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Some subsets of points in the plane associated to truncated Reed–Muller codes with good parameters

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Abstract

We gives some examples of subsets of points in the projective plane associated to truncated generalized projective Reed–Muller codes with good parameters, of dimensions 6 and 10 over $GF(7)$, $GF(8)$ and $GF(9)$.

Keywords: Arcs in finite plane; Bézout's theorem; Generalized Reed–Muller codes

1. Introduction

We use the standard notation $[n, k, d]$ to denote the parameters of a linear code C over $GF(q)$. As usual n stands for its length, k its dimension and d its minimal distance. We say that C is an $[n, k, d]$-code over $GF(q)$.

We say that the code C of parameters $[n, k, d]$ is optimal if there is no $[n, k, d+1]$-code. We will refer to E. Brouwer's table [2] to get the known lower and upper bounds for the minimal distance (given n and k). We will say that a code meets (resp. beats) the record if it reaches the lower bound of Brouwer (resp. if it gives a better lower bound).

In this paper, we consider truncated Reed–Muller codes obtained by evaluating polynomials at a given subset of points in the projective plane.

Let $PG(m, q)$ be the m-dimensional projective space over $GF(q)$ and let $H_q(m, l)$ be the $GF(q)$-vector space of all homogeneous polynomials of degree l in $m+1$-variables. Let $\Omega \subset PG(m, q)$ be a subset of cardinality $|\Omega| = \omega$. We consider an arbitrary ordering of the points of Ω, say $\Omega = \{A_1, \ldots, A_\omega\}$. Then we define a $GF(q)$-linear evaluation map

$$\begin{array}{rccc} \Phi_\Omega : & H_q(m, l) & \rightarrow & GF(q)^\omega \\ & P & \mapsto & (P(A_1), \ldots, P(A_\omega)). \end{array}$$

Its image $\Phi_\Omega(H_q(m, l))$ is a linear code $C_\Omega(m, l)$ over $GF(q)$ of length ω. Moreover, if Φ_Ω is injective, then $C_\Omega(m, l)$ has dimension $\binom{m+l}{m}$. To shorten notations in the planar case, we will denote $C_\Omega(2, l)$ by $C_\Omega(l)$.

When $\Omega = PG(m, q)$ we get the so-called projective Reed–Muller codes [3]. If Ω is an algebraic subset then Bézout's Theorem gives a bound on the minimal distance as we see in Section 1 (following [5]).

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The idea of the following sections is to take for Ω a (κ, v)-arc in the projective plane which is a subset of κ points in $PG(2, q)$ such that some v but no $v+1$ are collinear. Then, we generalize this idea and introduce the notion of a $(\kappa, v, 2)$-arc in the projective plane and show how it produces some new codes over $GF(7)$, $GF(8)$, $GF(9)$ of dimensions 6 and 10.

2. Codes from Bézout's Theorem

Let χ be an absolutely irreducible projective curve of genus g over $GF(q)$. The Hasse–Weil bound says that its number of $GF(q)$-rational points satisfy $|\chi(GF(q))| \leqslant q + 1 + 2g\sqrt{q}$. Curves which reach the Hasse–Weil bound are called maximal.

Table 1 gives, for small q, the maximum number of $GF(q)$-rational points of a projective absolutely irreducible curve of given genus g:

Let $F \in H_q(2, l)$ and denote by $Z(F)$ the locus of zeros of F in $PG(2, q)$ and let $N(F) = |Z(F)|$ their number. If F is absolutely irreducible of degree l over $GF(q)$, we have

$$N(F) \leqslant q + 1 + \frac{(l-1)(l-2)}{2} \lfloor 2\sqrt{q} \rfloor,$$

since the genus g of the algebraic projective plane curve given by the equation $F = 0$ of degree l is such that $g \leqslant (l-1)(l-2)/2$ (equality holds if the curve is non-singular).

