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Regularizations in Abelian Complete Ordered Groups

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Notions about Φ-convexity are extended to abelian complete partially ordered group-valued mappings in an attempt to unify in a general theory notions of Φ-convex sets and Φ-convex mappings. We obtain some group specific results and particularly a characterization of support functions.

INTRODUCTION

The origin of this work is in convex analysis. Let E be a locally convex space, H an ordered topological vector space with closed positive cone, and A a subset of E. We know ([9, p. 75]) that the upper-envelope of affine continuous maps from E to H majorized by the indicator function of A is exactly the indicator function of the closed convex hull of A.

We analyse the fact that such an upper-envelope is also an indicator function and leads to the situation where the following properties hold: E_1 is a set, $(H, +, \leqslant)$ an abelian complete ordered group, and E_2 a non-empty part of the group of all maps from E_1 to H with the property

$$\forall x_2, x_2 \in E_2 \qquad \text{and} \qquad \forall n, n \in \mathbb{N}\colon nx_2 = x_2 + \overset{n\text{ times}}{\cdots} + x_2 \in E_2.$$

In this frame, the upper-envelope of maps $x_1 \mapsto x_2(x_1) - z$, with $x_2 \in E_2$ and $z \in H$, majorized by the indicator function of a subset $A \subset E_1$ is also the indicator function of a subset $A^{\gamma(E_1,E_2)} \subset E_1$ (Theorem II.2.1). This situation occurs frequently; for example, if E_1 is an ordered set, we can successively take for E_2 the increasing, decreasing, monotone functions from E_1 to H. In the last case we prove that $A^{\gamma(E_1,E_2)}$ is the order convex hull of A (Corollary II.3.2). In order to study this situation we extend some notions of Φ-convexity introduced by Fan ([5]), Dolecki and Kurcyusz ([2, 3]) to abelian complete ordered group-valued functions. We give other results in a purely set duality frame. This is contained in Section I.

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In Section II we unify notions of sets and function envelopes through the medium of indicator functions (Theorem II.2.1). We also obtain a very simple characterization of support functions (Theorem II.1.2). Two basic properties of vectorial support functions hold (Theorems II.1.3 and II.2.3). With another hypothesis we obtain purely algebraic results, some of which are inspired by analysis convex theorems (Theorems II.4.1, 2, 3, 5).

Definition and Notations

Let $(H, +, \leqslant)$ be an abelian complete ordered group (that is, an abelian ordered group in which all non-empty and majorized subsets have a least upper bound). We adjoin to H a largest element $+\infty$ and a smallest element $-\infty$ and we set $\bar{H} = H \cup \{-\infty, +\infty\}$. Every subset of $\bar{H}$ has an infimum and a supremum (we set $\sup \varnothing = -\infty$ and $\inf \varnothing = +\infty$).

We extend the addition of H to $\bar{H}$ by setting

$$\forall z, z \in \bar{H}: \qquad z + (+\infty) = +\infty,$$

$$\forall z, z \in H: \qquad z + (-\infty) = -\infty,$$

$$(-\infty) + (-\infty) = -\infty, \qquad -(+\infty) = -\infty, \qquad -(-\infty) = +\infty.$$

If A is a subset of E, the indicator function of A is noted ψ_A and defined by $\psi_A : E \to \bar{H}$, $\psi_A(x) = 0$ if $x \in A$, $\psi_A(x) = +\infty$ if $x \notin A$. If f is a constant map from E to H with value z we note $f = z^E$.

Let E_1 and E_2 be two sets, $\langle\ ,\ \rangle$: $E_1 \times E_2 \to H$ a map, and $(x_1, x_2) \in E_1 \times E_2$. We shall note respectively $\langle x_1,\ \rangle$ and $\langle\ , x_2\rangle$ the maps defined by

$$\langle x_1,\ \rangle: E_2 \to H, \qquad \langle\ , x_2\rangle: E_1 \to H,$$

$$x_2' \mapsto \langle x_1, x_2'\rangle, \qquad x_1' \mapsto \langle x_1', x_2\rangle.$$

If $f, g: E \to \bar{H}$ are maps, $f \leqslant g$ will mean that $\forall x, x \in E$: $f(x) \leqslant g(x)$. Moreover, if $A \subset E$ we note $\sup_A f = \sup_{x \in A} f(x)$.

Note. We shall frequently use the fact that every abelian complete ordered group $(H, +, \leqslant)$ is completely integrally closed ([6, p. 90]). This means that if we have $z, t \in H$ and $nt \leqslant z$ for every $n \in \mathbb{N}$ then $t \leqslant 0$.

I. $\gamma(E_i, E_j)$-Regularization

In this section $(H, +, \leqslant)$ will be an abelian complete ordered group with elements > 0. Let E_1 and E_2 be two non-empty sets and $\langle\ ,\ \rangle: E_1 \times E_2 \to H$ $(x_1, x_2) \mapsto \langle x_1, x_2\rangle$ a map. For $i, j \in \{1, 2\}$ and $i \neq j$ we note $[x_i, x_j]$ instead of $\langle x_1, x_2\rangle$. So, $[x_i, \ ]$ will be the map $\langle x_1, \ \rangle$ or $\langle \ , x_2\rangle$ according as $i = 1$ or 2. Idem for $[\ , x_j]$.

