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A local smoothing estimate for the Schrödinger equation

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Abstract

We show that the Schrödinger operator $e^{it\Delta}$ is bounded from $W^{\alpha,q}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n \times [0,1])$ for all $\alpha > 2n(1/2 - 1/q) - 2/q$ and $q \geqslant 2 + 4/(n+1)$. This is almost sharp with respect to the Sobolev index. We also show that the Schrödinger maximal operator $\sup_{0<t<1} |e^{it\Delta} f|$ is bounded from $H^s(\mathbb{R}^n)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ when $s > s_0$ if and only if it is bounded from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ when $s > 2s_0$. A corollary is that $\sup_{0<t<1} |e^{it\Delta} f|$ is bounded from $H^s(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ when $s > 3/4$.

Keywords: Schrödinger equation; Local smoothing; Pointwise convergence; Restriction

1. Introduction

The solution to the wave equation, $\partial_{tt} u = \Delta u$, with initial data $u(\cdot,0) = f$ and $u'(\cdot,0) = 0$, can be formally written as the real part of

$$e^{it\sqrt{-\Delta}} f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i (x\cdot\xi - t|\xi|)}\, d\xi. \tag{1}$$

Let $\|\cdot\|_{q,\alpha}$ denote the inhomogeneous Sobolev norm with α derivatives in $L^q(\mathbb{R}^n)$. J.C. Peral [24] proved that for any fixed time t and $q \in (1,\infty)$,

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^q(\mathbb{R}^n)} \leqslant C_{t,q} \|f\|_{q,\alpha}$$

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for all $\alpha \geqslant (n-1)|\frac{1}{2}-\frac{1}{q}|$, and this is sharp. Sogge [30] conjectured that

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^q(\mathbb{R}^n\times[1,2])}\leqslant C_{q,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha > (n-1)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}$ and $q > 2+\frac{2}{n-1}$. This is known as the local smoothing conjecture due to the potential gain of $1/q$ derivatives.

In two spatial dimensions, Mockenhaupt, Seeger and Sogge [22] showed that the local smoothing estimate holds at the critical exponent $q=4$ for all $\alpha > 1/8$, and this was improved by Bourgain [2], Tao and Vargas [33], and Wolff [38] to $\alpha > 5/44$.

Moving away from the critical exponent, but remaining in two spatial dimensions, Wolff [37] proved the (almost) sharp estimate in the range $q > 74$, and Łaba and Wolff [16] generalised this to higher dimensions. Garrigós and Seeger [12] have recently refined their arguments, so that, in higher dimensions for example, the (almost) sharp estimate holds in the range

$$q > 2+\frac{8}{n-3}\left(1-\frac{1}{n+1}\right).$$

The Schrödinger equation, $i\partial_t u+\Delta u=0$, with initial datum f has solution $e^{it\Delta}f$ which can be formally written as

$$e^{it\Delta}f(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}\,d\xi. \tag{2}$$

Miyachi [21] (see also [11]) proved that for any fixed time t and $q\in(1,\infty)$,

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n)}\leqslant C_{t,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha\geqslant 2n|\frac{1}{2}-\frac{1}{q}|$, and this is sharp. When $n\geqslant 2$, square function estimates (see [3,17,20]) yield

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n\times[1,2])}\leqslant C_{q,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha > 2n(\frac{1}{2}-\frac{1}{q})-\frac{2}{q}$ and $q > 2+4/n$. We see that averaging locally in time yields a gain of $2/q$ derivatives.

We extend the range of q by taking advantage of all $n+1$ dimensions of curvature. This also allows us to treat the $n=1$ case for which we obtain almost sharp estimates. In higher dimensions, it may be possible to extend the range to $q > 2+2/n$, and we shall see later that this would follow from the restriction conjecture for paraboloids.

Theorem 1. *Let $q > 2+\frac{4}{n+1}$ and $\alpha > 2n(\frac{1}{2}-\frac{1}{q})-\frac{2}{q}$. Then there exists a constant $C_{q,\alpha}$ such that*

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n\times[0,1])}\leqslant C_{q,\alpha}\|f\|_{q,\alpha}.$$

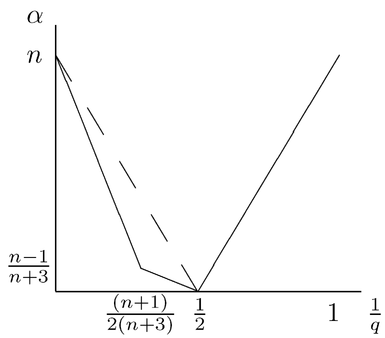

Fig. 1. Region of local smoothing in Corollary 2.

Although there is a formal similarity between this and the estimates of Wolff et al., the question for the Schrödinger equation is not as deep, and the arguments will bear no resemblance. An obvious difference is that the wave operator, for finite time, is a local operator, whereas the Schrödinger operator is not. We will see however, that one can decompose the initial data so that the Schrödinger operator, for finite time, may essentially be treated as a local operator.

Before proceeding further, we should mention that there are estimates for the Schrödinger equation, independently due to Sjölin [27], Vega [35,36], and Constantin and Saut [7], which are more deserving of the description 'local smoothing.' They proved that

$$\left\| e^{it\Delta} f \right\|_{L^2(\mathbb{B}^n\times[0,1])} \leqslant C_s \|f\|_{H^{-1/2}(\mathbb{R}^n)},$$

where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, and $\|\cdot\|_{H^s(\mathbb{R}^n)}$ denotes $\|\cdot\|_{2,\alpha}$. Thus, the solution is locally half a derivative smoother than the initial datum. We will see later that this is equivalent up to endpoints with the global estimate

$$\left\| e^{it\Delta} f \right\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant C_s \|f\|_{L^2(\mathbb{R}^n)},$$

which we will refer to as simply the conservation of charge.

Interpolating between this and the bound in Theorem 1 yields the following corollary. In one spatial dimension, it is almost sharp in the range $q \in [1,\infty]$, and in higher dimensions it is almost sharp in the ranges $q \in [1,2]$ and $q \in [2+\frac{4}{n+1},\infty]$.

