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Boundary feedback stabilization of the undamped Timoshenko beam with both ends free ☆

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Abstract

In this paper, we study a boundary feedback system of a class of nonuniform undamped Timoshenko beam with both ends free. We give some sufficient conditions and some necessary conditions for the system to have exponential stability. Our method is based on the operator semigroup technique, the multiplier technique, and the contradiction argument of the frequency domain method.

Keywords: C_0-semigroup; Timoshenko beam; Exponential stability; Frequency domain method

1. Introduction

In recent years, there has been much interest in the problems of stability for elastic beam. The exponential stability of the boundary feedback system of a Timoshenko beam with one or two ends fixed has been studied extensively during the past decade. But little attention has been paid to the case of the beam with both ends free. In this paper, we shall consider the system of nonhomogeneous undamped Timoshenko beam with both ends free. More precisely, we consider the following initial and boundary value problem:

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$$
\begin{cases}
\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x}\left(K\left(\varphi - \frac{\partial w}{\partial x}\right)\right) = 0, \quad 0 < x < l,\ t > 0, \\
I_\rho \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial}{\partial x}\left(EI \frac{\partial \varphi}{\partial x}\right) + K\left(\varphi - \frac{\partial w}{\partial x}\right) = 0, \quad 0 < x < l,\ t > 0, \\
K\left(\varphi - \frac{\partial w}{\partial x}\right)\big|_{x=0} = -EI \frac{\partial \varphi}{\partial x}\big|_{x=0} = 0, \quad t > 0, \\
K\left(\varphi - \frac{\partial w}{\partial x}\right)\big|_{x=l} = k_1 \frac{\partial w}{\partial t}(l,t) + k_2 w(l,t), \quad t > 0, \\
-EI \frac{\partial \varphi}{\partial x}\big|_{x=l} = k_3 \frac{\partial \varphi}{\partial t}(l,t) + k_4 \varphi(l,t), \quad t > 0, \\
w(x,0) = w_0(x),\ w_t(x,0) = z_0(x),\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \psi_0(x), \\
\quad 0 < x < l,
\end{cases}
\tag{1.1}
$$

here a nonuniform beam of length l moves in w–x plane, $\rho(x)$ is the mass density, $w(x,t)$, the deflection of the beam from its equilibrium, $\varphi(x,t)$, the total rotatory angle of the beam at x, $I_\rho(x)$, the mass moment of inertia, $EI(x)$ the rigidity coefficient of the cross section, $K(x)$ is the shear modulus of elasticity, and $k_j \geqslant 0$ $(j = 1,2,3,4)$ the feedback coefficients. We refer to [3,4,10,11] for the precise description of the problem and for more technique details.

In this paper, we are interested in the following feedback stabilization problem: Under what conditions on k_j $(j = 1,2,3,4)$ does the energy $E(t)$ (see (2.2) for its definition) of the system (1.1) exponentially decay?

Our main approach is based on the operator semigroup technique, the multiplier technique with contradiction argument of a frequency domain method. Recall that multiplier techniques were developed in the work of Lagnese [6], Liu and Liu [7,8] for various PDEs and control problems. On the other hand, the frequency domain method is based on the boundedness on the imaginary axis of resolvent of a C_0-semigroup generator to establish the exponential stability of the C_0-semigroup on Hilbert space (see Huang [5]).

The plan of this paper is as follows. In Section 2, we will state our main results. In Section 3, we show the well-posedness of the system and derive some spectral properties of the underlying semigroup. The proof of the main results is given in Section 4.

2. Statement of the main results

Throughout this paper, we need the following natural hypothesis:

$$
\begin{cases}
\rho(\cdot), I_\rho(\cdot) \in C^{0,1}[0,l], \qquad K(\cdot), EI(\cdot) \in C^1[0,l], \\
\rho(x), I_\rho(x), K(x), EI(x) \geqslant C > 0, \quad x \in [0,l],
\end{cases}
\tag{2.1}
$$

where C is a positive constant.