We extend to this situation notions of Φ-convexity, introduced in [2, 3, 5], that we shall need in Section II.

I.1. $\gamma(E_i, E_j)$-*Envelopes for Subsets of* E_i

We say that a subset $B \subset E_i$ is a *$\gamma(E_i, E_j)$-set* if there is a family $(x_k, z_k)_{k\in K}$ of $E_j \times H$ elements with $B = \bigcap_{k\in K} \{x \in E_i \mid [x, x_k] \leqslant z_k\}$.

Taking $K = \varnothing$ we see that E_i is always a $\gamma(E_i, E_j)$-set. E_i with the Moore $\gamma(E_i, E_j)$-sets family ([1, p. 4]) constitute a *cyrtological-space* ([2]) a basis of which is given by the sets $\{x \in E_i \mid [x, x_j] \leqslant z\}$, where $(x_j, z) \in E_j \times H$.

By definition, the *$\gamma(E_i, E_j)$-envelope* of $A \subset E_i$ is the intersection of all $\gamma(E_i, E_i)$-sets which contain A and *we note it* $A^{\gamma(E_i, E_j)}$.

PROPOSITION I.1.1. *The empty set is always a $\gamma(E_i, E_j)$-set.*

Proof. Let $z \in H$, $z > 0$, and $x_j \in E_j$. For $n \in \mathbb{N}$ we set

$$A_n = \{x \in E_i \mid [x, x_j] \leqslant -nz\}.$$

We have $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. In fact, if not we would have $x \in E_i$ such that for all $n \in \mathbb{N}$: $[x, x_j] \leqslant -nz$, that is, $nz \leqslant -[x, x_j]$ and at last $z \leqslant 0$.

THEOREM I.1.2. *For every $A \subset E_i$ we have*

$$A^{\gamma(E_i, E_j)} = \{x \in E_i \mid x_j \in E_j \Rightarrow [x, x_j] \leqslant \sup_A [\ , x_j]\}.$$

Proof. Let

$$B = \{x \in E_i \mid x_j \in E_j \Rightarrow [x, x_j] \leqslant \sup_A [\ , x_j]\}.$$

If $A = \varnothing$ we have $\sup_A [\ , x_j] = -\infty$ and by this $B = \varnothing$. In this case we conclude with Proposition I.1.1.

If $A \neq \varnothing$ let J be the set of $x_j \in E_j$ such that $\sup_A [\ , x_j] \in H$. It is easy to see that $B = \bigcap_{x_j\in J} \{x \in E_i \mid [x, x_j] \leqslant \sup_A [\ , x_j]\}$. Therefore, B is a $\gamma(E_i, E_j)$-set containing A. Inversely, if $C = \bigcap_{k\in K} \{x \in E_i \mid [x, x_k] \leqslant z_k\}$ is a $\gamma(E_i, E_j)$-set containing A, we have to prove that $B \subset C$. Since $A \subset C$ we have $[x, x_k] \leqslant z_k$ for all $(x, k) \in A \times K$ and therefore $\sup_A [\ , x_k] \leqslant z_k$. Now for all $(x, k) \in B \times K$: $[x, x_k] \leqslant \sup_A [\ , x_k] \leqslant z_k$. This completes the proof.

COROLLARY I.1.3. *If $A \subset E_i$, $A^{\gamma(E_i, E_j)}$ is the largest part of E_i such that*

$$x_j \in E_j \Rightarrow \sup_A [\ , x_j] = \sup_{A^{\gamma(E_i, E_j)}} [\ , x_j].$$

Proof. We first show that $A^{\gamma(E_i,E_j)}$ verifies this property. If $x \in A^{\gamma(E_i,E_j)}$ and $x_j \in E_j$ we have by Theorem I.1.2, $[x, x_j] \leqslant \sup_A [\ , x_j]$. Therefore $\sup_{A^{\gamma(E_i,E_j)}} [\ , x_j] \leqslant \sup_A [\ , x_j]$. In fact we obtain an equality because $A \subset A^{\gamma(E_i,E_j)}$. Now, let $B \subset E_i$ such that

$$x_j \in E_j \Rightarrow \sup_A [\ , x_j] = \sup_B [\ , x_j].$$

For $x \in B$ and $x_j \in E_j$ we have

$$[x, x_j] \leqslant \sup_B [\ , x_j] = \sup_A [\ , x_j]$$

then $x \in A^{\gamma(E_i,E_j)}$ and therefore $B \subset A^{\gamma(E_i,E_j)}$.