Corollary 2. *Let $q \in [1,\infty]$ and $\alpha > \max\{2n(\frac{1}{q}-\frac{1}{2}), (n-1)(\frac{1}{2}-\frac{1}{q}), 2n(\frac{1}{2}-\frac{1}{q})-\frac{2}{q}\}$. Then there exists a constant $C_{T,\alpha}$ such that*

$$\left\| e^{it\Delta} f \right\|_{L^q(\mathbb{R}^n\times[-T,T])} \leqslant C_{T,\alpha} \|f\|_{q,\alpha}$$

(*see Fig.* 1).

In the second part of the article, we will consider the minimal value of s for which

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L^2(\mathbb{B}^n)} \leqslant C_{n,s} \|f\|_{H^s(\mathbb{R}^n)} \tag{3}$$

holds. By standard arguments, the estimate implies the almost everywhere convergence of $e^{it\Delta}f$ to f, as t tends to zero. The minimal s for which the global bound

$$\Big\| \sup_{0<t<1} \big|e^{it\Delta}f\big| \Big\|_{L^2(\mathbb{R}^n)} \leqslant C_{n,s}\|f\|_{H^s(\mathbb{R}^n)} \tag{4}$$

holds, has also been considered in connection with the well-posedness of certain initial value problems (see [14]).

In one spatial dimension, Carleson, Kenig and Ruiz [6,13] showed that (3) holds when $s \geqslant 1/4$, and Dahlberg and Kenig [9] showed that this is sharp. Vega [14,35] (see also [28]) showed that the global bound (4) holds when $s > 1/2$, and this is also sharp.

In higher dimensions, it was independently proven by Sjölin [27] and Vega [36] that (3) holds when $s > 1/2$, and the bound cannot hold when $s < 1/4$. Carbery [4] and Cowling [8] independently showed that (4) holds when $s > 1$, and in this case, the bound cannot hold when $s < 1/2$. It is conjectured that, the minimal value of s for which (3) holds is $1/4$, and the minimal value for which (4) holds is $1/2$.

We will put these results and conjectures in context by proving the following theorem.

Theorem 3. (3) *holds for* $s > s_0 \Leftrightarrow$ (4) *holds for* $s > 2s_0$.

In two spatial dimensions, more was known for the local bound than for the global bound. Bourgain [1] showed that there exists an s strictly less that $1/2$ for which (3) holds, and this was improved by Moyua, Vargas and Vega [23], and Tao and Vargas [32,33]. The best known result is due to S. Lee [19], who showed that (3) holds when $s > 3/8$.

Therefore, as a consequence of the equivalence, we have the following corollary, which improves the result of Carbery and Cowling in two spatial dimensions.

Corollary 4. *For all* $s > 3/4$, *there exists a constant* C_s *such that*

$$\Big\| \sup_{0<t<1} \big|e^{it\Delta}f\big| \Big\|_{L^2(\mathbb{R}^2)} \leqslant C_s\|f\|_{H^s(\mathbb{R}^2)}.$$

The result of Cowling also holds when the Laplacian is replaced by a more general class of operators that includes

$$\Box = \partial_{x_1}^2 - \partial_{x_2}^2 \pm \partial_{x_3}^2 \pm \cdots \pm \partial_{x_n}^2.$$

For physical applications of the nonelliptic Schrödinger equation, see for example [31]. We will also prove the equivalence in this case, so that, by a local result of Vargas, Vega and the author [26], the global result of Cowling is almost sharp. We state this as a corollary.

Corollary 5. *For all* $s > 1$, *there exists a constant* C_s *such that*

$$\Big\| \sup_{0<t<1} \big|e^{it\Box}f\big| \Big\|_{L^2(\mathbb{R}^n)} \leqslant C_s\|f\|_{H^s(\mathbb{R}^n)},$$

and this is not true when $s < 1$.

Throughout, c and C will denote positive constants that may depend on the dimensions and exponents of the Sobolev spaces. It will be made explicit when they depend on other factors like, for example, the Sobolev index. Their values may change from line to line. The following are notations that will be used frequently:

$L^q_x(\mathbb{R}^n, L^r_t(I))$: the Lebesgue space with norms $(\int_{\mathbb{R}^n} |\int_I |f(x,t)|^r\,dt|^{q/r}\,dx)^{1/q}$.
$W^{\alpha,q}(\mathbb{R}^n)$: the inhomogeneous Sobolev space with α derivatives in $L^q(\mathbb{R}^n)$.
$\|\cdot\|_{q,\alpha}$: the inhomogeneous Sobolev norm with α derivatives in $L^q(\mathbb{R}^n)$.
$H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$.
$\Box := \partial^2_{x_1} - \partial^2_{x_2} \pm \partial^2_{x_3} \pm \cdots \pm \partial^2_{x_n}$.
$\mathbb{B}^n := \{x \in \mathbb{R}^n\colon\ |x| \leqslant 1\}$.
$\mathbb{A}^n := \{x \in \mathbb{R}^n\colon\ 1/2 \leqslant |x| \leqslant 1\}$.
$B_R := \{x \in \mathbb{R}^n\colon\ |x| \leqslant R\}$.
$A_R := \{x \in \mathbb{R}^n\colon\ R/2 \leqslant |x| \leqslant R\}$.
χ_{B_R}: the indicator function of B_R.
$\varphi_{R^2}(x) := R^{-2n}(1 + \frac{|x|}{R^2})^{-2n}$.
$L_{R^2} f := \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|$.
v_j: a member of the lattice $R^{-2}\mathbb{Z}^n$.
x_k: a member of the lattice $R^2\mathbb{Z}^n$.
$T_{jk} := \{(x,t) \in \mathbb{R}^n \times [0, R^4]\colon\ |x - (x_k + 4\pi t v_j)| \leqslant R^2\}$.
$\{Q_l\}_{l\in\mathbb{N}}$: a partition of $\mathbb{R}^n$ into cubes of side R^2, centred at $x_l \in R^2\mathbb{Z}^n$.
$\hat{\psi}$: a positive and smooth function, supported in $B_{\sqrt{n}}$.
$\hat{\eta}$: a positive and smooth function, supported in $\mathbb{B}^n$, and equal to 1 at the origin.