Denote the energy of system (1.1) by

$$
E(t) = \frac{1}{2}\left[\int_0^l \left(K\left|\varphi - \frac{\partial w}{\partial x}\right|^2 + EI\left|\frac{\partial \varphi}{\partial x}\right|^2 + \rho\left|\frac{\partial w}{\partial t}\right|^2 + I_\rho\left|\frac{\partial \varphi}{\partial t}\right|^2\right) dx \right.
$$

$$
\left. + k_2\left|w(l,t)\right|^2 + k_4\left|\varphi(l,t)\right|^2\right], \tag{2.2}
$$

where $k_2|w(l,t)|^2 + k_4|\varphi(l,t)|^2$ represents the energy of the rigid motion of elastic system. Simple calculations yield that

$$\begin{aligned}\frac{dE}{dt}(t) = \int_0^l \Big[& K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\cdot\Big(\frac{\partial\varphi}{\partial t} - \frac{\partial^2 w}{\partial x\partial t}\Big) + EI\frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial x\partial t} \\ & + \rho\frac{\partial w}{\partial t}\cdot\frac{\partial^2 w}{\partial t^2} + I_\rho\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial t^2}\Big]dx \\ & + k_2 w(l,t)\frac{\partial w}{\partial t}(l,t) + k_4\varphi(l,t)\frac{\partial\varphi}{\partial t}(l,t) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & \int_0^l \Big(\rho\frac{\partial w}{\partial t}\cdot\frac{\partial^2 w}{\partial t^2} + I_\rho\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial t^2}\Big)dx \\ & = \int_0^l \Big\{-\frac{\partial w}{\partial t}\cdot\frac{\partial}{\partial x}\Big[K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\Big] + \frac{\partial\varphi}{\partial t}\cdot\Big[\frac{\partial}{\partial x}\Big(EI\frac{\partial\varphi}{\partial x}\Big) - K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\Big]\Big\}dx \\ & = -\frac{\partial w}{\partial t}\Big[K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\Big]\Big|_0^l + \int_0^l K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\frac{\partial^2 w}{\partial x\partial t}dx \\ & \quad + \frac{\partial\varphi}{\partial t}\cdot EI\frac{\partial\varphi}{\partial x}\Big|_0^l - \int_0^l EI\frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial x\partial t}dx - \int_0^l \frac{\partial\varphi}{\partial t}\cdot K\Big(\varphi - \frac{\partial w}{\partial x}\Big)dx \\ & = -\frac{\partial w}{\partial t}\Big[K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\Big]\Big|_{x=l} + \int_0^l K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\cdot\Big(\frac{\partial^2 w}{\partial x\partial t} - \frac{\partial\varphi}{\partial t}\Big)dx \\ & \quad + \frac{\partial\varphi}{\partial t}\cdot EI\frac{\partial\varphi}{\partial x}\Big|_{x=l} - \int_0^l EI\frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial x\partial t}dx \\ & = -\frac{\partial w}{\partial t}(l,t)\cdot\Big(k_1\frac{\partial w}{\partial t}(l,t) + k_2 w(l,t)\Big) - \frac{\partial\varphi}{\partial t}(l,t)\cdot\Big(k_3\frac{\partial\varphi}{\partial t}(l,t) + k_4\varphi(l,t)\Big) \\ & \quad - \int_0^l K\Big(\varphi - \frac{\partial w}{\partial x}\Big)\cdot\Big(\frac{\partial\varphi}{\partial t} - \frac{\partial^2 w}{\partial x\partial t}\Big)dx - \int_0^l EI\frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial x\partial t}dx. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\frac{dE}{dt}(t) = -k_1\Big|\frac{\partial w}{\partial t}(l,t)\Big|^2 - k_3\Big|\frac{\partial\varphi}{\partial t}(l,t)\Big|^2, \tag{2.5}$$

which implies that $k_1 \geqslant 0$ and $k_3 \geqslant 0$ are necessary for the energy $E(t)$ to be not increasing.

For simplicity, we will denote $\frac{\partial u}{\partial x}$ by u'. Let $L^2_\rho(0,l)$ and $L^2_{I_\rho}(0,l)$ with the norm $\|u\|_{L^2_\rho(0,l)} = (\int_0^l \rho|u|^2\,dx)^{\frac{1}{2}}$ and $\|u\|_{L^2_{I_\rho}(0,l)} = (\int_0^l I_\rho|u|^2\,dx)^{\frac{1}{2}}$, respectively. Let $H = L^2_\rho(0,l)\times L^2_{I_\rho}(0,l)$ with the norm

$$\|(u,v)\|_H = \left(\|u\|^2_{L^2_\rho(0,l)} + \|v\|^2_{L^2_{I_\rho}(0,l)}\right)^{\frac{1}{2}}$$

and let $V = H^1(0,l)\times H^1(0,l)$ with the equivalent norm

$$\|(u,v)\|_V = \left(k_2|u(l)|^2 + k_4|v(l)|^2 + \int_0^l \left(K|v-u'|^2 + EI|v'|^2\right)dx\right)^{\frac{1}{2}} \quad (k_2, k_4 > 0),$$

where $H^k(0,l)$ is the Sobolev space of order k (see [1]).