We illustrate the construction given in [5, Theorem 2.27], writing down the result obtained by Bézout's Theorem:

Theorem 2.1. *Let $F \in H_q(3, l')$ be such that $F = 0$ is the equation of an irreducible non-singular plane curve. Let $\Omega = Z(F)$ and let l be an integer such that $|\Omega| > ll'$. Then, $C_\Omega(l)$ is a linear code over $GF(q)$ with parameters*:

- $n = |\Omega|$,
- $d \geqslant n - ll'$
- $k = \begin{cases} \binom{l+2}{2} & \text{if } l < l' \\ ll' + 1 - \binom{l'-1}{2} & \text{if } l \geqslant l' \end{cases}$

To use Theorem 2.1, we are obviously interested in curves with many points (maximal curves for instance) in order to get codes with good parameters.

Example. We take as an example, $\Omega = Z(X^3Y + Y^3Z + Z^3X)$ where $X^3Y + Y^3Z + Z^3X = 0$ is the equation of a projective non-singular maximal plane curve over $GF(8)$. Then $C_\Omega(l)$, with $l = 2, 3, 4, 5$, are codes of parameters [24, 6, $\geqslant$16], [24, 10, $\geqslant$12], [24, 14, $\geqslant$8], [24, 18, $\geqslant$4]. To compare with the parameters of records [24, 6, 16], [24, 10, 12], [24, 14, 8], [24, 18, 5].

Table 1
Maximal number of $GF(q)$-rational points

q	2	3	4	5	7	8	9	11	13	16
$g=1$	5	7	9	10	13	14	16	18	21	25
$g=2$	6	8	10	12	16	18	20	24	26	33
$g=3$	7	10	14	16	20	24	28	28	32	38

3. Configuration of lines in the plane

3.1. Arcs in the plane

Concerning all the notions of this section we refer to [1,4,6] for a survey. A κ-arc in $PG(2,q)$ is a set of κ points no three of which are colinear. The maximum number of points in a κ-arc is denoted by $m(2,q)$. A maximal plane κ-arc is called an *oval*. We have

$$m(2,q) = \begin{cases} q+1 & \text{for } q \text{ odd,} \\ q+2 & \text{for } q \text{ even.} \end{cases}$$

More generally, a (κ, v)-arc in $PG(2,q)$ is a subset of κ points such that some v but no $v+1$ are colinear. Again, we denote by $m_v(2,q)$ the maximum number of points in a (κ, v)-arc. We have the trivial values: $m_2(2,q) = m(2,q)$, $m_{q+1}(2,q) = q^2 + q + 1$ and $m_q(2,q) = q^2$ [4]. And for $v \leqslant q-1$ "Table 2" is the table of values $m_v(2,q)$ for small q:

We have also the inequality: $m_v(2,q) \leqslant (v-1)q + v$, and there are many upper bounds when we add some conditions on v. A (κ, v)-arc which satisfies $m_v(2,q) = (v-1)q + v$ is said to be *maximal*. However, we are mostly interested in lower bound, and for instance let us state the following:

Proposition 3.1. We have

(1) *if* $q = 2^h$, $v = q-2$, *then* $m_v(2,q) \geqslant (v-1)q + 2$,
(2) *if* q *is a square, then* $m_v(2,q) \geqslant (q + \sqrt{q} + 1)(v - \sqrt{q})$,
(3) *if* q *is a square and* $v = q - \sqrt{q}$, *then* $m_v(2,q) \geqslant (v-1)q + \sqrt{q}$.

3.2. Truncated Reed–Muller codes

Let $N_q(l, \Omega)$ be the maximal number of zeros in $\Omega \subset PG(2,q)$ of a polynomial in $H_q(2,l)$. We also define arc (Ω) to be the lowest integer v such that Ω does not contain any $(\kappa, v+1)$-arc. We have the following:

Proposition 3.2. *Let* $\Omega \subset PG(2,q)$ *and set* $\omega = |\Omega|$. *If* $N_q(l,\Omega) < \omega$, *then the evaluation map* Φ_Ω *is injective and its image* $C_\Omega(l)$ *is a code of parameters*

$$\left[\omega, \frac{(l+1)(l+2)}{2}, \omega - N_q(l,\Omega)\right]$$

over $GF(q)$.

