$. The *Caratheodory number* $c(T)$ is the size of a largest C-independent set. Equivalently, it is the smallest number c such that for every $S \subseteq V$ and $p \in C(S)$, there exists $F \subseteq S$ with $|F| \leq c$ such that $p \in C(F)$. F is *R-independent* if F does not have a *Radon partition*; that is, there is no partition $F = A \cup B$ with $C(A) \cap C(B) \neq \emptyset$. The Radon number $r(T)$ is the size of a largest R-independent set. We caution the reader that the Radon number is often defined as the smallest r for which every set of size r is R-dependent. This is one larger than the definition given here.

We say F is *convexly independent* if, for each $p \in F$, we have $p \notin C(F - \{p\})$. The rank $d(T)$ is the size of a largest convexly independent set. Any set that is H-, C-, or R-independent must also be convexly independent, so rank is an upper bound for the Helly, Caratheodory, and Radon numbers. Finally, F is a hull set if $C(F) = V$. The *hull number* $\mathrm{hul}(T)$ is the size of a smallest hull set and is also bounded by the rank.

Let $T = (V, E)$ be a digraph with vertex set V and arc set E. We denote an arc $(v, w) \in E$ by $v \to w$ and say that v dominates w. If $U, W \subseteq V$, then we write $U \to W$ to indicate that every vertex in U dominates every vertex in W. We denote by T^* the digraph with the same vertex set as T, and where (v, w) is an arc of T^* if and only if (w, v) is an arc of T. Recall that, for $p \geq 2$, T is a p-partite tournament if one can partition V into p partite sets such that every two vertices in different partite sets have precisely one arc between them and no arcs exist between vertices in the same partite set. Two vertices are clones if they have identical insets and outsets, and T is *clone-free* if it has no clones. If $u, v, w \in V$ with $u \to v \to w$, we say that v *distinguishes* the vertices u and w. Note that in a clone-free multipartite tournament, for every pair of vertices u, w in the same partite set there is at least one vertex (not in that partite set) that distinguishes u and w. If $A, B \in \mathcal{C}(T)$, we denote the convex hull of $A \cup B$ by $A \vee B$. If $v, w \in V$, we drop the set notation and write $\{v\} \vee \{w\}$ as $v \vee w$.

One can construct the convex hull of a set $U \subseteq V$ in the following way. Define $C_k(U)$ inductively by $C_0(U) = U$ and $C_k(U) = C_{k-1}(U) \cup \{w \in V : x \to w \to y \text{ for some } x, y \in C_{k-1}(U)\}$, $k \geq 1$. Then $C_\infty(U) = C(U)$.

To facilitate our study of bipartite tournaments, we introduce the following matrix notation. Let $P_1 = \{x_1, \ldots, x_k\}$ and $P_2 = \{y_1, \ldots, y_\ell\}$ be the partite sets of T. For i and j with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let $m_{i,j} = 1$ if $x_i \to y_j$ and let $m_{i,j} = 0$ otherwise. We call $M = (m_{i,j})$ the matrix of T.

2. Inequalities involving the Caratheodory number

In this section, we explore the Caratheodory number of a multipartite tournament T. The following two results show $c(T) \leq 3$.

Lemma 2.1. *Let T be a multipartite tournament. Suppose $U \subseteq V$ and $p \in C(U)$.*
(i) *There exists $F \subseteq U$, $|F| \leq 3$ with $p \in C(F)$.*
(ii) *If U lies in a single partite set of T then there exists $F \subseteq U$, $|F| \leq 2$ with $p \in C(F)$.*

Proof. If $|U| \leq 2$ or if $p \in U$, the result is trivial, so assume $|U| \geq 3$ and $p \notin U$. Since $p \in C(U)$ and $p \notin U$ then there is a smallest positive integer k such that $p \in C_k(U)$. If U does not lie in a single partite set of T, then there exist $u, v \in U$ with u and v in different partite sets. Since k is the smallest positive integer with $p \in C_k(U)$ then there exist $x_1, y_1 \in C_{k-1}(U)$ with

$x_1 \to p \to y_1$. At least one of u or v is not in the same partite set as p, so $u \to p$, $v \to p$, $p \to u$ or $p \to v$. In any case, $p \in u \vee v \vee x_1$ or $p \in u \vee v \vee y_1$ so $p \in u \vee v \vee z_1$ for some $z_1 \in C_{k-1}(U)$. Since k was minimal, $z_1 \notin C_{k-2}(U)$, and so there exist $x_2, y_2 \in C_{k-2}(U)$ with $x_2 \to z_1 \to y_2$. At least one of u or v is not in the same partite set as z_1, so $u \to z_1$, $v \to z_1$, $z_1 \to u$ or $z_1 \to v$. Thus $z_1 \in u \vee v \vee x_2$ or $z_1 \in u \vee v \vee y_2$, so $z_1 \in u \vee v \vee z_2$ for some $z_2 \in C_{k-2}(U)$. Now $p \in u \vee v \vee z_1$ implies $p \in u \vee v \vee z_2$. Continuing in this way we generate a sequence of vertices, $z_1, z_2, \ldots, z_k$ with $p \in u \vee v \vee z_i$ and $z_i \in C_{k-i}(U)$ for each i. In particular, $z_k \in C_0(U) = U$ and $p \in u \vee v \vee z_k$.

Now suppose U lies in a single partite set of T. Since $C(U) \neq U$, there exist $u_1, u_2 \in U$ and $v \in V$ such that $u_1 \to v \to u_2$. Repeat the above argument with u_1 and v to create a sequence $z_1, z_2, \ldots, z_k$ such that $z_i \in u_1 \vee v \vee z_{i+1}$ for $1 \leq i \leq k-1$, $p \in u_1 \vee v \vee z_i$ and $z_i \in C_{k-i}(U)$ for each i. Let $u_3 = z_k \in U$. Then $p \in C(\{u_1, v, u_3\}) \subseteq C(\{u_1, u_2, u_3\})$. By construction, either $u_1 \to z_{k-1} \to u_3$, $u_3 \to z_{k-1} \to u_1$, $v \to z_{k-1} \to u_3$ or $u_3 \to z_{k-1} \to v$.