Define $\mathcal{H} = V\times H$ with the norm

$$\|(w,\varphi,z,\psi)\|_{\mathcal{H}} = \left(\|(w,\varphi)\|^2_V + \|(z,\psi)\|^2_H\right)^{\frac{1}{2}}.$$

To formulate (1.1) as an abstract Cauchy problem on $\mathcal{H}$, we define a linear operator $\mathcal{A}$ as follows:

$$\begin{aligned} D(\mathcal{A}) = \{&(w,\varphi,z,\psi) \mid (w,\varphi)\in H^2(0,l)\times H^2(0,l),\ (z,\psi)\in V,\ K(\varphi-w')|_{x=0}=0,\\ &-EI\varphi'|_{x=0}=0,\ K(\varphi-w')|_{x=l} = k_1 z(l) + k_2 w(l),\\ &-EI\varphi'|_{x=l} = k_3\psi(l) + k_4\varphi(l)\} \end{aligned} \tag{2.6}$$

and

$$\mathcal{A}(w,\varphi,z,\psi) = \left(z,\psi,-\rho^{-1}\left(K(\varphi-w')\right)', I_\rho^{-1}\left[(EI\varphi')' - K(\varphi-w')\right]\right),\quad (w,\varphi,z,\psi)\in D(\mathcal{A}).$$

Then the system (1.1) can be formulated as following Cauchy problem on $\mathcal{H}$:

$$\frac{dY}{dt} = \mathcal{A}Y, \quad Y(0) = Y_0,$$

where $Y = (w,\varphi,z,\psi)$ and $Y_0 = (w_0,\varphi_0,z_0,\psi_0)$.

Next, it is easy to see that $k_2 \neq 0$ and $k_4 \neq 0$ are necessary for the energy $E(t)$ to uniform exponentially decay. In fact, if $k_2 = 0$, then $w(x) = 1$, $\varphi(x) = 0$, $z(x) = 0$, $\psi(x) = 0$, $x\in[0,l]$ is an eigenvector belonging to eigenvalue $\lambda = 0$. Likewise, if $k_4 = 0$, then $w(x) = x - l$, $\varphi(x) = 1$, $z(x) = 0$, $\psi(x) = 0$, $x \in [0,l]$ is an eigenvector belonging to eigenvalue $\lambda = 0$.

Now we can state our main results as follows.

Theorem 2.1. *Let* (2.1) *hold and* $k_j > 0$, $j = 1,2,3,4$. *Then the* C_0*-semigroup* $e^{t\mathcal{A}}$ *is uniformly exponentially stable*; *i.e., there exist positive constants* M, ω *such that*

$$\|e^{t\mathcal{A}}\| \leqslant Me^{-\omega t}, \quad t \geqslant 0.$$

Theorem 2.2. *Assume that* $k_2, k_4 > 0$ *and the* C_0*-semigroup* $e^{t\mathcal{A}}$ *decays uniformly exponentially. Then* $k_1 > 0$ *and* $k_3 > 0$.

3. Preliminaries

In this section, we will prove that $\mathcal{A}$ generates a contraction C_0-semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$, which shows the well-posedness of system (1.1), and give some spectral properties of the generator $\mathcal{A}$.

Theorem 3.1. *Let $k_j > 0$ $(j = 1, 2, 3, 4)$. Then $\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$.*

Proof. It is easy to see that $\mathcal{A}$ is density defined in $\mathcal{H}$. Furthermore, for any $(w, \varphi, z, \psi) \in D(\mathcal{A})$, integrating by parts, we have

$$\operatorname{Re}\langle \mathcal{A}(w, \varphi, z, \psi), (w, \varphi, z, \psi)\rangle_{\mathcal{H}} = -k_1 |z(l)|^2 - k_3 |\psi(l)|^2, \tag{3.1}$$

which implies that $\mathcal{A}$ is dissipative in $\mathcal{H}$.