2. Necessary conditions for local smoothing

Let $\hat{\eta}$ be a positive and smooth function supported in $\mathbb{B}^n$, and denote by $\hat{\eta}_{R^{-1}}$ the scaled version $\hat{\eta}(\frac{\cdot}{R})$. Correspondingly, we let $\eta_{R^{-1}}$ denote its inverse Fourier transform $R^n\eta(R\cdot)$. We consider initial data f_R defined by

$$\hat{f}_R(\xi) = e^{2\pi^2 i|\xi|^2} \frac{\hat{\eta}_{R^{-1}}(\xi)}{(1+|\xi|^2)^{\alpha/2}}.$$

We note that

$$\|f_R\|_{r,\alpha} = \big\| e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}\big\|_{L^r(\mathbb{R}^n)},$$

and by a change of variables,

$$e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x) = R^n \int_{\mathbb{R}^n} \hat{\eta}(\xi) e^{2\pi i(Rx\cdot\xi + R^2\pi|\xi|^2)}\,d\xi.$$

When $|x| > 2\pi R$, by repeated integration by parts, there exists constants C_N such that

$$\big|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x)\big| \leqslant C_N \left(\frac{|x|}{2\pi R}\right)^{-N} \tag{5}$$

for all $N \in \mathbb{N}$. When $|x| \leqslant 2\pi R$, by the dispersive estimate,

$$\left|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x)\right| \leqslant C\|\eta_{R^{-1}}\|_{L^1(\mathbb{R}^n)} \leqslant C. \tag{6}$$

Combining these two bounds, we see that

$$\|f_R\|_{r,\alpha} = \left\|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}\right\|_{L^r(\mathbb{R}^n)} \leqslant CR^{\frac{n}{r}}. \tag{7}$$

On the other hand, by a change of variables,

$$\begin{aligned}\left|e^{it\Delta}f_R(x)\right| &= \left|\int_{\mathbb{R}^n} \frac{\hat{\eta}(\frac{\xi}{R})}{(1+|\xi|^2)^{\alpha/2}} e^{2\pi i(x\cdot\xi - 2\pi(t-\frac{1}{2})|\xi|^2)}\,d\xi\right| \\ &= \left|R^{n-\alpha}\int_{\mathbb{R}^n} \frac{\hat{\eta}(\xi)}{(\frac{1}{R^2}+|\xi|^2)^{\alpha/2}} e^{2\pi i(Rx\cdot\xi - 2\pi R^2(t-\frac{1}{2})|\xi|^2)}\,d\xi\right|,\end{aligned}$$

so when $|x| \leqslant \frac{1}{10R}$ and $|t-\frac{1}{2}| \leqslant \frac{1}{20\pi R^2}$, we have $|e^{it\Delta}f_R(x)| \geqslant CR^{n-\alpha}$. Thus,

$$\left\|e^{it\Delta}f_R\right\|_{L^q(\mathbb{R}^n\times[0,1])} \geqslant CR^{n-\alpha}R^{-\frac{n+2}{q}},$$

and combining this with (7), we see that for

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n\times[0,1])} \leqslant C_\alpha\|f\|_{r,\alpha} \tag{8}$$

to hold, it is necessary that $\alpha \geqslant n(1-\frac{1}{q}-\frac{1}{r})-\frac{2}{q}$.

By considering f_R defined by $\hat{f}_R = \hat{\eta}_{R^{-1}}$, we reverse the previous focusing example. Note that the rapid decay (5) and upper bound (6) remain true for all $t \in [1/2, 1]$. This forces $|e^{it\Delta}f_R| \geqslant c$ on a set of measure cR^n as otherwise the conservation of charge would be violated. We see that

$$\left\|e^{it\Delta}f_R\right\|_{L^q(\mathbb{R}^n\times[0,1])} \geqslant CR^{\frac{n}{q}},$$

and as $\|f_R\|_{r,\alpha} \leqslant CR^\alpha R^{n-\frac{n}{r}}$, for (8) to hold it is also necessary that $\alpha \geqslant n(\frac{1}{q}+\frac{1}{r}-1)$.

Finally, we consider initial data f_R defined by $\hat{f}_R(\xi) = \hat{\eta}(R^\lambda(\xi-(R,\dots,R)))$, where $\lambda \geqslant 1$, so that

$$e^{it\Delta}f_R(x) = \int \hat{\eta}\big(R^\lambda\big(\xi-(R,\dots,R)\big)\big)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}\,d\xi.$$

One can calculate that $|2\pi\nabla_\xi(x\cdot\xi-2\pi t|\xi|^2)| \leqslant \frac{R^\lambda}{10}$ in the region defined by

$$|x| \leqslant \frac{R^\lambda}{100}, \qquad |t| \leqslant \frac{1}{1000}, \quad \text{and} \quad \left|\xi-(R,\dots,R)\right| \leqslant \frac{1}{R^\lambda},$$

so that the phase is almost constant for each pair (x, t) in the region. Thus,

$$\|e^{it\Delta} f_R\|_{L^q(\mathbb{R}^n\times[0,1])} \geqslant CR^{-n\lambda} R^{\frac{n\lambda}{q}},$$

and combining this with

$$\|f_R\|_{r,\alpha} \leqslant R^\alpha R^{-n\lambda+\frac{n\lambda}{r}},$$

we see that

$$\alpha \geqslant \lambda n\left(\frac{1}{q} - \frac{1}{r}\right).$$

Setting $\lambda = 1$ and letting $\lambda \to \infty$ yield the necessary conditions $\alpha \geqslant n(\frac{1}{q} - \frac{1}{r})$ and $q \geqslant r$, respectively.

In particular, ignoring endpoint issues, one may hope that

$$\|e^{it\Delta} f\|_{L^q(\mathbb{R}^n\times[0,1])} \leqslant C_\alpha \|f\|_{q,\alpha}$$

for all $\alpha > \max\{2n(\frac{1}{q} - \frac{1}{2}), 0, 2n(\frac{1}{2} - \frac{1}{q}) - \frac{2}{q}\}$.

3. Localising lemmas

As in the arguments of Fefferman [10], Bourgain [2], Wolff [38], Tao [32], and others, we decompose the solution of the Schrödinger equation into wave packets at scale $R^2 \gg 1$.