Assume that $u_1 \to z_{k-1} \to u_3$. If $v \to u_3$ then $v \in u_1 \vee u_3$ and $p \in u_1 \vee u_3$, so assume $u_3 \to v$. Similarly, if $z_{k-1} \to u_2$ then $z_{k-1} \in u_1 \vee u_2$ and $p \in u_1 \vee u_2$ so assume $u_2 \to z_{k-1}$. Then $u_3 \to v \to u_2$ and $u_2 \to z_{k-1} \to u_3$ imply $v, z_{k-1} \in u_2 \vee u_3$. We next show that $z_{k-2} \in u_2 \vee u_3$. If z_{k-2} is in the same partite set as U then, by construction, either $v \to z_{k-2} \to z_{k-1}$ or $z_{k-1} \to z_{k-2} \to v$. On the other hand, if z_{k-2} is not in the same partite set as U then z_{k-2} is comparable to u_1 and u_3. If $u_1 \to z_{k-2} \to u_3$ or $u_3 \to z_{k-2} \to u_1$ then $p \in C_{k-2}(U)$ which is impossible. Thus, $u_1, u_3 \to z_{k-2}$ or $z_{k-2} \to u_1, u_3$. By construction, either $z_{k-1} \to z_{k-2} \to u_1$, $u_1 \to z_{k-2} \to z_{k-1}$, $z_{k-1} \to z_{k-2} \to v$ or $v \to z_{k-2} \to z_{k-1}$. In any case we obtain $z_{k-2} \in u_2 \vee u_3$. Continuing in this way, we obtain $p \in u_2 \vee u_3$ proving (ii). The case $u_3 \to z_{k-1} \to u_1$ is similar.

If $v \to z_{k-1} \to u_3$ then by the above argument we may assume $z_{k-1} \to u_1$. Since $v \in C(\{u_1, u_2\})$, we have $z_{k-1}, p \in C(\{u_1, u_2\})$. The case $u_3 \to z_{k-1} \to v$ is similar. □

Theorem 2.2. *Let T be a multipartite tournament. Then $c(T) \leq 3$.*

Since singleton subsets are convex, the Radon number of a multipartite tournament with $|V| \geq 2$ is at least 2. If $r(T) = 2$, then every triple $\{u, v, w\} \subseteq V$ has a Radon partition, which is, without loss of generality, $\{u, v\} \cup \{w\}$. Then $w \in u \vee v$, so $\{u, v, w\}$ is convexly dependent. Thus, $c(T) \leq d(T) = 2 = r(T)$. In general, we have the following.

Corollary 2.3. *Let T be a multipartite tournament. Then $c(T) \leq h(T) \leq r(T)$.*

Proof. Levi's inequality gives us $h(T) \leq r(T)$ (see e.g., [11, p. 169]), so we need only prove $c(T) \leq h(T)$. The case $h(T) = 1$ is trivial, and if $h(T) \geq 3$, the result follows from Theorem 2.2. If $h(T) = 2$, let $U \subseteq V$ with $p \in C(U)$. If U lies in a single partite set of T, then $p \in x \vee y$ for some $x, y \in U$ by Lemma 2.1(ii). If U does not lie in a single partite set, then we need only show that there is $F \subset U$ with $|F| = 2$ such that $U \subseteq C(F)$. By Lemma 2.1(i), we need only consider U with $|U| = 3$. Let $U = \{x, y, z\}$. If each vertex is in a different partite set, then the graph induced by U has a two-path, and we let F be the set of the two endpoints of this two-path. If the vertices lie in two different partite sets, we assume without loss of generality that x and y lie in the same partite set. Thus, $x \vee z = \{x, z\}$ and $y \vee z = \{y, z\}$. Since $h(T) = 2$, $(x \vee z) \cap (y \vee z) \cap (x \vee y) \neq \emptyset$, implying that $z \in x \vee y$. This completes the proof. □

Note that $c(T) = 1$ precisely when all subsets of V are convex. This occurs when T is bipartite and every vertex in one partite set dominates all the vertices in the other partite set. The following helps identify bipartite tournaments of Caratheodory number 3.

Theorem 2.4. *Let T be a bipartite tournament with Caratheodory number 3. Then there exist $a, \overline{a}, b_i, \overline{b_i} \in \{0, 1\}$ with $a \neq \overline{a}$, $b_i \neq \overline{b_i}$ such that T has an induced bipartite subtournament with one of the following matrices.*

$$\begin{bmatrix} a & a & \overline{a} & a & a & a & \cdots & a \\ b_0 & b_0 & b_1 & b_3 & b_5 & b_7 & \cdots & b_{2m-1} \\ b_2 & b_2 & \overline{b}_1 & \overline{b}_2 & b_2 & b_2 & \cdots & b_2 \\ b_4 & b_4 & b_1 & \overline{b}_3 & \overline{b}_4 & b_4 & \cdots & b_4 \\ b_6 & b_6 & b_1 & b_3 & \overline{b}_5 & \overline{b}_6 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & b_{2m-4} \\ b_{2m-2} & b_{2m-2} & b_1 & b_3 & b_5 & \ddots & \ddots & \overline{b}_{2m-2} \\ b_{2m} & \overline{b}_{2m} & b_1 & b_3 & b_5 & \cdots & b_{2m-3} & \overline{b}_{2m-1} \end{bmatrix},$$

$$\begin{bmatrix} a & \overline{a} & a & a & \cdots & a & a \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & b_{2m+1} \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & \overline{b}_{2m+1} \\ b_2 & \overline{b}_1 & \overline{b}_2 & b_2 & \cdots & b_2 & b_2 \\ b_4 & b_1 & \overline{b}_3 & \overline{b}_4 & b_4 & \cdots & b_4 \\ b_6 & b_1 & b_3 & \overline{b}_5 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \overline{b}_{2m-2} & b_{2m-2} \\ b_{2m} & b_1 & \cdots & \cdots & b_{2m-3} & \overline{b}_{2m-1} & \overline{b}_{2m} \end{bmatrix}.$$