Finally, we show that $\lambda = 0 \in \rho(\mathcal{A})$ (the resolvent set of $\mathcal{A}$). For any $(f_1, g_1, f_2, g_2) \in \mathcal{H}$, we are going to solve the following equation:

$$\mathcal{A}(w, \varphi, z, \psi) = (f_1, g_1, f_2, g_2), \quad (w, \varphi, z, \psi) \in D(\mathcal{A}). \tag{3.2}$$

This implies

$$\begin{cases} z = f_1, \\ \psi = g_1, \\ (K(\varphi - w'))' = -\rho f_2, \\ (EI\varphi')' - K(\varphi - w') = I_\rho g_2. \end{cases} \tag{3.3}$$

Integrating from 0 to x and using boundary conditions at $x = 0$, we have

$$K(\varphi - w') = -\int_0^x \rho(y) f_2(y)\, dy. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$(EI\varphi')' = -\int_0^x \rho(y) f_2(y)\, dy + I_\rho g_2 \tag{3.5}$$

and

$$EI\varphi' = -\int_0^x \int_0^y \rho(y_1) f_2(y_1)\, dy_1\, dy + \int_0^x I_\rho(y) g_2(y)\, dy, \tag{3.6}$$

or equivalently,

$$\varphi'(x) = -\frac{1}{EI(x)} \int_0^x \int_0^y \rho(y_1) f_2(y_1)\, dy_1\, dy + \frac{1}{EI(x)} \int_0^x I_\rho(y) g_2(y)\, dy, \quad x \in [0, l]. \tag{3.7}$$

Therefore, we have

$$\varphi(x)=\varphi(l)+\int_x^l \frac{1}{EI(\tau)}\int_0^\tau\int_0^y \rho(y_1)f_2(y_1)\,dy_1\,dy\,d\tau$$
$$-\int_x^l \frac{1}{EI(\tau)}\int_0^\tau I_\rho(y)g_2(y)\,dy\,d\tau,\quad x\in[0,l]. \tag{3.8}$$

Let $x=l$ in (3.6); we can obtain

$$k_3\psi(l)+k_4\varphi(l)=\int_0^l\int_0^y \rho(y_1)f_2(y_1)\,dy_1\,dy-\int_0^l I_\rho(y)g_2(y)\,dy,$$

which, using (3.3), yields

$$\varphi(l)=-\frac{1}{k_4}\left(k_3g_1(l)+\int_0^l I_\rho(y)g_2(y)\,dy\right)+\frac{1}{k_4}\int_0^l\int_0^y \rho(y_1)f_2(y_1)\,dy_1\,dy. \tag{3.9}$$

From (3.8), (3.9), we can assert that

$$\varphi(x)=-\frac{1}{k_4}\left(k_3g_1(l)+\int_0^l I_\rho(y)g_2(y)\,dy\right)+\frac{1}{k_4}\int_0^l\int_0^y \rho(y_1)f_2(y_1)\,dy_1\,dy$$
$$+\int_x^l \frac{1}{EI(\tau)}\int_0^\tau\int_0^y \rho(y_1)f_2(y_1)\,dy_1\,dy\,d\tau$$
$$-\int_x^l \frac{1}{EI(\tau)}\int_0^\tau I_\rho(y)g_2(y)\,dy\,d\tau,\quad x\in[0,l]. \tag{3.10}$$

From (3.4), we can obtain

$$Kw'=K\varphi+\int_0^x \rho(y)f_2(y)\,dy, \tag{3.11}$$

and consequently,

$$w(x)=w(l)-\int_x^l \varphi(y)\,dy-\int_x^l \frac{1}{K(\tau)}\int_0^\tau \rho(y)f_2(y)\,dy\,d\tau,\quad x\in[0,l]. \tag{3.12}$$

Let $x=l$ in (3.4); we can obtain

$$k_1z(l)+k_2w(l)=-\int_0^l \rho(y)f_2(y)\,dy, \tag{3.13}$$

which, using (3.3), yields

$$w(l)=-\frac{1}{k_2}\left(k_1f_1(l)+\int_0^l \rho(y)f_2(y)\,dy\right). \tag{3.14}$$

From (3.10), (3.12), and (3.14), we can assert that

$$
\begin{aligned}
w(x) = {} & -\frac{1}{k_2}\left(k_1 f_1(l) + \int_0^l \rho(y) f_2(y)\,dy\right) - \int_x^l \frac{1}{K(\tau)} \int_0^\tau \rho(y) f_2(y)\,dy\,d\tau \\
& + \frac{1}{k_4}(l-x)\left(k_3 g_1(l) + \int_0^l I_\rho(y) g_2(y)\,dy\right) - \frac{1}{k_4}(l-x) \int_0^l \int_0^y \rho(y_1) f_2(y_1)\,dy_1\,dy \\
& - \int_x^l \int_\tau^l \frac{1}{EI(\tau)} \int_0^{y_1} \int_0^{y_2} \rho(y) f_2(y)\,dy\,dy_2\,dy_1\,d\tau \\
& + \int_x^l \int_\tau^l \frac{1}{EI(\tau)} \int_0^{y_1} \int_0^{y_2} I_\rho(y) g_2(y)\,dy\,dy_2\,dy_1\,d\tau, \quad x \in [0,l]. \qquad (3.15)
\end{aligned}
$$