Fix a positive and smooth function $\hat{\psi}$, supported in $B_{\sqrt{n}}$, such that

$$\sum_j \hat{\psi}\big(\xi - R^2 v_j\big) = 1,$$

where $v_j \in R^{-2}\mathbb{Z}^n$. We also fix a positive and smooth $\hat{\eta}$, supported in $\mathbb{B}^n$, that satisfies $\hat{\eta}(0) = 1$, so that, by the Poisson summation formula,

$$\sum_k \eta\left(x - \frac{x_k}{R^2}\right) = 1,$$

where $x_k \in R^2\mathbb{Z}^n$. Now for any Schwartz function f, we define f_j and f_{jk} implicitly in the following decomposition:

$$\hat{f}(\xi) = \sum_j \hat{f}_j(\xi) = \sum_j \hat{\psi}\big(R^2(\xi - v_j)\big)\hat{f}(\xi), \tag{9}$$

$$f(x) = \sum_{j,k} f_{jk}(x) = \sum_{j,k} \eta\left(\frac{x - x_k}{R^2}\right) f_j(x). \tag{10}$$

Note that $\hat{f}_{jk}$ is supported in the ball of radius $(\sqrt{n} + 1)R^{-2}$ with centre v_j.

We also partition $\mathbb{R}^n$ into cubes Q_l of side R^2, centred at $x_l \in R^2\mathbb{Z}^n$, and define the function φ_{R^2} by

$$\varphi_{R^2}(x) = R^{-2n}\left(1 + \frac{|x|}{R^2}\right)^{-2n},$$

and the operator L_{R^2} by

$$L_{R^2} f = \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|.$$

We state a slightly refined version of a lemma which can be found in [32], or more explicitly in [18], where we replace the Hardy–Littlewood maximal operator by a convolution operator. It is clear from their proofs that this is permissible.

Lemma 6. *Let $t \in [0, R^4]$. Then for all $N \in \mathbb{N}$ there exists a constant C_N such that*

$$\left|e^{it\Delta} f_{jk}(x)\right| \leqslant C_N \varphi_{R^2} * \left|f_j(x_k)\right| \left(1 + \frac{|x - (x_k + 4\pi t v_j)|}{R^2}\right)^{-N}.$$

We note that when $t \in [0, R^4]$, the wave packets $e^{it\Delta} f_{jk}$ are essentially supported in the tubes T_{jk} defined by

$$T_{jk} = \left\{(x,t) \in \mathbb{R}^n \times \left[0, R^4\right]\colon \left|x - (x_k + 4\pi t v_j)\right| \leqslant R^2\right\}.$$

Lemma 7. *For all f frequency supported in $\mathbb{B}^n$ and $\varepsilon > 0$, there exists functions f_l, $\widetilde{f}_l$ satisfying*

(i) $\|f_l\|_{L^p(\mathbb{R}^n)} \leqslant C R^{2n(\frac{1}{p}-\frac{1}{q})+\varepsilon} \|\widetilde{f}_l\|_{L^q(\mathbb{R}^n)}$

for all $p \leqslant q$,

(ii) $\sum_l \|\widetilde{f}_l\|^q_{L^q(\mathbb{R}^n)} \leqslant C R^{\varepsilon} \|f\|^q_{L^q(\mathbb{R}^n)}$,

and for all $l, N \in \mathbb{N}$ and $(x,t) \in Q_l \times [0, R^2]$,

(iii) $|e^{it\Delta} f(x)| \leqslant |e^{it\Delta} f_l(x)| + C_N R^{-N} L_{R^2} f(x)$.

Proof. We decompose the solution into wave packets, $e^{it\Delta} f = \sum_{j,k} e^{it\Delta} f_{jk}$, at scale R^2. We recall that

$$f_{jk}(x) = \eta\left(\frac{x - x_k}{R^2}\right) f_j(x),$$

and we define $\tilde{f}_{jk}$ by

$$\tilde{f}_{jk}(x) = |\eta|^{1/2}\left(\frac{x - x_k}{R^2}\right) f_j(x).$$

As η decays rapidly and $\sum_k \eta(x-\frac{x_k}{R^2})=1$, it is easy to see that

$$\sum_k |\eta|^{1/2}\left(x-\frac{x_k}{R^2}\right)\leqslant C,$$

so that

$$\sum_k \Big\|\sum_j \tilde f_{jk}\Big\|^q_{L^q(\mathbb{R}^n)} \leqslant C\Big\|\sum_{j,k} \tilde f_{jk}\Big\|^q_{L^q(\mathbb{R}^n)} \leqslant C\|f\|^q_{L^q(\mathbb{R}^n)}. \tag{11}$$

As supp $\hat f\subset\mathbb{B}^n$, we have that the v_j's are contained in a slight enlargement of $\mathbb{B}^n$. Thus, the tubes T_{jk} make angles with the spatial hyperplane which are uniformly bounded below. Letting $R^\varepsilon Q_l$ denote the cube of side $R^{2+\varepsilon}$ with centre x_l, we write

$$f_l=\sum_{k\colon Q_k\cap R^\varepsilon Q_l\neq\emptyset}\ \sum_j f_{jk},$$

so that $e^{it\Delta}f_l$ consists of the wave packets that pass through or near to $Q_l\times[0,R^2]$. Similarly, we define $\widetilde{f_l}$ by

$$\widetilde{f_l}=\sum_{k\colon Q_k\cap R^\varepsilon Q_l\neq\emptyset}\ \sum_j \tilde f_{jk}.$$

To prove property (i), we note that

$$\begin{aligned}
\big|f_l(x)\big| &= \Bigg|\sum_{k\colon Q_k\cap R^\varepsilon Q_l\neq\emptyset}\eta\left(\frac{x-x_k}{R^2}\right)f(x)\Bigg| \\
&\leqslant C\left(1+\frac{|x-x_l|}{R^{2+2\varepsilon}}\right)^{-M}\Bigg|\sum_{k\colon Q_k\cap R^\varepsilon Q_l\neq\emptyset}|\eta|^{1/2}\left(\frac{x-x_k}{R^2}\right)f(x)\Bigg| \\
&= C\left(1+\frac{|x-x_l|}{R^{2+2\varepsilon}}\right)^{-M}\big|\widetilde{f_l}(x)\big|
\end{aligned}$$

for some large $M\in\mathbb{N}$, so that, by Hölder,

$$\|f_l\|_{L^p(\mathbb{R}^n)}\leqslant CR^{2(1+\varepsilon)n(\frac1p-\frac1q)}\|\widetilde{f_l}\|_{L^q(\mathbb{R}^n)}.$$