Proof. Since $c(T) = 3$, there must exist a set $U = \{u_1, u_2, u_3\}$ and $p \in C(U)$ with u_1, u_2 in the same partite set and $p \notin u_1 \vee u_2$. If $p = z_0$ is in the same partite set as u_3, then, as in the proof of Lemma 2.1, there exist vertices $z_1, \cdots, z_{2m}$ such that z_i distinguishes u_1 and z_{i+1} if i is even, z_i distinguishes u_3 and z_{i+1} if i is odd, and z_{2m} distinguishes u_1 and u_2. Let m be minimal with this property. We order the rows and columns of the matrix of T as follows. We let z_0 be the first row, u_3 the second row, with the remaining rows $z_2, z_4, \ldots, z_{2m}$. The first column is u_1, the second column is u_2, and the remaining columns are $z_1, z_3, \ldots, z_{2m-1}$. Denote the matrix $M = [a_{ij}]$.

Let $a = a_{11}$, $b_{2(k-2)} = a_{k1}$ for each $2 \leq k \leq m + 2$, and $b_{2(\ell-3)+1} = a_{2\ell}$ for each $3 \leq \ell \leq m + 2$. By the arcs already given, we have $a_{13} = \overline{a}$, $a_{ss} = \overline{b}_{2s-5}$, $a_{t(t+1)} = \overline{b}_{2t-4}$, and $a_{(2m+2)2} = \overline{b}_{2m}$, where $3 \leq s \leq m + 2$ and $3 \leq t \leq m + 1$. If u_1 and u_2 were to distinguish any vertex represented by a row of M besides z_{2m}, then either $p \in u_1 \vee u_2$ (if $a_{12} = \overline{a}$ or $a_{22} = \overline{b}_0$) or the minimality of m is violated. Thus, $a_{12} = a$ and $a_{r2} = b_{2(r-2)}$ for all $2 \leq r \leq m + 1$. Also, if any z_i is distinguished by some u_j and z_k, where $i < k$, then the minimality of m is violated. This determines the rest of the entries of M, and gives us the result. The case of z_0 in the same partite set as u_1 and u_2 is similar. □

3. Convex independence in multipartite tournaments

Since rank is an upper bound for the Helly, Radon, and hull numbers, it is helpful to better understand convexly independent sets.

Lemma 3.1. *Let T be a multipartite tournament, and suppose A is a convexly independent set.*

(i) *Let P_1 and P_2 be partite sets of T whose intersection with A is nonempty. Then either $(A \cap P_1) \to (A \cap P_2)$ or $(A \cap P_2) \to (A \cap P_1)$.*
(ii) *A has a nonempty intersection with at most 2 partite sets of T.*

Proof. For (i), let $x \in A \cap P_1$ and $y \in A \cap P_2$. Without loss of generality, assume $x \to y$. Suppose $x' \in A \cap P_1$ and $y' \in A \cap P_2$ with $y' \to x'$. If $x \to y'$, we have $x \to y' \to x'$, which contradicts convex independence. The case $y' \to x$ is similar, so $(A \cap P_1) \to (A \cap P_2)$.

For (ii), any three vertices in distinct partite sets must have a 2-path between two of the vertices, which makes them convexly dependent. □

We then say that A and B *form a convexly independent set* if $A \cup B$ is convexly independent and A and B are in distinct partite sets. The following is immediate.

Corollary 3.2. *Let T be a tournament, $|V| \geq 2$. Then $d(T) = 2$.*

For a general multipartite tournament T, a trivial upper bound for $d(T)$ is $|V|$ and this bound is achieved precisely when T is bipartite and one partite set dominates the other. In this case, any two vertices in the same partite set are clones. In a clone-free multipartite tournament, every pair of vertices in a given partite set is distinguished by at least one other vertex. We are interested in the vertices that distinguish pairs of vertices in convexly independent sets. Given $A \subseteq V$, define

$$D_A^{\rightarrow} = \{z \in V : z \to x \text{ for some } x \in A, y \to z \text{ for all } y \in A - \{x\}\}$$
$$D_A^{\leftarrow} = \{z \in V : z \leftarrow x \text{ for some } x \in A, z \to y \text{ for all } y \in A - \{x\}\}.$$

Lemma 3.3. *Let A and B form a convexly independent set in a multipartite tournament T, and in the case $B \neq \emptyset$ suppose $A \to B$.*
(i) *If $|A| \geq 3$, then each of $D_A^{\rightarrow}$ and $D_A^{\leftarrow}$ intersects at most one partite set nontrivially.*
(ii) *If $|A| \geq 2$ and $B \neq \emptyset$, then $D_A^{\rightarrow}$ is a subset of the same partite set as B. If $|B| \geq 2$ and $A \neq \emptyset$, then $D_B^{\leftarrow}$ is a subset of the same partite set as A.*
(iii) *If $|A|, |B| \geq 2$, then $D_B^{\leftarrow} \to D_A^{\rightarrow}$.*

Proof. For (i), we prove the result for $D_A^{\rightarrow}$. The case of $D_A^{\leftarrow}$ is similar. Suppose $z_1, z_2 \in D_A^{\rightarrow}$ with $z_1 \to z_2$. Then there exist $x_1, x_2 \in A$ with $z_1 \to x_1$ and $z_2 \to x_2$. Since $|A| \geq 3$, there exists $x_3 \in A - \{x_1, x_2\}$. We have $x_3 \to z_2 \to x_2$, giving us $z_2 \in x_2 \vee x_3$. Similarly, $x_3 \to z_1 \to z_2$, and so $z_1 \in x_2 \vee x_3$. But $z_1 \to x_1 \to z_2$, so $x_1 \in x_2 \vee x_3$, a contradiction.