Since hypothesis (2.1) is satisfied, we can easily deduce that $(w,\varphi) \in V$ and that $(w,\varphi,z,\psi) \in D(\mathcal{A})$ is the unique solution of (3.2) $z = f_1$, $\psi = g_1$. Hence, $\lambda = 0 \in \rho(\mathcal{A})$. Finally, from the above discussion and the Lumer–Phillips theorem [9, Theorem 1.4.3], it follows that $\mathcal{A}$ generates a contraction C_0-semigroup. The proof has been completed. □

Proposition 3.2. *Assume that* $k_j > 0$, $j = 1,2,3,4$. *Then* $i\mathbf{R} \subset \rho(\mathcal{A})$.

Proof. From the proof of Theorem 3.1 we have $\lambda = 0 \in \rho(\mathcal{A})$ and we can prove

$$\{\lambda \in \sigma(\mathcal{A}) \mid \operatorname{Im}\lambda \neq 0\} \subset \sigma_P(\mathcal{A})$$

in a similar way as in [2, Lemma 4.1]. Therefore it suffices to show $i\omega \,\overline{\in}\, \sigma_P(\mathcal{A})$. Indeed, if it is not true, then there exists $\omega \in \mathbf{R}$, $\omega \neq 0$ such that $i\omega \in \sigma_P(\mathcal{A})$. Hence, there exists $(w,\varphi,z,\psi) \in D(\mathcal{A})$, $(w,\varphi,z,\psi) \neq 0$, such that

$$(i\omega - \mathcal{A})(w,\varphi,z,\psi) = 0, \qquad (3.16)$$

which implies

$$\operatorname{Re}\langle (i\omega - \mathcal{A})(w,\varphi,z,\psi), (w,\varphi,z,\psi)\rangle_{\mathcal{H}} = k_1 |z(l)|^2 + k_3 |\psi(l)|^2 = 0.$$

Using $k_1, k_3 > 0$, we conclude that

$$z(l) = \psi(l) = 0. \qquad (3.17)$$

From (3.16), we can obtain

$$
\begin{cases}
z = i\omega w, \\
\psi = i\omega\varphi, \\
(K(\varphi - w'))' = -i\omega\rho z, \\
(EI\varphi')' - K(\varphi - w') = i\omega I_\rho \psi.
\end{cases} \qquad (3.18)
$$

From (3.17) and (3.18), using $\omega \neq 0$, we can obtain

$$w(l) = \varphi(l) = 0. \qquad (3.19)$$

From (2.6), (3.17), and (3.19), we can easily obtain

$$\begin{cases} K(\varphi - w')|_{x=0} = K(\varphi - w')|_{x=l} = 0, \\ -EI\varphi'|_{x=0} = -EI\varphi'|_{x=l} = 0. \end{cases} \tag{3.20}$$

From (3.18) and (3.20), we can conclude that

$$\begin{cases} (K(\varphi - w'))' + \omega^2 \rho w = 0, \\ (EI\varphi')' - K(\varphi - w') - \omega^2 I_\rho \varphi = 0, \\ K(\varphi - w')|_{x=0} = K(\varphi - w')|_{x=l} = 0, \\ -EI\varphi'|_{x=0} = -EI\varphi'|_{x=l} = 0. \end{cases} \tag{3.21}$$

The uniqueness theorem of ODEs shows that $(w, \varphi, z, \psi) \equiv 0$. This is in contradiction with $(w, \varphi, z, \psi) \neq 0$, and the proof is completed. □

4. Proof of the main results

Proof of Theorem 2.1. Clearly, if $w = w(x,t)$, $\varphi = \varphi(x,t)$ is the solution of the system (1.1), then

$$(w, \varphi, z, \psi) = e^{t\mathcal{A}}(w_0, \varphi_0, z_0, \psi_0) \quad \text{and} \quad \|(w, \varphi, z, \psi)\|_{\mathcal{H}}^2 = 2E(t), \quad t > 0.$$