Table 2
$m_v(2,q)$

v	q					
	3	4	5	7	8	9
2	4	6	6	8	10	10
3		9	11	15	15	17
4			16	22	28	28
5				29	33	37
6				36	42	48
7					49	55
8						65

Since it is difficult to compute $N_q(l, \Omega)$ in general, we may bound it. Let $I_q(l)$ be the maximal numbers of zeros in $PG(2, q)$ of an absolutely irreducible polynomial in $H_q(2, l)$. Note that we clearly have

$$I_q(l) \leqslant q + 1 + (l-1)(l-2)\sqrt{q}$$

by the Hasse–Weil bound.

The following result handle the situation where the maximal numbers of zeros of a polynomial on Ω is bounded by those of a product of linear factors.

Lemma 3.3. *Let* $\Omega \subset PG(2, q)$ *and set* $\omega = |\Omega|$. *Let* $\mathrm{arc}\,(\Omega) = a$. *If* $I_q(j) \leqslant ja$ *for all* $j \in \{2, \ldots, l\}$, *then* $N_q(l, \Omega) \leqslant la$.

Proof. Let $F \in H_q(2, s)$ with $s \leqslant l$ and let $N(F)$ be its number of zeros. If $s = 1$ then we obviously have $N(F) \leqslant a$ since $N_q(1, \Omega) = \mathrm{arc}\,(\Omega) = a$. Assume now that $s > 1$.

It is enough to show that $N(F) \leqslant sa$. We proceed by induction on s. Let $F = f_1 \cdots f_r$ be the decomposition of F into absolutely irreducible factors. If $r = 1$ then $N_q(F) \leqslant I_q(s) \leqslant sa$ by assumption. Then, we may assume that $r > 1$.

The polynomial f_i has coefficients in a given extension $GF(q^{u(i)})$ of degree $u(i)$ of $GF(q)$. Considering a basis of the $GF(q)$-vector space $GF(q^{u(i)})$, the equation $f_i = 0$ splits into a system of $u(i)$ polynomial equations over $GF(q)$ which are either the zero equation or equations of degree equal to $\deg f_i$. Pick one such nonzero equation $\widetilde{f_i} = 0$. We obviously have $N_q(f_i) \leqslant N_q(\widetilde{f_i})$ and hence by the induction hypothesis $N(f_i) \leqslant N(\widetilde{f_i}) \leqslant a \deg f_i$. This concludes the proof. □

In the following, the difference between our use of Proposition 3.2 in place of Theorem 2.1 is that, instead of taking Ω to be all the $GF(q)$-rational points of a maximal curve, we consider for Ω a (κ, v)-arc with κ as big as possible, namely $\kappa = m_v(2, q)$.

For instance, when $l = 1$, we have an easy bound for $N_q(1, \Omega)$. Thus, when $l = 1$, we may compare codes of dimension 3 obtained by Theorem 2.1 (the Bézout construction with $l' = 2$) and those obtained from Proposition 3.2 (the arc construction with $v = 2$).

Examples. 1. Over $GF(q)$, the Bézout construction gives $[q + 1, 3, q - 1]$-codes, whereas the arc construction gives $[q + 1, 3, q - 1]$-codes for q odd and $[q + 2, 3, q]$-codes for q even.

2. For greater length, we can produce a lot of examples where the arc construction (together with Table 2) give better result than the Bézout construction (together with Table 1).

For instance, over $GF(7)$, the Bézout construction yields [13, 3, 10] and [20, 3, 16]-codes, whereas the arc construction yields [15, 3, 12] and [22, 3, 18]-codes.

4. Quadric-arcs and codes

As an application of Proposition 3.2 to codes of dimension 6, we have to bound $N_q(2, \Omega)$, namely to bound the number of zeros of a polynomial P of degree 2 in a subset Ω of $PG(2, q)$.