For (ii), suppose that $z \in D_A^{\rightarrow}$ is not in the same partite set as B. Clearly, z is also not in the same partite set as A. Since $|A| \geq 2$, there exist $x_1, x_2 \in A$ with $x_1 \to z \to x_2$. Let $y \in B$. If $z \to y$, then $x_1 \to z \to y$ and $z \to x_2 \to y$ imply $x_2 \in x_1 \vee y$, which contradicts convex independence. If instead $y \to z$, we have $z \in x_1 \vee x_2$, and so $x_2 \to y \to z$ implies $y \in x_1 \vee x_2$, which contradicts convex independence. This implies z and y are incomparable and are thus in the same partite set. The argument for $D_B^{\leftarrow}$ is similar.

For (iii), suppose $z_1 \in D_A^{\rightarrow}$, $z_2 \in D_B^{\leftarrow}$ with $z_1 \to z_2$. Since $|A|, |B| \geq 2$, then there exist $x_1, x_2 \in A$, $y_1, y_2 \in B$ with $x_1 \to z_1 \to x_2$ and $y_1 \to z_2 \to y_2$. Then $z_2 \in y_1 \vee y_2$. We get $x_1 \to z_1 \to z_2$ and $z_1 \to x_2 \to y_1$, which implies $x_2 \in y_1 \vee y_2 \vee x_1$, a contradiction. □

We now explore lower bounds on $|D_A^{\rightarrow}|$ and $|D_B^{\leftarrow}|$.

Theorem 3.4. *Let T be a clone-free multipartite tournament, and suppose A is a convexly independent subset of a partite set. If $A = \{x_1, x_2, \ldots, x_r\}$, one can order the elements in A such that either there exist $y_2, \ldots, y_r \in D_A^{\rightarrow}$ with $y_i \to x_i$ or there exist $y_2, \ldots, y_r \in D_A^{\leftarrow}$ with $x_i \to y_i$.*

Proof. If we look at A as a set of vertices in both T and T^*, then $D_A^{\leftarrow}$ in T is the same set as $D_A^{\rightarrow}$ in T^*. Thus, we need only show the result in either T or T^*.

The case $r = 1$ is trivial. If $r = 2$, let y_2 be any vertex distinguishing x_1 and x_2. By relabelling x_1 and x_2, if necessary, we have $x_1 \to y_2 \to x_2$. If $r = 3$, let y_2 distinguish x_1 and x_2. By relabelling and considering T^*, if necessary, we may assume $x_1 \to y_2 \to x_2$, and that $x_3 \to y_2$. Since T is clone-free, there is some y_3 that distinguishes x_1 and x_3. By switching x_1 and x_3 if necessary, we may assume that $x_1 \to y_3 \to x_3$. It suffices to show that $x_2 \to y_3$ If $y_3 \to x_2$, then $x_1 \to y_2 \to x_2$ and $x_1 \to y_3 \to x_2$, so $y_2, y_3 \in x_1 \vee x_2$. But then $y_3 \to x_3 \to y_2$, so $x_3 \in x_1 \vee x_2$, a contradiction.

Inducting on m, assume the result for $r = m \geq 3$. For $r = m + 1$, there exist $y_2, \ldots, y_m$ such that $y_i \to x_i$ for all $2 \leq i \leq m$ and $x_i \to y_j$ for all $i \neq j$. Clearly, $x_i \vee x_j = y_i \vee y_j$ for all $2 \leq i \neq j \leq m$. For a contradiction, suppose $y_i \to x_{m+1}$ for some $i \leq m$. If $x_{m+1} \to y_j$, then $x_{m+1} \in y_i \vee y_j = x_i \vee x_j$, which contradicts convex independence. Thus, $y_i \to x_{m+1}$ for all $i \leq m$. Since $m \geq 3$, $y_2, y_3 \to x_{m+1}$. We have $x_1 \to \{y_2, y_3\} \to x_{m+1}$, and so $x_2 \vee x_3 = y_2 \vee y_3 \subseteq x_1 \vee x_{m+1}$, a contradiction. Thus, $x_{m+1} \to y_i$ for all $i \leq m$. Now let y_{m+1} be a vertex distinguishing x_1 and x_{m+1}. By switching x_1 and x_{m+1}, if necessary, we can assume that $x_1 \to y_{m+1} \to x_{m+1}$. Arguments similar to the $r = 3$ case give us $x_i \to y_{m+1}$ for $2 \leq i \leq m$, which completes the proof. $\square$

The following lemma shows that these distinguishing sets contain all vertices that distinguish vertices in A and B.

Lemma 3.5. *Suppose A and B form a convexly independent set, with $A \to B$ when $A, B \neq \emptyset$.*
(i) *If $|A| \geq 3$, then either $D_A^{\rightarrow} = \emptyset$ or $D_A^{\leftarrow} = \emptyset$. Moreover, any vertex that distinguishes two vertices in A must be in $D_A^{\rightarrow} \cup D_A^{\leftarrow}$.*
(ii) *If $|A| \geq 2$ and $B \neq \emptyset$, then any vertex that distinguishes two vertices in A is in $D_A^{\rightarrow}$.*
(iii) *If $A \neq \emptyset$ and $|B| \geq 2$, then any vertex that distinguishes two vertices in B must be in $D_B^{\leftarrow}$.*

Proof. For (i), let $u \in D_A^{\rightarrow}$, $v \in D_A^{\leftarrow}$. Let $x_1, x_2 \in A$ with $u \to x_1$ and $x_2 \to v$. Then $A - \{x_1\} \to u$ and $v \to A - \{x_2\}$. In the case $x_1 = x_2$, let $x_3, x_4 \in A - \{x_1\}$. In the case $x_1 \neq x_2$, let $x_3 = x_2$, and let $x_4 \in A - \{x_1, x_3\}$. In either case, $u, v \in x_1 \vee x_3$. Then $v \to x_4 \to u$ implies $x_4 \in x_1 \vee x_3$, a contradiction.