Hence, the uniformly exponential decay of the energy $E(t)$ is equivalent to the uniform exponential stability of C_0-semigroup $e^{t\mathcal{A}}$. It follows from Proposition 3.2 and the frequency domain results (see [5]) that we need only to prove

$$\sup_{\lambda \in i\mathbf{R}} \|(\lambda - \mathcal{A})^{-1}\| < +\infty. \tag{4.1}$$

If (4.1) is false, from the resonance theorem, there are two sequences $\{\lambda_n\} \subset \mathbf{R}$, and $\{(w_n, \varphi_n, z_n, \psi_n)\} \subset D(\mathcal{A})$ such that

$$\|(w_n, \varphi_n, z_n, \psi_n)\|_{\mathcal{H}} = 1, \quad |\lambda_n| \to \infty \tag{4.2}$$

and

$$(i\lambda_n - \mathcal{A})(w_n, \varphi_n, z_n, \psi_n) = (f_{1n}, g_{1n}, f_{2n}, g_{2n}) \to 0 \quad \text{in } \mathcal{H}. \tag{4.3}$$

From (4.3), it follows that

$$(i\lambda_n w_n - z_n, i\lambda_n \varphi_n - \psi_n) = (f_{1n}, g_{1n}) \to 0 \quad \text{in } V, \tag{4.4}$$

$$\begin{aligned} &\big(i\lambda_n z_n + \rho^{-1}\big(K(\varphi_n - w_n')\big)', i\lambda_n \psi_n - I_\rho^{-1}\big[(EI\varphi_n')' - K(\varphi_n - w_n')\big]\big) \\ &\quad = (f_{2n}, g_{2n}) \to 0 \quad \text{in } H. \end{aligned} \tag{4.5}$$

Since

$$\mathrm{Re}\big\langle\big((i\lambda_n - \mathcal{A})(w_n, \varphi_n, z_n, \psi_n), (w_n, \varphi_n, z_n, \psi_n)\big)\big\rangle_{\mathcal{H}} \to 0,$$

by (3.1), we can obtain

$$k_1 |z_n(l)|^2 + k_3 |\psi_n(l)|^2 \to 0,$$

which implies

$$z_n(l) \to 0, \qquad \psi_n(l) \to 0. \tag{4.6}$$

By using a simple interpolation inequality, (4.4), (4.6) and $|\lambda_n| \to \infty$, we can deduce that

$$\begin{cases} \lambda_n w_n(l) \to 0, & \lambda_n \varphi_n(l) \to 0, \\ w_n'(l) \to 0, & \varphi_n'(l) \to 0. \end{cases} \tag{4.7}$$

From (4.4), we have

$$\|(f_{1n}, g_{1n})\|_V^2 = k_2 |f_{1n}(l)|^2 + k_4 |g_{1n}(l)|^2 + \int_0^l \left(K|g_{1n} - f_{1n}'|^2 + EI|g_{1n}'|^2\right) dx \to 0,$$

which yields that

$$f_{1n}(l) \to 0, \qquad g_{1n}(l) \to 0 \quad \text{and} \quad g_{1n} - f_{1n}' \to 0, \qquad g_{1n}' \to 0 \quad \text{in } L^2(0,l). \tag{4.8}$$

From (4.5), we have

$$\|(f_{2n}, g_{2n})\|_H^2 = \int_0^l \rho |f_{2n}|^2 \, dx + \int_0^l I_\rho |g_{2n}|^2 \, dx \to 0,$$

which yields that

$$f_{2n} \to 0, \qquad g_{2n} \to 0 \quad \text{in } L^2(0,l). \tag{4.9}$$

Again by (4.4), we have that $\langle (f_{1n}, g_{1n}), (z_n, \psi_n) \rangle_H \to 0$, and consequently,

$$i\lambda_n \int_0^l \rho w_n \overline{z_n} \, dx + i\lambda_n \int_0^l I_\rho \varphi_n \overline{\psi_n} \, dx - \int_0^l \left(\rho |z_n|^2 + I_\rho |\psi_n|^2\right) dx \to 0. \tag{4.10}$$

By (4.5) and (4.7), we have that $\langle (f_{2n}, g_{2n}), (w_n, \varphi_n) \rangle_H \to 0$, and consequently,

$$i\lambda_n \int_0^l \rho \overline{w_n} z_n \, dx + i\lambda_n \int_0^l I_\rho \overline{\varphi_n} \psi_n \, dx + \int_0^l \left(K|\varphi_n - w_n'|^2 + EI|\varphi_n'|^2\right) dx \to 0. \tag{4.11}$$