If P is absolutely irreducible (P is a conic) then we know that it has at most $q + 1$ zeros in $PG(2, q)$. And if P is reducible, namely a product of two linear factors, then the number of its zeros in Ω is bounded by $2\,\mathrm{arc}\,(\Omega)$.

So we get

$$N_q(2, \Omega) \leqslant \max(q + 1, 2\,\mathrm{arc}\,(\Omega)),$$

which leads to the following result:

Proposition 4.1. *If* $2v \geqslant q + 1$ *then* $m_v(2, q) \geqslant q + 1$ *and there is a code with parameters* $[m_v(2, q), 6, \geqslant m_v(2, q) - 2v]$ over $GF(q)$.

Example. We have

1. Let Ω be a (29, 5)-arc in $PG(2,7)$ (such an arc exists by Table 2). By the Hasse–Weil bound, we have $I_7(2) \leqslant 7+1=8$ and hence $I_7(2) \leqslant 2\,\mathrm{arc}\,(\Omega) = 10$. Which defines a [29, 6, 19]-code over $GF(7)$ meeting the record.
2. Together with Proposition 3.1, Proposition 4.1 yields the existence of codes over $GF(q)$ with parameters: $[(q+\sqrt{q}+1)(v-\sqrt{q}), 6, (q+\sqrt{q}+1)(v-\sqrt{q})-2v]$ if q is a square.

Next, to get a more precise bound on $N_q(2, \Omega)$ we introduce the notion of quadric-arc.

Definition. A quadric arc or a $(\kappa, v, 2)$-arc is a set of κ points in $PG(2,q)$ such that some v but no $v+1$ are the zeros (not counted with multiplicity) of a polynomial of degree 2. Let $m_{v,2}(2,q)$ be the maximal number of points in a $(\kappa, v, 2)$-arc.

Of course, we have the following result, which can be seen as a straightforward generalization of Proposition 4.1:

Theorem 4.2. *There is a code of parameters* $[m_{v,2}(2,q), 6, \geqslant m_{v,2}(2,q)-v]$ *over* $GF(q)$.

Since it is difficult to get exact values for $m_{v,2}(2,q)$ in general, we give the following simple ones:

Proposition 4.3. *For all* q, *we have* $m_{4,2}(2,q)=4$, $m_{2q,2}(2,q)=q^2$ *and* $m_{2q+1,2}(2,q)=q^2+q+1$. *Furthermore, for very small* q, *the values of* $m_{v,2}(2,q)$ *are given by* Table 3:

Proof. The values of $m_{4,2}(2,q)$ and $m_{2q+1,2}(2,q)$ are obvious. So we will focus on $m_{2q,2}(2,q)$.
We need the following elementary result:

Lemma 4.4. *Let* a *be such that* $m_{v-1}(2,q) < a \leqslant m_v(2,q)$ *and* $a - v > m_{v'}(2,q)$. *Then* $m_{v+v',2}(2,q) < a$.

Proof. Let $\Omega \subset PG(2,q)$ be such that $|\Omega| = a$. Then, there is a line L containing v points of Ω. Since $|\Omega \backslash L| = a - v$, there is a line K containing $v'+1$ points of $\Omega \backslash L$. Thus, the union of the two lines $L \cup K$ contains at least $v+v'+1$ points of Ω. □

By the inequality $m_v(2,q) \leqslant (v-1)q+v$ and Lemma 4.4 (with $(a, v, v') = (q^2+1, q+1, q-1)$) we deduce that $m_{2q,2} \leqslant q^2$.

For the converse inequality, we consider q parallel lines in the affine plane, and obviously get $m_{2q,2}(2,q) \geqslant q^2$. In fact it is the generic example since all (q^2, q)-arcs are projectively equivalent to subsets $\Omega = PG(2,4) \backslash l$ where l is a line in $PG(2,4)$ [4, 12.2.1(ii)].