For (ii), let $x, y \in A$, $z \in V$ with $x \to z \to y$, and let $w \in B$. Then $z \in x \vee y$. If $z \notin D_A^{\rightarrow}$ then there exists $v \in A - \{y\}$ with $z \to v$. Then $z \to v \to w$ implies $v \in x \vee y \vee w$, which contradicts convex independence. Thus, $z \in D_A^{\rightarrow}$. Part (iii) follows similarly. $\square$

An immediate extension of the lemma is

Corollary 3.6. *Suppose A and B form a convexly independent set, and $A \to B$.*
(i) *If $|A| \geq 3$ and $B \neq \emptyset$ then $D_A^{\leftarrow} = \emptyset$.*
(ii) *If $|B| \geq 3$ and $A \neq \emptyset$ then $D_B^{\rightarrow} = \emptyset$.*

When T is clone-free, Corollary 3.6 and Theorem 3.4 give us the following.

Corollary 3.7. *Suppose that $A \neq \emptyset$ and $B \neq \emptyset$ form a convexly independent set and that $A \to B$. Then $|D_A^{\rightarrow}| \geq |A| - 1$ and $|D_B^{\leftarrow}| \geq |B| - 1$.*

Corollary 3.7 leads to our main theorem of the section.

Theorem 3.8. *Let T be a clone-free multipartite tournament. Then*
(i) *$d(T)$ is at most one greater than the order of the second largest partite set in T.*
(ii) *$d(T) \leq \lfloor \frac{|V|}{2} + 1 \rfloor$.*

Proof. Let A and B form a maximum convexly independent set of T with $A \to B$. Also, let $P_1 \supseteq A$ and $P_2 \supseteq B$ be partite sets. For (i), if A and B are both nonempty, then Lemma 3.3(ii) gives us $D_A^{\to} \subseteq P_2$ and $D_B^{\leftarrow} \subseteq P_1$. We then have $|P_1| \geq |A|+|D_B^{\leftarrow}| \geq |A|+|B|-1 = d(T)-1$. Thus, $d(T) \leq |P_1|+1$. Similarly, $d(T) \leq |P_2|+1$. If $B = \emptyset$, the case $d(T) = 1$ or $d(T) = 2$ is clear. If $d(T) \geq 3$, then Lemma 3.3(i) implies each of $D_A^{\to}$ and $D_A^{\leftarrow}$ lies in one partite set. Theorem 3.4 implies that $|D_A^{\to}| \geq |A|-1$ or $|D_A^{\leftarrow}| \geq |A|-1$. In either case, some partite set $P_2 \neq P_1$ has at least $|A|-1$ elements. Then $d(T) = |A| \leq |P_1|$ and $d(T) = |A| \leq |P_2|+1$. For (ii), the second largest partite set of T has at most $\frac{|V|}{2}$ vertices so $d(T) \leq \frac{|V|}{2}+1$ by (i). □

Corollary 3.9. *Let T be a clone-free multipartite tournament, and let A and B form a maximum convexly independent set of T. Then*
(i) *If $d(T) = \lfloor \frac{|V|}{2}+1 \rfloor$, and if B is empty, then $|V|$ is odd.*
(ii) *Every convex subset of T is the convex hull of at most $\lfloor \frac{|V|}{2}+1 \rfloor$ vertices.*

Proof. For (i), we have $|D_A^{\to} \cup D_A^{\leftarrow}| \geq |A|-1$ by Theorem 3.4. We then have $|V| \geq |A|+|A|-1 = 2d(T)-1$. This gives us $d(T) \leq \frac{|V|+1}{2}$. But this can happen only if $|V|$ is odd. Part (ii) follows from Theorem 3.8(ii) and the definition of rank. □

Since rank is an upper bound for the Helly, Radon, and Caratheodory numbers, we have the following.

Corollary 3.10. *Let T be a clone-free multipartite tournament. Then*
(i) *$h(T)$, $r(T)$, and* $\mathrm{hul}(T)$ *are at most one larger than the cardinality of the second largest partite set.*
(ii) $h(T), r(T), \mathrm{hul}(T) \leq \lfloor \frac{|V|}{2}+1 \rfloor$.

We then say that a clone-free multipartite tournament T has *maximum rank* (resp. *maximum Helly number, maximum Radon number, maximum hull number*) if the rank (resp. the Helly number, Radon number, hull number) is $\lfloor \frac{|V|}{2}+1 \rfloor$.

4. Classifying clone-free multipartite tournaments with maximum convexity numbers

Let T be a clone-free multipartite tournament. As before, let A and B form a convexly independent set of T with $A \to B$, and let $d = d(T)$. We begin with some examples of clone-free multipartite tournaments with maximum convexity numbers.

Example 4.1. Tournaments have rank at most 2, so all tournaments with $|V| \leq 3$ have maximum rank, Helly number, Radon number, and hull number. This includes C_3, the cyclic tournament on three vertices.

Example 4.2. Let B_{2d-1} have partite sets $P_1 = \{x_1, \ldots, x_d\}$, $P_2 = \{y_2, \ldots, y_d\}$, with $y_i \to x_i$ for all $2 \leq i \leq b$ and $x_i \to y_j$ otherwise. Since P_1 is H-, R-, and convexly independent, B_{2d-1} has maximum rank, Helly number, and Radon number. Every hull set must include x_1 and either x_i or y_i for $2 \leq i \leq d$, so B_{2d-1} has maximum hull number.

Let B'_{2d-1} have partite sets $P_1 = \{z, x_1, \ldots, x_{d-1}\}$ and $P_2 = \{y_1, \ldots, y_{d-1}\}$, with $P_2 \to z$, $y_i \to x_i$ for $i \geq 2$, and $x_i \to y_j$ otherwise. Then $\{x_1, \ldots, x_{d-1}, y_1\}$ is H-, R-, and convexly independent, and $x_1 \vee z = V$, so $h(B'_{2d-1}) = r(B'_{2d-1}) = d(B'_{2d-1}) = d$ and $\mathrm{hul}(B'_{2d-1}) = 2$.