Taking the real parts of the sum of (4.10) and (4.11), we obtain

$$\int_0^l \left(K|\varphi_n - w_n'|^2 + EI|\varphi_n'|^2\right) dx - \int_0^l \left(\rho |z_n|^2 + I_\rho |\psi_n|^2\right) dx \to 0. \tag{4.12}$$

By (4.2) and (4.12), we conclude that

$$\begin{cases} \int_0^l (K|\varphi_n - w_n'|^2 + EI|\varphi_n'|^2) \, dx \to \frac{1}{2}, \\ \int_0^l (\rho |z_n|^2 + I_\rho |\psi_n|^2) \, dx \to \frac{1}{2}. \end{cases} \tag{4.13}$$

By (4.4) and (4.13), we can obtain

$$w_n \to 0, \qquad \varphi_n \to 0 \quad \text{in } L^2(0,l). \tag{4.14}$$

From (4.4) and (4.5), we can obtain

$$\lambda_n^2 \rho w_n - \left(K(\varphi_n - w_n')\right)' = -(i\lambda_n f_{1n} + f_{2n})\rho, \tag{4.15}$$

$$\lambda_n^2 I_\rho \varphi_n + (EI\varphi_n')' - K(\varphi_n - w_n') = -(i\lambda_n g_{1n} + g_{2n}) I_\rho. \tag{4.16}$$

Now let $q(x) = e^{\eta x} - 1$, where η is a positive constant to be determined after soon. Multiplying (4.15) by $q\overline{w_n'}$, respectively, integrating from 0 to l, we obtain

$$\int_0^l \left[\lambda_n^2 \rho w_n - \left(K(\varphi_n - w_n')\right)'\right] q\overline{w_n'}\, dx = -\int_0^l (i\lambda_n f_{1n} + f_{2n})\rho q \overline{w_n'}\, dx, \tag{4.17}$$

combining this with (4.4), (4.7), and (4.8), we can obtain

$$\int_0^l \left[\lambda_n^2 \rho w_n - \left(K(\varphi_n - w_n')\right)'\right] q\overline{w_n'}\, dx \to 0. \tag{4.18}$$

From (4.17) and (4.18), integrating by parts and taking the real parts, we have

$$\begin{aligned} &\operatorname{Re}\left\{\int_0^l \left[\lambda_n^2 \rho w_n - \left(K(\varphi_n - w_n')\right)'\right] q\overline{w_n'}\, dx\right\} \\ &\quad = -\frac{1}{2}\int_0^l (\rho q)'|\lambda_n w_n|^2\, dx - \frac{1}{2}\int_0^l (Kq' - K'q)|w_n'|^2\, dx \\ &\qquad - \operatorname{Re}\left(\int_0^l Kq\varphi_n'\overline{w_n'}\, dx\right) \to 0. \end{aligned} \tag{4.19}$$

Multiplying (4.16) by $q\overline{\varphi_n'}$, respectively, integrating from 0 to l, we obtain

$$\int_0^l \left[\lambda_n^2 I_\rho \varphi_n + (EI\varphi_n')' - K(\varphi_n - w_n')\right] q\overline{\varphi_n'}\, dx = -\int_0^l (i\lambda_n g_{1n} + g_{2n}) I_\rho q\overline{\varphi_n'}\, dx, \tag{4.20}$$

combining this with (4.4), (4.7), and (4.9), we can obtain

$$\int_0^l \left[\lambda_n^2 I_\rho \varphi_n + (EI\varphi_n')' - K(\varphi_n - w_n')\right] q\overline{\varphi_n'}\, dx \to 0. \tag{4.21}$$

From (4.20) and (4.21), integrating by parts and taking the real parts, we conclude that

$$\begin{aligned} &\operatorname{Re}\left\{\int_0^l \left[\lambda_n^2 I_\rho \varphi_n + (EI\varphi_n')' - K(\varphi_n - w_n')\right] q\overline{\varphi_n'}\, dx\right\} \\ &\quad = -\frac{1}{2}\int_0^l (I_\rho q)'|\lambda_n \varphi_n|^2\, dx - \frac{1}{2}\int_0^l (EIq' - EIK'q)|\varphi_n'|^2\, dx \\ &\qquad + \operatorname{Re}\left(\int_0^l Kq\overline{\varphi_n'} w_n'\, dx\right) \to 0. \end{aligned} \tag{4.22}$$