Now, we compute the values of $m_{v,2}(2,4)$ for $v \in \{5,6,7\}$. By Lemma 4.4 with $(a, v, v') = (10, 4, 1)$ we have $m_{5,2}(2,4) \leqslant 9$. Likewise $m_{6,2}(2,4) \leqslant 10$ (take $(a, v, v') = (10, 4, 1)$) and $m_{7,2}(2,4) \leqslant 13$ (take $(a, v, v') = (10, 4, 1)$). Let

$$\Omega_5 = \{(0,0,1), (0,1,1), (1,a,1), (a,1,1), (a,0,1), (a+1,a,1), (1,1,1), (0,1,0)\},$$

$$\Omega_6 = \{(0,0,1), (0,1,1), (1,0,1), (0,a,1), (a,0,1), (1,1,1), (a+1,a,1), (1,1,0), (1,a,0), (a,1,0)\},$$

$$\Omega_7 = \{(0,0,1), (0,1,1), (1,0,1), (0,a,1), (a,0,1), (1,1,1), (a+1,a,1), (1,a,1), (a,1,1), (a,1,1), (1,1,0), (1,a,0), (a,1,0)\}.$$

By a Maple computation on $\Omega_5, \Omega_6, \Omega_7$, we respectively deduce that $m_{5,2}(2,4) \geqslant 8$, $m_{6,2}(2,4) = 10$ and $m_{7,2}(2,4) = 13$. So it only remains to show that $m_{5,2}(2,4) \neq 9$.

The full list of projectively distinct (9, 3)-arcs in $PG(2,4)$ is given in [4, pp. 355]: a (9, 3)-arc is either of the form $PG(2,4) \backslash (l_1 \cup l_2 \cup l_3)$ or $PG(2,4) \backslash (O \cup l \cup l')$ where l_1, l_2, l_3 are distinct lines, O an oval, $l = (PQ)$, $l' = (PQ')$

Table 3
$m_{v,2}(2, q)$

v	q		
	2	3	4
5	7	7	8
6		9	10
7		13	13
8			16
9			21

with $P, Q, Q' \in O$.

- First case: $\Omega = PG(2, 4)\backslash(l_1 \cup l_2 \cup l_3)$. Send l_3 to infinity. Then consider the parallel lines of l_1 in the affine plane $AG(2, 4) = PG(2, 4)\backslash\{l_3\}$. They intersect $l_1 \cup l_2 \cup l_3$ at most in one point in $AG(2, 4)$. And hence a product of two of them contains at least 6 points of Ω, a contradiction.
- Second case: $\Omega = PG(2, 4)\backslash(O \cup l \cup l')$. After a change of coordinates, we may assume that $P = (0, 0, 1)$, $Q = (0, 1, 0)$ and (QQ') is the line at infinity (i.e. of equation $z = 0$). Let D_i be the line of equation $x = iz$, $i \in GF(4)$. Then the oval O contains exactly 3 points in $AG(2, 4)\backslash D_0$, where $AG(2, 4)$ is the affine plane $PG(2, 4)\backslash\{(QQ')\}$. Necessarily we can find a union of two lines D_i, D_j, where $i, j \in GF(4)\backslash\{0\}$, which contains six points of $\Omega \cap AG(2, 4)$.

This concludes the proof. □

5. Further examples and results

5.1. Codes of dimension 6

All the codes of this section are constructed using Theorem 4.2. Our estimates of $m_{2v,2}(2, q)$ mainly comes from Lemma 4.4, and sometimes using Bézout's Theorem (to get lower bounds), or even by a direct explicit computation.

Note that we have explicit generating matrices of the codes given below, since the construction of maximal (κ, v)-arcs and $(\kappa, v, 2)$-arcs of Tables 2 and 3 can be made explicit.