Let B_{2d-2} have partite sets $P_1 = \{x_1, \ldots, x_{d-1}\}$ and $P_2 = \{y_1, y_2, \ldots, y_{d-1}\}$, with $y_i \to x_i$ for all $i \geq 2$, and $x_i \to y_j$ otherwise. Then $\{x_1, \ldots, x_{d-1}, y_1\}$ is H-, R-, and convexly independent, so B_{2d-2} has maximum rank, Helly number, and Radon number. Moreover, B_{2d-2} has maximum hull number, and $B_{2d-2} \cong B^*_{2d-2}$.

Example 4.3. Let $T_{2d-1} = B_{2d-2} \cup \{z\}$, where $z \to B_{2d-2}$, and let $T'_{2d-1} = B_{2d-2} \cup \{z\}$, where $P_1 \to z \to P_2$. The maximum convexly independent sets of B_{2d-2} are maximum convexly independent in T_{2d-1} and T'_{2d-1}, so both are of maximum rank. In T_{2d-1}, the maximum convexly independent sets are also H- and R-independent, so T_{2d-1} has maximum Helly and Radon number. However, every convex subset of T'_{2d-1} with more than one vertex contains z, so $h(T'_{2d-1}) = 2$. Once can show that $r(T'_3) = 2$ and $r(T'_{2d-1}) = 3$ for $d \geq 3$. Moreover, T'_{2d-1} has maximum hull number, and $T'_{2d-1} \cong (T'_{2d-1})^*$.

A final example is $T''_5 = B_4 \cup \{z\}$ where $z \to P_1$, $y_2 \to z$, and $z \to y_1$, which has maximum rank, Helly number, Radon number, and hull number.

We begin our classification with the case of $B = \emptyset$.

Theorem 4.4. *Let T be a clone-free multipartite tournament of maximum rank, and let A and $B = \emptyset$ form a maximum convexly independent set. Then $T \cong B_{2d-1}$ or $T \cong B^*_{2d-1}$.*

Proof. By Corollary 3.9(i), n must be odd. Let $A = \{x_1, \ldots, x_d\}$. Theorem 3.4 implies that, by reordering the x_i's and looking at T^* if necessary, there exists $C = \{y_2, \ldots, y_d\} \subseteq D^{\to}_A$ such that $y_i \to x_i$. Furthermore, we have that $y_2, \ldots, y_d$ are all in the same partite set (if $d = 2$, it follows trivially; the $d \geq 3$ case follows from Lemma 3.3(i). Since $n = 2d - 1$, $V = A \cup C$, and so $T \cong B_{2d-1}$ (or, if we had to take T^* to get the y_i, then $T \cong B^*_{2d-1}$). □

We now pursue the case of $A, B \neq \emptyset$. We first consider B_{2d-2}.

Lemma 4.5. *Suppose that A and B form a maximum convexly independent set of B_{2d-2}. Let the $x_i, y_j \in B_{2d-2}$ be as in Example 4.2.*
(i) *For all $i \geq 2$, we cannot have both $x_i \in A$ and $y_i \in B$*
(ii) *If $A \to B$, then $x_1 \in A$ and $y_1 \in B$.*

Proof. For (i), suppose $x_i \in A$, $y_i \in B$. Since $i \geq 2$, we have $d \geq 3$, so $|A| \geq 2$ or $|B| \geq 2$. If $|A| \geq 2$, we have some $x_j \in A$, $j \neq i$. Thus, $x_j \to y_i \to x_i$, contradicting convex independence. The case $|B| \geq 2$ follows similarly.

For (ii), the case of $d = 2$ is obvious. For $d \geq 3$, suppose $x_1 \notin A$. Since each y_i dominates at most one x_j, we have $A \subseteq P_1$ and $B \subseteq P_2$. Let $r = |A|$, so $|B| = d - r$. We have $d - r - 2$ vertices among $x_2, \ldots, x_{d-1}$ that are not in A, and one of the vertices in B can be y_1, which leaves at least $d - r - 1$ vertices to be chosen from $y_2, \ldots, y_{d-1}$. But there are only $d - r - 2 y_i$'s for which $x_i \notin A$. Thus, $y_i \in B$ and $x_i \in A$ for some $i \geq 2$, which contradicts (i). The case $y_1 \in B$ follows similarly. □

We consider the cases of $|V|$ even and $|V|$ odd separately.

Lemma 4.6. *Let T be a clone-free multipartite tournament of maximum rank.*
(i) *If $|V|$ is even, then $V = A \cup B \cup D^{\to}_A \cup D^{\leftarrow}_B$, $|D^{\to}_A| = |A| - 1$, and $|D^{\leftarrow}_B| = |B| - 1$.*
(ii) *If $|V|$ is odd and $V \neq A \cup B \cup D^{\to}_A \cup D^{\leftarrow}_B$, then there exists a unique $z \notin A \cup B \cup D^{\to}_A \cup D^{\leftarrow}_B$.*

Proof. If $|V|$ is even, we have $|V| = 2d - 2$. Using Corollary 3.7, we obtain

$$\begin{aligned} |V| &\geq |A| + |B| + |D^{\to}_A| + |D^{\leftarrow}_B| \\ &\geq |A| + |B| + (|A| - 1) + (|B| - 1) = 2d - 2 = |V| \end{aligned}$$

so all inequalities must be equalities, and (i) follows. If $|V|$ is odd, we still have $|A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}| \geq 2d - 2$. This leaves one other possible vertex z, which proves (ii). □

Theorem 4.7. *If T is a clone-free multipartite tournament of maximum rank, and if $|V| = 2d - 2$, then $T \cong B_{2d-2}$.*

Proof. The case of $|V| = 2$ is obvious, so we can assume $|V| \geq 4$ and $d \geq 3$. Since $B_{2d-2} \cong B_{2d-2}^*$, we consider T^* if necessary. Let $A = \{x_1, x_2, \ldots, x_r\}$ and $B = \{y_1, y_2, \ldots, y_s\}$. Without loss of generality, $r \geq 2, s \geq 1$ and $r + s = d$. If $s \geq 2$, then by Lemma 3.3(ii), Theorem 3.4, and Lemma 4.6(i), $D_A^{\rightarrow} = \{z_2, \ldots, z_r\} \subseteq P_2$ and $D_B^{\leftarrow} = \{w_2, \ldots, w_s\} \subseteq P_1$. We can fill in the arcs using Lemma 3.3, which gives us $T \cong B_{2b-2}$. If $s = 1$ we have $B = \{y_1\}$ and $D_B^{\leftarrow} = \emptyset$. We similarly conclude that $T \cong B_{2d-2}$. □