Taking the sum of (4.19) and (4.22), we can obtain

$$\int_0^l (\rho q)'|\lambda_n w_n|^2\,dx + \int_0^l (Kq' - K'q)|w_n'|^2\,dx + \int_0^l (I_\rho q)'|\lambda_n \varphi_n|^2\,dx + \int_0^l (EIq' - EI'q)|\varphi_n'|^2\,dx \to 0. \tag{4.23}$$

It is easy to see that hypothesis (2.1) guarantees the existence of η as above. We choose η large enough and positive constant C_1, such that

$$\begin{cases} \eta e^{\eta x}\rho + \rho'(e^{\eta x} - 1) > C_1 > 0, & \eta e^{\eta x} I_\rho + I_\rho'(e^{\eta x} - 1) > C_1 > 0, \\ \eta e^{\eta x} K - K'(e^{\eta x} - 1) > C_1 > 0, & \eta e^{\eta x} EI - EI'(e^{\eta x} - 1) > C_1 > 0, \end{cases} \tag{4.24}$$

combining this with (4.23), we can obtain

$$\begin{cases} \lambda_n w_n \to 0, & w_n' \to 0 \quad \text{in } L^2(0,l), \\ \lambda_n \varphi_n \to 0, & \varphi_n' \to 0 \quad \text{in } L^2(0,l). \end{cases} \tag{4.25}$$

From (4.14) and (4.25), we conclude that

$$\int_0^l \left(K|\varphi_n - w_n'|^2 + EI|\varphi_n'|^2\right) dx \to 0, \tag{4.26}$$

which is in contradiction with (4.13), and the proof is completed. □

Proof of the Theorem 2.2. First it is impossible that $k_1 = k_3 = 0$. Indeed, it follows from $k_1 = k_3 = 0$ and (2.5) that

$$\begin{aligned} E(t) &= \frac{1}{2}\left\| e^{t\mathcal{A}}(w_0, \varphi_0, z_0, \psi_0)\right\|_{\mathcal{H}}^2 \\ &\equiv E(0) \\ &= \frac{1}{2}\left[\int_0^l \left(K|\varphi_0 - w_0'|^2 + EI|\varphi_0'|^2 + \rho|z_0|^2 + I_\rho|\psi_0|^2\right) dx \right. \\ &\quad \left. + k_2\big|w_0(l)\big|^2 + k_4\big|\varphi_0(l)\big|^2\right], \quad t \geqslant 0, \end{aligned} \tag{4.27}$$

for any $(w_0, \varphi_0, z_0, \psi_0) \in D(\mathcal{A})$. Thus the energy $E(t)$ of the system (1.1) does not uniformly exponentially decay.

Next, let $k_1 = 0$ and $k_3 > 0$, we can define

$$V_0 = \{(w, \varphi) \in V \mid \varphi(l) = 0\}, \qquad \mathcal{H}^0 = V_0 \times H, \qquad \mathcal{A}^0 = \mathcal{A}|_{\mathcal{H}^0}.$$

That is

$$D(\mathcal{A}^0) = \{(w, \varphi, , z, \psi) \in D(\mathcal{A}) \cap \mathcal{H}^0 \mid \mathcal{A}(w, \varphi, z, \psi) \in \mathcal{H}^0\}$$

and

$$\mathcal{A}^0(w, \varphi, z, \psi) = \mathcal{A}(w, \varphi, z, \psi), \quad (w, \varphi, , z, \psi) \in D(\mathcal{A}^0).$$

By the same argument as in the proof of Theorem 3.1, we can conclude that $\mathcal{A}^0$ is dissipative and $0 \in \rho(\mathcal{A}^0)$. Hence, $\mathcal{A}^0$ is m-dissipative and generates a contraction C_0-semigroup $e^{t\mathcal{A}^0}$. Since $e^{t\mathcal{A}^0}(w_0, \varphi_0, z_0, \psi_0) \in D(\mathcal{A}^0)$, the same proof of (2.5) yields

$$\frac{d}{dt}\left(\frac{1}{2}\left\|e^{t\mathcal{A}^0}(w_0, \varphi_0, z_0, \psi_0)\right\|^2_{\mathcal{H}^0}\right) = \frac{d}{dt}E(t) = 0,$$

which implies that C_0-semigroup $e^{t\mathcal{A}^0}$ is an isometric semigroup. Therefore the energy of the system (1.1) does not exponentially decay. Finally, let $k_1 > 0$ and $k_3 = 0$, we only need to define $V_0 = \{(w, \varphi) \in V \mid w(l) = 0\}$, and the similar proof follows. This completes the proof of the theorem. $\square$

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