To prove property (ii), we note that a cube Q_k can intersect $R^\varepsilon Q_l$ for at most $2R^{n\varepsilon}$ different cubes Q_l, so that

$$\begin{aligned}
\sum_l \|\widetilde{f_l}\|^q_{L^q(\mathbb{R}^n)} &\leqslant C\sum_l\ \sum_{k\colon Q_k\cap R^\varepsilon Q_l\neq\emptyset}\Big\|\sum_j \tilde f_{jk}\Big\|^q_{L^q(\mathbb{R}^n)} \\
&\leqslant CR^{n\varepsilon}\sum_k\Big\|\sum_j \tilde f_{jk}\Big\|^q_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

Thus, by (11), we see that

$$\sum_l \|\widetilde{f}_l\|_{L^q(\mathbb{R}^n)}^q \leqslant C R^{n\varepsilon} \|f\|_{L^q(\mathbb{R}^n)}^q.$$

To prove property (iii), we consider the pointwise bound

$$\left|e^{it\Delta} f\right| \leqslant \left|e^{it\Delta} f_l\right| + \left|\sum_{k:\, Q_k \cap R^{\varepsilon} Q_l = \emptyset} \sum_j e^{it\Delta} f_{jk}\right|. \tag{12}$$

By construction and Lemma 6,

$$\left|\sum_{k:\, Q_k \cap R^{\varepsilon} Q_l = \emptyset} \sum_j e^{it\Delta} f_{jk}(x)\right| \leqslant C_{N'} R^{2N'} \sum_{j=1}^{cR^{2n}} \sum_{k:\, |x_k - x_l| \geqslant \frac{1}{2} R^{2+\varepsilon}} \frac{\varphi_{R^2} * |f_j|(x_k)}{|x_k - x_l|^{N'}}$$

for all $(x,t) \in Q_l \times [0, R^2]$, and all $N' \in \mathbb{N}$. Choosing an $N' > (4n+N)/\varepsilon + 2n$, we have

$$\left|\sum_{k:\, Q_k \cap R^{\varepsilon} Q_l = \emptyset} \sum_j e^{it\Delta} f_{jk}(x)\right| \leqslant C_N R^{-N} \sum_{j=1}^{cR^{2n}} \sum_{k:\, |x_k - x_l| \geqslant \frac{1}{2} R^{2+\varepsilon}} \frac{\varphi_{R^2} * |f_j|(x_k)}{|x_k - x_l|^{2n}}$$

for all $N \in \mathbb{N}$. Now, by (9),

$$|f_j| \leqslant \left|R^{-2n} \psi\left(R^{-2} \cdot\right)\right| * |f| \leqslant C \varphi_{R^2} * |f|,$$

so that

$$\left|\sum_{k:\, Q_k \cap R^{\varepsilon} Q_l = \emptyset} \sum_j e^{it\Delta} f_{jk}(x)\right| \leqslant C_N R^{-N} \sum_{j=1}^{cR^{2n}} \sum_{k:\, |x_k - x_l| \geqslant \frac{1}{2} R^{2+\varepsilon}} \frac{\varphi_{R^2} * \varphi_{R^2} * |f|(x_k)}{|x_k - x_l|^{2n}}.$$

Now, it is easy to see that

$$\varphi_{R^2} * \varphi_{R^2} * |f|(x) \approx \varphi_{R^2} * \varphi_{R^2} * |f|(x')$$

when $|x - x'| \leqslant \sqrt{n} R^2$, so that

$$\begin{aligned} \sum_{k:\, |x_k - x_l| \geqslant \frac{1}{2} R^{2+\varepsilon}} \frac{\varphi_{R^2} * \varphi_{R^2} * |f|(x_k)}{|x_k - x_l|^{2n}} &\leqslant C \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|(x_l) \\ &\leqslant C \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|(x) \end{aligned}$$

for all $x \in Q_l$. Substituting into (12), we have

$$\left|e^{it\Delta} f(x)\right| \leqslant \left|e^{it\Delta} f_l(x)\right| + C_N R^{-N} \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|(x)$$

for all $(x,t) \in Q_l \times [0, R^2]$, and we are done. □

Lemma 8. *Let $q \geqslant p_1 \geqslant p_0$ and $I \subset [0, R^2]$. Suppose that*

$$\left\| e^{it\Delta} f \right\|_{L^q_x(B_{R^2}, L^r_t(I))} \leqslant C R^s \| f \|_{L^{p_0}(\mathbb{R}^n)}$$

whenever $R \gg 1$, and f is frequency supported in $\mathbb{B}^n$. Then for all $\varepsilon > 0$,

$$\left\| e^{it\Delta} f \right\|_{L^q_x(\mathbb{R}^n, L^r_t(I))} \leqslant C_\varepsilon R^{s+2n(\frac{1}{p_0}-\frac{1}{p_1})+\varepsilon} \| f \|_{L^{p_1}(\mathbb{R}^n)}.$$

Proof. By Lemma 7, for all $\varepsilon > 0$, there exists functions f_l and $\widetilde{f}_l$ such that

$$\| f_l \|_{L^{p_0}(\mathbb{R}^n)} \leqslant C R^{2n(\frac{1}{p_0}-\frac{1}{p_1})+\varepsilon} \| \widetilde{f}_l \|_{L^{p_1}(\mathbb{R}^n)}, \tag{13}$$

$$\sum_l \| \widetilde{f}_l \|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \leqslant C R^{\varepsilon} \| f \|^{p_1}_{L^{p_1}(\mathbb{R}^n)}, \tag{14}$$

and for all $N, l \in \mathbb{N}$ and $(x, t) \in Q_l \times [0, R^2]$,

$$\left| e^{it\Delta} f(x) \right| \leqslant \left| e^{it\Delta} f_l(x) \right| + C_N R^{-N} L_{R^2} f(x).$$

We use these pointwise bounds on cubes, to obtain an $L^q(\mathbb{R}^n, L^r_t(I))$ bound. We have

$$\begin{aligned} \left\| e^{it\Delta} f \right\|^q_{L^q(\mathbb{R}^n, L^r_t(I))} &= \sum_l \left\| e^{it\Delta} f \right\|^q_{L^q(Q_l, L^r_t(I))} \\ &\leqslant \sum_l \left\| \left| e^{it\Delta} f_l \right| + C_N R^{-N} L_{R^2} f \right\|^q_{L^q(Q_l, L^r_t(I))}, \end{aligned}$$

and using the fact that $\|g + h\|^q \leqslant 2^q(\|g\|^q + \|h\|^q)$, we see that

$$\left\| e^{it\Delta} f \right\|^q_{L^q(\mathbb{R}^n, L^r_t(I))} \leqslant C \sum_l \left\| e^{it\Delta} f_l \right\|^q_{L^q(Q_l, L^r_t(I))} + C_N R^{-N} \sum_l \| L_{R^2} f \|^q_{L^q(Q_l, L^r_t(I))}.$$