The final case is $|V| = 2d - 1$. By Lemma 4.6(ii), there is at most one vertex $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$. When such a z exists, let T' be the subtournament induced by $V' = V - \{z\}$. Then A and B form a maximum convexly independent set of T', so $d(T') = d$, and $T' \cong B_{2d-2}$ by Theorem 4.7. Thus, T has at least two partite sets $P_1 \supseteq \{x_1, \ldots, x_{d-1}\}$ and $P_2 \supseteq \{y_1, \ldots, y_{d-1}\}$ with arcs as in Example 4.2, and $V - V' = \{z\}$.

Lemma 4.8. *Let T be a clone-free multipartite tournament with $d(T) = d \geq 3$ and $|V| = 2d - 1$. Let P_1 and P_2 be partite sets of T, and let A and B form a convexly independent set with $A \subseteq P_1$, $B \subseteq P_2$ as above. Finally, assume that $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$.*
(i) *If $z \notin P_2$, then $z \to B$ or $B \to z$.*
(ii) *If $z \notin P_1$, then $z \to A$ or $A \to z$.*
(iii) *If $z \notin P_1 \cup P_2$, then we cannot have $B \to z \to A$.*
(iv) *If $z \notin P_2$, then either $z \to P_2$, $P_2 \to z$, or there is a unique $u \in P_2$ with $u \to z$.*
(v) *If $z \notin P_1$, then either $z \to P_1$, $P_1 \to z$, or there is a unique $u \in P_1$ with $z \to u$.*

Proof. For (i), if it were not true that $z \to B$ or $B \to z$, then Lemma 3.5(iii) would imply $z \in D_B^{\leftarrow}$, a contradiction. Part (ii) follows similarly.

For (iii), since $d \geq 3$, $|A| \geq 2$ or $|B| \geq 2$. If $u, v \in A$, and $B \to z \to A$, then $B \to z \to u$ and $z \to v \to B$, so $v \in B \vee u$, a contradiction. The case $|B| \geq 2$ follows similarly.

For (iv), suppose it is not true that $z \to P_2$ or $P_2 \to z$. Then there exist $u, v \in P_2$ with $u \to z \to v$. For contradiction, assume that there is some $w \in P_2 - \{u\}$ with $w \to z$. In the case $z \to B$, we have $u, w \in P_2 - B = D_A^{\rightarrow}$, and without loss of generality, $v \in B$. Then there exist $x_u, x_w \in A$ with $u \to x_u$ and $w \to x_w$. By Lemma 4.5, $x_1 \in A$. Thus, $x_1 \to u \to x_u$ and $u \to z \to v$, so $z \in x_1 \vee x_u \vee v$. But then $x_1 \to w \to z$ and $w \to x_w \to v$, which implies $x_w \in x_1 \vee x_u \vee v$, a contradiction. In the case $B \to z$, we have $v \in P_2 - B = D_A^{\rightarrow}$ and without loss of generality $u \in B$. Let $x_v \in A$ with $v \to x_v$. If $w \in B$, then $x_1 \to v \to x_v$ and $u \to z \to v$, so $z \in x_1 \vee x_v \vee u$. But then $x_1 \to w \to z$, so $w \in x_1 \vee x_v \vee u$, a contradiction. If $w \in P_2 - B = D_A^{\rightarrow}$, let $x_w \in A$ with $w \to x_w$. We have $x_1 \to v \to x_v$ and $u \to z \to v$, so $z \in x_1 \vee x_v \vee u$. But then $x_1 \to w \to z$ and $w \to x_w \to u$, so $x_w \in x_1 \vee x_v \vee u$, a contradiction. Thus, $z \to P_2 - \{u\}$.

In either case, $u \to z$ for precisely one $u \in P_2$, and (iv) follows. Part (v) is similar. □

Corollary 4.9. *Let T be a clone-free bipartite tournament with $d(T) = d$, $|V| = 2d - 1$. Then T is isomorphic to either B_{2d-1}, B_{2d-1}^*, B'_{2d-1}, or $(B'_{2d-1})^*$.*

Proof. For the case $V = A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$, since $|V| = 2d - 1$, Corollary 3.7 implies that $|D_A^{\rightarrow}| = |A|$ or $|D_B^{\leftarrow}| = |B|$. In the first case, $T \cong B^*_{2d-1}$, and in the second, $T \cong B_{2d-1}$.

In the case $V \neq A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$, we have a unique $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$ by Lemma 4.6(ii). If $z \in P_1$, then we cannot have $z \rightarrow P_2$ because z and x_1 would be clones. If $P_2 \rightarrow z$, then $T \cong B'_{2d-1}$. Otherwise $v \rightarrow z$ for precisely one $v \in P_2$ by Lemma 4.8(iv). We cannot have $v = y_i$ for $i \geq 2$ because z would be a clone of x_i. Thus, $v = y_1$ and $z \in D_B^{\leftarrow}$, a contradiction. Arguments are similar if $z \in P_2$, where we get $T \cong (B'_{2d-1})^*$. □

This brings us to the main theorem.

Theorem 4.10. *Let T be a clone-free multipartite tournament with $d(T) = \lfloor \frac{|V|}{2} + 1 \rfloor$. Then T is isomorphic to one of B_{2d-2}, B_{2d-1}, B^*_{2d-1}, B'_{2d-1}, $(B'_{2d-1})^*$, T_{2d-1}, T^*_{2d-1}, T'_{2d-1}, T''_5, $(T''_5)^*$, or C_3.*

Proof. We have already proven the case where T is bipartite. Since A, B, $D_A^{\rightarrow}$, and $D_B^{\leftarrow}$ are all contained in two partite sets, and there is at most one other vertex, only the case of T tripartite remains. In this case, $|V|$ is odd, the third partite set is $P_3 = \{z\}$, and the bipartite tournament induced by $V - \{z\}$ is isomorphic to B_{2d-2}. Thus, the other partite sets are $P_1 = \{x_1, \ldots, x_{d-1}\}$ and $P_2 = \{y_1, \ldots, y_{d-1}\}$ with $y_i \rightarrow x_i$ for $i \geq 2$ and $x_i \rightarrow y_j$ otherwise. By Lemma 4.5, $x_1 \in A$ and $y_1 \in B$.