- Over $GF(7)$:We have the elementary inequalities $13 \leqslant m_{6,2}(2, 7) \leqslant 15$. Since there is no code of parameters [15, 6, 9] we deduce that $m_{6,2}(2, 7) \leqslant 14$. Furthermore, if $m_{6,2}(2, 7) = 14$ then we would get a new code [14, 6, 8]. Although, all the computation we have done show only $m_{6,2}(7, 2) \geqslant 13$.
  We have also $m_{8,2}(2, 7) \geqslant 22$ since $I_7(2) = 8$.
  Results: We construct 10 codes of dimension 6 and length $\leqslant 29$ which meet the record.
- Over $GF(8) = GF(2)[b]$ with $b^3 = b + 1$: We have $14 \leqslant m_{6,2}(2, 8) \leqslant 15$.
  We have also the elementary inequalities $24 \leqslant m_{8,2}(2, 8) \leqslant 28$. In fact, a computation on the set

$$\{(0, 1, 0), (b^2, b, 1), (b, 1, 1), (b + b^2, b, 1), (1, b + b^2, 1), (1 + b^2, 1 + b^2, 1),$$
$$(1 + b, b + b^2, 1), (1 + b + b^2, 1 + b^2, 1), (1, 1, 0), (0, 1 + b, 1),$$
$$(b^2, b + b^2, 1), (b, 1 + b, 1), (b + b^2, b + b^2, 1), (1, 1, 1), (1 + b^2, b^2, 1),$$
$$(1 + b, 1, 1), (1 + b + b^2, b^2, 1), (0, b^2, 1), (b^2, 1 + b, 1),$$
$$(b, b^2, 1), (b + b^2, 1 + b, 1), (1, 1 + b^2, 1), (1 + b^2, b, 1),$$
$$(1 + b, 1 + b^2, 1), (1 + b + b^2, b, 1), (b^2 + b, 1, 0), (b^2 + b + 1, 1, 0)\}$$

  shows that $m_{8,2}(2, 8) \geqslant 27$.
  Results: We construct codes with parameters $[27 - i, 6, 19 - i]$ which beat the record for $i \in \{0, 1, 2\}$,
- Over $GF(9)$: We have the elementary inequalities $16 \leqslant m_{6,2}(2, 9) \leqslant 17$.

Table 4
$m_{v,2}(2, q)$

v	q						
	2	3	4	5	7	8	9
6		9	10	11	13–14	14–15	16–17
7		13	13	12–15	14–19	15–19	17–21
8			16	16	22	27–28	28
9			21	17–24	23–27	29–33	29–33
10				25	29	33–34	37
11				31	30–35	34–39	38–43
12					36	42	48

We have also $m_{8,2}(2, 9) = 28$ and $m_{12,2}(2, 9) = 48$, $m_{13,2}(2, 9) \geqslant 49$.
Results: We construct codes of parameters $[48 - i, 6, 36 - i]$ for $i \in \{0, 1, 2\}$, and also [49, 6, 36], which beat the record.

We end this section by a table of very loose possible ranges of values for $m_{v,2}(2, q)$ (Table 4):
To get these values, we mainly used Lemma 4.4 and Table 2.

5.2. Codes of dimension 10

Using Proposition 3.2 with $l = 3$, we may construct codes of dimension 10. To estimate the minimal distance, we have to bound $N_q(3, \Omega)$ for $\Omega \subset PG(2, q)$. For instance, the Hasse–Weil bound gives

$$N_q(3, \Omega) \leqslant \max(q + 1 + 2\sqrt{q}, \text{arc}\,(\Omega) + N_q(2, \Omega)).$$

and also

$$N_q(3, \Omega) \leqslant \max(q + 1 + 2\sqrt{q}, \text{arc}\,(\Omega) + q + 1, 3\,\text{arc}\,(\Omega)).$$

Results: Over $GF(8)$, we construct $[27 - i, 10, 15 - i]$-codes which beat the record for $i \in \{0, 1, 2\}$.
We construct also, over $GF(7)$, $GF(8)$ and $GF(9)$, few other codes of dimension 10 which meet the record.
It would be natural to try to apply Proposition 3.2 to get codes of higher dimensions (namely of dimension $(l + 1)(l + 2)/2$ when $l \geqslant 4$). But when l grows, the Hasse–Weil bound gives a poor bound on $N_q(l)$, and we get codes far from the record. This is in contrast to the Bézout construction where the algebraic nature of the subset Ω ensures a good bound on $N_q(l, \Omega)$ by Bézout's Theorem.

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