Now, by Young's inequality,

$$\begin{aligned} \sum_l \| L_{R^2} f \|^q_{L^q(Q_l, L^r_t(I))} &\leqslant R^{2q} \left\| \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f| \right\|^q_{L^q(\mathbb{R}^n)} \\ &\leqslant C R^{2q} \| f \|^q_{L^q(\mathbb{R}^n)}, \end{aligned}$$

so that

$$\left\| e^{it\Delta} f \right\|^q_{L^q(\mathbb{R}^n, L^r_t(I))} \leqslant C \sum_l \left\| e^{it\Delta} f_l \right\|^q_{L^q(Q_l, L^r_t(I))} + C_N R^{-N} \| f \|^q_{L^q(\mathbb{R}^n)}. \tag{15}$$

By translation invariance and the hypothesis,

$$\left\| e^{it\Delta} f_l \right\|_{L^q(Q_l, L^r_t(I))} \leqslant C R^s \| f_l \|_{L^{p_0}(\mathbb{R}^n)}$$

for all $l \in \mathbb{N}$, and combining this with (13),

$$\left\| e^{it\Delta} f_l \right\|_{L^q(Q_l, L_t^r(I))} \leqslant C R^{s+2n(\frac{1}{p_0}-\frac{1}{p_1})+\varepsilon} \|\widetilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}. \tag{16}$$

On the other hand, as supp $\hat{f} \subset \mathbb{B}^n$ and $p_1 \leqslant q$, by Bernstein's inequality,

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{17}$$

Substituting (16) and (17) into (15), we see that

$$\left\| e^{it\Delta} f \right\|^q_{L^q(\mathbb{R}^n, L_t^r(I))} \leqslant C R^{q(s+2n(\frac{1}{p_0}-\frac{1}{p_1})+\varepsilon)} \sum_l \|\widetilde{f}_l\|^q_{L^{p_1}(\mathbb{R}^n)} + C_N R^{-N} \|f\|^q_{L^{p_1}(\mathbb{R}^n)}.$$

Finally, as $q \geqslant p_1$, by convexity,

$$\sum_l \|\widetilde{f}_l\|^q_{L^{p_1}(\mathbb{R}^n)} \leqslant \left(\sum_l \|\widetilde{f}_l\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \right)^{q/p_1},$$

so that, by (14), we can sum to obtain the required bound. □

4. Restriction implies local smoothing

We denote by $LS(q \to q)$ the estimate

$$\left\| e^{it\Delta} f \right\|_{L^q(\mathbb{R}^n \times [0,1])} \leqslant C_\alpha \|f\|_{q,\alpha}$$

for all $\alpha > 2n(\frac{1}{2} - \frac{1}{q}) - \frac{2}{q}$.

We denote by $R^*(p \to q)$ the (adjoint) restriction estimate

$$\left\| e^{it\Delta} f \right\|_{L^q(\mathbb{R}^{n+1})} \leqslant C\|\hat{f}\|_{L^p(\mathbb{R}^n)},$$

where $p' = \frac{nq}{n+2}$. It is conjectured that $R^*(p \to q)$ holds for all $q > 2 + \frac{2}{n}$, and it has been proven in the affirmative by Tao [32] in the range $q > 2 + \frac{4}{n+1}$.

Theorem 9. $R^*(p \to q) \Rightarrow LS(q \to q)$.

Proof. Suppose first that supp $\hat{f} \subset \mathbb{B}^n$. Considering (2), we see that $e^{it\Delta} f$ can be viewed as the convolution of f with the Fourier transform of $e^{-4\pi^2 i|\xi|^2 t}$, so that we can also write

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{\frac{i|x-y|^2}{4t}}\, dy. \tag{18}$$

As in [5], we 'complete the square' in (2), and compare the representations, so that

$$\left| e^{it\Delta} f(x) \right| = \left| \frac{c^{n/2}}{t^{n/2}} e^{-ic^2 \frac{1}{t}\Delta} \hat{f}\left(\frac{cx}{t}\right) \right|. \tag{19}$$

Making a 'pseudo-conformal' change of variables, we have

$$\|e^{it\Delta}f\|_{L^q(B_{R^2}\times[R^2/2,R^2])} \leqslant CR^{-n}\left\|e^{i\frac{1}{t}\Delta}\hat{f}\left(\frac{\cdot}{t}\right)\right\|_{L^q(B_{R^2}\times[R^2/2,R^2])}$$
$$\leqslant CR^{-n+\frac{2(n+2)}{q}}\|e^{it\Delta}\hat{f}\|_{L^q(\mathbb{B}^{n+1})}.$$

Now, by hypothesis,

$$\|e^{it\Delta}\hat{f}\|_{L^q(\mathbb{B}^{n+1})} \leqslant C\|f\|_{L^p(\mathbb{R}^n)},$$

where $p' = \frac{nq}{n+2}$, so that

$$\|e^{it\Delta}f\|_{L^q(B_{R^2}\times[R^2/2,R^2])} \leqslant CR^{-n+\frac{2(n+2)}{q}}\|f\|_{L^p(\mathbb{R}^n)}.$$

Thus, by Lemma 8,

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[R^2/2,R^2])} \leqslant CR^{-n+\frac{2(n+2)}{q}+2n(\frac{1}{p}-\frac{1}{q})+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$
$$= CR^{n(1-\frac{2}{q})+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}.$$

Finally we scale, so that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[2^{-k},2^{-k+1}])} \leqslant C2^{-\frac{2k}{q}}R^{n(1-\frac{2}{q})-\frac{2}{q}+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$

whenever $\operatorname{supp}\hat{f} \subset B_{2^kR}$ with $k \geqslant 0$. Summing, we see that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[0,1])} \leqslant CR^{n(1-\frac{2}{q})-\frac{2}{q}+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$

whenever $\operatorname{supp}\hat{f} \subset B_R$, and the proof is completed with the standard Littlewood–Paley arguments. □