Suppose T is not isomorphic to any of T_{2d-1}, T^*_{2d-1}, or T'_{2d-1}. By Lemma 4.8(iii), we cannot have $P_2 \rightarrow z \rightarrow P_1$ unless $d = 2$. In this case, $|V| = 3$ and so $T \cong C_3$. Thus, we can assume $d \geq 3$. By Lemma 4.8(iv), (v), either there exists a unique $v \in P_2$ with $v \rightarrow z$ or there exists a unique $v \in P_1$ with $z \rightarrow v$. In the first case, we assume, for contradiction that $v \in B$. Then $B \rightarrow z$, so $B = \{v\}$. Thus, $|A| \geq 2$, and there is some $u \in D_A^{\rightarrow}$. Let $x_u \in A$ with $u \rightarrow x_u$. If $A \rightarrow z$, then $x_1 \rightarrow u \rightarrow x_u$ and $x_u \rightarrow z \rightarrow u$, so $z \in x_1 \vee x_u$. But then $x_1 \rightarrow v \rightarrow z$, so $v \in x_1 \vee x_u$, a contradiction. Thus, $z \rightarrow A$ by Lemma 4.8(ii). But then $B \rightarrow z \rightarrow A$, contradicting Lemma 4.8(iii). This leaves us with $v \in P_2 - B = D_A^{\rightarrow}$. Since $D_A^{\rightarrow} \neq \emptyset$, this implies that $|A| \geq 2$.

Let $x_v \in A$ with $v \rightarrow x_v$. As before, either $A \rightarrow z$ or $z \rightarrow A$. If $A \rightarrow z$, then $z \rightarrow B$ implies $x_1 \rightarrow z \rightarrow y_1$ and $x_1 \rightarrow v \rightarrow z$, so $v \in x_1 \vee y_1$. But then $v \rightarrow x_v \rightarrow y_1$, so $x_v \in x_1 \vee y_1$, a contradiction. Thus, $z \rightarrow A$. Since $|A| \geq 2$ and $z \rightarrow A$, Lemma 4.8(v), implies that $z \rightarrow P_1$.

We now claim that $|A| = 2$. Suppose that $|A| \geq 3$, and let $x \in A - \{x_1, x_v\}$. Then $x_1 \rightarrow v \rightarrow x_v$, $v \rightarrow z \rightarrow x_1$, and $z \rightarrow x \rightarrow y_1$ imply $x \in x_1 \vee x_v \vee y_1$, a contradiction. Thus, $|A| = 2$. This along with Lemma 4.8(v), imply $z \rightarrow P_1$.

Suppose $|B| \geq 2$. Then there exists $y \in B - \{y_1\}$ and an $x_y \in D_B^{\leftarrow}$ such that $y \rightarrow x_y$. As above $z \in x_1 \vee x_v$. Then $z \rightarrow x_y \rightarrow y_1$ and $z \rightarrow y \rightarrow x_y$, so $y \in x_1 \vee x_v \vee y_1$, a contradiction. Thus $d = 3$ and $|V| = 5$, so $T \cong T''_5$.

In the case of a unique $v \in P_1$ with $z \rightarrow v$, apply the above to T^*. Then $T^* \cong T''_5$, and so $T \cong (T''_5)^*$. □

We get a similar result for the, Helly, Radon, and hull numbers.

Theorem 4.11. *Let T be a clone-free multipartite tournament with n vertices.*
(i) *If $h(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then T or T^* is isomorphic to B_{2d-1}, B'_{2d-1}, B_{2d-2}, T_{2d-1}, T''_5 or C_3.*
(ii) *If $r(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then one of T or T^* is isomorphic to B_{2d-1}, B'_{2d-1}, B_{2d-2}, T_{2d-1}, T'_5, T''_5 or C_3.*
(iii) *If $\operatorname{hul}(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then T or T^* is isomorphic to B_{2d-1}, B'_3, B_{2d-2}, T'_{2d-1} or C_3.*

References

[1] G. Chartrand, A. Chichisan, P. Zhang, On convexity in graphs, Congr. Numer. 148 (2001) 33–41.

[2] G. Chartrand, J.F. Fink, P. Zhang, Convexity in oriented graphs, Discrete Appl. Math. 116 (2002) 115–126.

[3] M. Changat, J. Mathew, On triangle path convexity in graphs, Discrete Math. 206 (1999) 91–95.

[4] P. Duchet, Convexity in graphs II. Minimal path convexity, J. Combin. Theory Ser. B 44 (1988) 307–316.

[5] P. Erdös, E. Fried, A. Hajnal, E.C. Milner, Some remarks on simple tournaments, Algebra Univers. 2 (1972) 238–245.

[6] F. Harary, J. Nieminen, Convexity in graphs, J. Differential Geom. 16 (1981) 185–190.

[7] J. Nieminen, On path- and geodesic-convexity in digraphs, Glas. Mat. 16 (1981) 193–197.

[8] J.L. Pfaltz, Convexity in directed graphs, J. Combin. Theory 10 (1971) 143–152.

[9] D.B. Parker, R.F. Westhoff, M.J. Wolf, Two-path convexity in clone-free regular multipartite tournaments, Australas. J. Combin. 36 (2006) 177–196.

[10] J.C. Varlet, Convexity in tournaments, Bull. Soc. Roy. Sci. Liege 45 (1976) 570–586.

[11] M.L.J. van de Vel, Theory of Convex Structures, North Holland, Amsterdam, 1993.