5. Equivalence of the conjectures for the maximal operator

We consider the local bound,

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{B}^n,L^r_t[0,1])} \leqslant C_s\|f\|_{H^s(\mathbb{R}^n)}, \tag{20}$$

and the global bound,

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{R}^n,L^r_t[0,1])} \leqslant C_s\|f\|_{H^s(\mathbb{R}^n)}. \tag{21}$$

Theorem 10. *Let $q, r \geqslant 2$. Then* (20) *holds for all $s > s_0$ if and only if* (21) *holds for all $s > 2s_0 - n(\frac{1}{2}-\frac{1}{q}) + \frac{2}{r}$.*

Letting $q = 2$ and $r = \infty$, we obtain Theorem 3. Letting $q = r = 2$, we see the equivalence up to endpoints of the conservation of charge and the local smoothing theorem of Sjölin, Vega, and Constantin and Saut, mentioned in Section 1.

We will need the following lemma due to Lee.

Lemma 11. *(See [19].) Let $q, r \geqslant 2$. Suppose that*

$$\left\|e^{it\Delta} f\right\|_{L_x^q(B_R, L_t^r[0,R])} \leqslant C R^s \|f\|_{L^2(\mathbb{R}^n)},$$

whenever $R \gg 1$, and f is frequency supported in $\mathbb{A}^n$. Then for all $\varepsilon > 0$,

$$\left\|e^{it\Delta} f\right\|_{L_x^q(B_R, L_t^r[0,R^2])} \leqslant C_\varepsilon R^{s+\varepsilon} \|f\|_{L^2(\mathbb{R}^n)}.$$

By the standard Littlewood–Paley arguments and scaling, to prove Theorem 10, it will suffice to prove the following theorem, where (ii) and (iii) correspond to (20) and (21), respectively.

Theorem 12. *Let $q, r \geqslant 2$, and consider functions f which are frequency supported in $\mathbb{A}^n$. Then the following bounds are equivalent*:

(i) $\|e^{it\Delta} f\|_{L_x^q(B_R, L_t^r[0,R])} \leqslant C R^s \|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$,*
(ii) $\|e^{it\Delta} f\|_{L_x^q(B_R, L_t^r[0,R^2])} \leqslant C R^s \|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$,*
(iii) $\|e^{it\Delta} f\|_{L_x^q(\mathbb{R}^n, L_t^r[0,R^2])} \leqslant C R^{2s} \|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$.*

Proof. By changing variables $R \to R^{1/2}$ in (iii), we see that (ii) and (iii) trivially imply (i). Thus, it will suffice to show that (i) implies (ii) and (iii). Now, (i) implies (ii) is precisely the content of Lemma 11. Similarly, by changing variables and letting $p_0 = p_1 = 2$ and $I = [0, R^2]$ in Lemma 8, we see that (i) implies (iii). □

By the local result of Lee [19], mentioned in Section 1, and Theorem 10 with q and r taken to be 2 and ∞, respectively, we obtain the following corollary.

Corollary 4. *For all $s > 3/4$, there exists a constant C_s such that*

$$\Big\| \sup_{0<t<1} \left|e^{it\Delta} f\right| \Big\|_{L^2(\mathbb{R}^2)} \leqslant C_s \|f\|_{H^s(\mathbb{R}^2)}.$$

We note that as (21) cannot hold for any value of s when $q < 2$ (see for example [25]), there can be no such equivalence when $q < 2$. Letting $r = \infty$, we also see that the necessary conditions for (21) to hold given in [25], are equivalent to the necessary conditions for (20) to hold given in [29].

6. The nonelliptic Schrödinger equation

The generalised Schrödinger equation, $i\partial_t u + \phi(D)u = 0$, where $\widehat{\phi(D)u} = \phi(\xi)\hat{u}(\xi)$ and $\phi(\xi)$ is real, has solution $e^{it\phi(D)} f$ which can be formally written as

$$e^{it\phi(D)} f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi + it\phi(\xi)}\, d\xi.$$

In the local case, Kenig, Ponce and Vega [15] showed that if there are at most $N \in \mathbb{N}$ solutions to

$$\phi(\xi_1, \dots, \xi_k, x, \xi_{k+1}, \dots, \xi_{n-1}) = r \tag{22}$$

for all $\xi \in \mathbb{R}^{n-1}$, $r \in \mathbb{R}$, $k = 0, \dots, n-1$, and

$$\frac{|\phi(\xi)|}{|\nabla\phi(\xi)|} \leqslant C\big(1 + |\xi|^2\big)^{s_0},$$

then for $s > s_0$,

$$\Big\| \sup_{0<t<1} \big|e^{it\phi(D)} f\big| \Big\|_{L^2(\mathbb{B}^n)} \leqslant C_{N,s} \|f\|_{H^s(\mathbb{R}^n)}. \tag{23}$$

In the global case, Cowling [8] showed that if $|\phi(\xi)| \leqslant C(1 + |\xi|^2)^{s_0}$, then for $s > s_0$,

$$\Big\| \sup_{0<t<1} \big|e^{it\phi(D)} f\big| \Big\|_{L^2(\mathbb{R}^n)} \leqslant C_s \|f\|_{H^s(\mathbb{R}^n)}. \tag{24}$$

In particular, both these results hold for smooth ϕ that are homogeneous of degree $m \geqslant 1$. The injectivity condition (22) is fulfilled and

$$\frac{|\phi(\xi)|}{|\nabla\phi(\xi)|} \leqslant C\big(1 + |\xi|^2\big)^{1/2},$$

so that (23) holds for all $s > 1/2$. On the other hand $|\phi(\xi)| \leqslant C(1 + |\xi|^2)^{m/2}$, so that (24) holds for all $s > m/2$.

For such ϕ, these results are again equivalent. Indeed, for any ϕ satisfying $|D^\alpha\phi(\xi)| \leqslant C_0|\xi|^{m-|\alpha|}$, where $|\alpha| \leqslant 2$, and $|\nabla\phi(\xi)| \geqslant C_0^{-1}|\xi|^{m-1}$, there is an equivalence.

We consider the local bound,

$$\big\|e^{it\phi(D)} f\big\|_{L^q_x(\mathbb{B}^n, L^r_t[0,1])} \leqslant C_s \|f\|_{H^s(\mathbb{R}^n)}, \tag{25}$$

and the global bound,

$$\big\|e^{it\phi(D)} f\big\|_{L^q_x(\mathbb{R}^n, L^r_t[0,1])} \leqslant C_s \|f\|_{H^s(\mathbb{R}^n)}. \tag{26}$$

By scaling, it will suffice to consider $e^{it\phi_R(D)} f$ defined by

$$e^{it\phi_R(D)} f = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi + tR^{-m}\phi(R\xi)}\, d\xi,$$

where $\phi_R = R^{-m}\phi(R\cdot)$, $\hat{f}$ is supported in $\mathbb{A}^n$ and $t \in [0, R^m]$. It is easy to see that $|D^\alpha\phi_R(\xi)| \leqslant C_0|\xi|^{m-|\alpha|}$ and $|\nabla\phi_R(\xi)| \geqslant C_0^{-1}|\xi|^{m-1}$ for all R, so that $|\nabla\phi_R(v_j)| \approx |v_j|^{m-1}$.

Now, Lemma 6 generalises to ϕ such that $|D^\alpha\phi(\xi)| \leqslant C_0|\xi|^{m-|\alpha|}$ for $|\alpha| \leqslant 2$ (see [18]). The $2v_j$ is replaced by $\nabla\phi(v_j)$, and the constants depend only on C_0.

To prove versions of Lemmas 7 and 8 with $e^{it\phi_R(D)}f$ in place of $e^{it\Delta}f$, only the numerology changes. The important point is that the tubes make angles with the spatial plane which are uniformly bounded away from zero, which we have insured by requiring that $|\nabla\phi_R(\xi)| \leqslant C_0$ for all $\xi \in \mathbb{A}^n$.

Lemma 11 can be similarly generalised. The important point there is that the tubes make angles with the t-axis which are uniformly bounded away from zero, which we have insured by requiring that $|\nabla\phi_R(\xi)| \geqslant \frac{1}{2}C_0^{-1}$ for all $\xi \in \mathbb{A}^n$.

Thus, considering f frequency supported in $\mathbb{A}^n$, and $q, r \geqslant 2$, the following bounds are equivalent:

(i) $\|e^{it\phi_R(D)}f\|_{L_x^q(B_R, L_t^r[0,R])} \leqslant CR^s\|f\|_{L^2(\mathbb{R}^n)}$ for all $R \gg 1$ and $s > s_0$,
(ii) $\|e^{it\phi_R(D)}f\|_{L_x^q(B_R, L_t^r[0,R^m])} \leqslant CR^s\|f\|_{L^2(\mathbb{R}^n)}$ for all $R \gg 1$ and $s > s_0$,
(iii) $\|e^{it\phi_R(D)}f\|_{L_x^q(\mathbb{R}^n, L_t^r[0,R^m])} \leqslant CR^{ms}\|f\|_{L^2(\mathbb{R}^n)}$ for all $R \gg 1$ and $s > s_0$.

By scaling and the usual arguments of Littlewood and Paley, this yields the following theorem.

Theorem 13. *Let $q, r \geqslant 2$. Suppose that $|D^\alpha\phi(\xi)| \leqslant C_0|\xi|^{m-|\alpha|}$ and $|\nabla\phi(\xi)| \geqslant C_0^{-1}|\xi|^{m-1}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $|\alpha| \leqslant 2$ and $m > 1$. Then* (25) *holds for all $s > s_0$ if and only if* (26) *holds for all $s > ms_0 - (m-1)(n(\frac{1}{2} - \frac{1}{q}) - \frac{m}{r})$.*

A corollary of this and the generalised result of Lee [19], is that Corollary 4 also holds for the generalised Schrödinger equation; where $|D^\alpha\phi(\xi)| \leqslant C|\xi|^{2-|\alpha|}$ and $|\nabla\phi(\xi)| \geqslant C^{-1}|\xi|$, and the Hessian of ϕ has two nonzero eigenvalues of the same sign.

For completeness, we note that when $m \leqslant 1$, we no longer need Lemma 11, so that we have the following theorem.

Theorem 14. *Let $q \geqslant 2$ and suppose that $|D^\alpha\phi(\xi)| \leqslant C_0|\xi|^{m-|\alpha|}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $|\alpha| \leqslant 2$ and $m \leqslant 1$. Then* (25) *holds for all $s > s_0$ if and only if* (26) *holds for all $s > s_0$.*

In particular, we consider $\phi(\xi) = (2\pi|\xi|)^m$ so that $\phi(D) = (-\Delta)^{m/2}$ with $m \in (0,1)$. The conditions of Theorem 14 are fulfilled, and we see that global bounds are equivalent to local bounds.

Finally, we consider the nonelliptic Schrödinger equation; where ϕ is defined by $\phi(\xi) = -4\pi^2(\xi_1^2 - \xi_2^2 \pm \xi_3^2 \pm \cdots \pm \xi_n^2)$, and

$$\phi(D) = \Box = \partial_{x_1}^2 - \partial_{x_2}^2 \pm \partial_{x_3}^2 \pm \cdots \pm \partial_{x_n}^2.$$

Note that the conditions of Theorem 13 are fulfilled with $m = 2$. Vargas, Vega and the author [26] showed that, in this case, the bound of Kenig, Ponce and Vega is almost sharp, in the sense that

$$\Big\| \sup_{0<t<1} \big|e^{it\Box}f\big| \Big\|_{L^2(\mathbb{B}^n)} \leqslant C_s\|f\|_{H^s(\mathbb{R}^n)}$$

does not hold when $s < 1/2$.

Therefore, by Theorem 13, we see that the bound of Cowling is similarly sharp, and we state this as a corollary.

Corollary 5. *For all $s > 1$, there exists a constant C_s such that*

$$\Big\| \sup_{0<t<1} \big| e^{it\Box} f \big| \Big\|_{L^2(\mathbb{R}^n)} \leqslant C_s \|f\|_{H^s(\mathbb{R}^n)},$$

and this is not true when $s < 1$.

Theorem 9 also generalises to the nonelliptic case, so the well-known Stein–Tomas–Strichartz estimate yields an almost sharp local smoothing estimate in the range $q \geqslant 2 + 4/n$. In two spatial dimensions, by a restriction theorem independently due to Vargas [34] and Lee [18], we have the result in the range $q \geqslant 10/3$.

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