I.2. $\gamma(E_i, E_j)$-*Regularization of Mappings from* E_i *to* $\bar{H}$

Let $f\colon E_i \to \bar{H}$ be a map. The $\gamma(E_i, E_j)$-*regularized* (*or* $\gamma(E_i, E_j)$-*envelope*) of f is defined by $f^{\gamma(E_i,E_j)} = \sup\{[\ , x_j] - z \mid (x_j, z) \in E_j \times H$ and $[\ , x_j] - z \leqslant f\}$. We say that f is a $\gamma(E_i, E_j)$-*map* if $f = f^{\gamma(E_i,E_j)}$. We note $\Gamma(E_i, E_j)$ is the set of maps from E_i to $\bar{H}$ which are upper-envelope of maps like $[\ , x_j] - z$ with $(x_j, z) \in E_j \times H$. If f is a map from E_i to $\bar{H}$ then f is a $\gamma(E_i, E_j)$-map if and only if $f \in \Gamma(E_i, E_j)$; $f^{\gamma(E_i,E_j)}$ is also the largest map in $\Gamma(E_i, E_j)$ majorized by f. We note that $f^{\gamma(E_i,E_j)} = (-\infty)^{E_i}$ if and only if f does not majorize any map $[\ , x_j] - z$ with $(x_j, z) \in E_j \times H$.

Proposition I.2.1. *The constant map* $(+\infty)^{E_i}$ *is always in* $\Gamma(E_i, E_j)$.

Proof. Let $x_j \in E_j$ and $z \in H$, $z > 0$. For $n \in \mathbb{N}$ we set $f_n = [\ , x_j] + nz$. We show that for $x \in E_i$ we have $\sup_{n \in \mathbb{N}} f_n(x) = +\infty$. Otherwise we would have $z' \in H$ with $[x, x_j] + nz \leqslant z'$ for every $n \in \mathbb{N}$, then $nz \leqslant z' - [x, x_j]$ and finally $z \leqslant 0$; that is impossible.

Proposition I.2.2. *Let f be a* $\gamma(E_i, E_j)$-*map and* $z \in H$. *The set* $\{x \in E_i \mid f(x) \leqslant z\}$ *is a* $\gamma(E_i, E_j)$-*set.*

Proof. By hypothesis, $f = \sup_{k \in K} [\ , x_k] - z_k$. Then

$$\{x \in E_i \mid f(x) \leqslant z\} = \bigcap_{k \in K} \{x \in E_i \mid [x, x_k] - z_k \leqslant z\}$$

$$= \bigcap_{k \in K} \{x \in E_i \mid [x, x_k] \leqslant z + z_k\}.$$

Corollary I.2.3. *Let* $A \subset E_i$ *and suppose that the indicator function of* A *is a* $\gamma(E_i, E_j)$-*map. Then* A *is a* $\gamma(E_i, E_j)$-*set.*

Proof. We note that $A = \{x \in E_i \mid \psi_A(x) \leqslant 0\}$ and we apply Proposition I.2.2.

Important Remark. The converse is false. Let $\langle\, ,\rangle: H \times H \to H$ with $\langle x_1, x_2\rangle = x_1 + x_2$ and $A = \{x \in H \mid x \leqslant 0\}$. Then A is a $\gamma(H, H)$-set. However,

$$(\langle\ , x_2\rangle - z \leqslant \psi_A) \Leftrightarrow (x + x_2 - z \leqslant 0 \text{ if } x \leqslant 0)$$
$$\Leftrightarrow z - x_2 \geqslant 0.$$

Now, for $x \in H$,

$$(\psi_A)^{\gamma(H,H)}(x) = \sup_{z - x_2 \geqslant 0} x + x_2 - z = x + \sup_{z - x_2 \geqslant 0} x_2 - z = x.$$

Therefore, $(\psi_A)^{\gamma(H,H)} = id_H$ and by this ψ_A is not a $\gamma(H, H)$-map. The aim of Section II is to define a frame where this phenomen does not occur.

THEOREM I.2.4. *Let f be a $\gamma(E_i, E_j)$-map and $A \subset E_i$. Then* $\sup_A f = \sup_{A^{\gamma(E_i,E_j)}} f$.

Proof. Let $f = \sup_{k \in K} [\ , x_k] - z_k$.

$$\sup_A f = \sup_A \sup_{k \in K} [\ , x_k] - z_k = \sup_{k \in K} \sup_A [\ , x_k] - z_k$$
$$= \sup_{k \in K} (\sup_A [\ , x_k]) - z_k.$$

By Corollary I.1.3 we have also

$$\sup_A f = \sup_{k \in K} (\sup_{A^{\gamma(E_i,E_j)}} [\ , x_k]) - z_k = \sup_{k \in K} \sup_{A^{\gamma(E_i,E_j)}} [\ , x_k] - z_k$$
$$= \sup_{A^{\gamma(E_i,E_j)}} \sup_{k \in K} [\ , x_k] - z_k = \sup_{A^{\gamma(E_i,E_j)}} f.$$

I.3. *Conjugates of Maps from E_i to $\bar{H}$*

Let f: $E_i \to \bar{H}$ be a map. The *conjugate mapping of f* is defined by

$$f^*: E_j \to \bar{H}; \qquad x_j \mapsto f^*(x_j) = \sup_{E_i} ([\ , x_j] - f).$$

We may also note $f^* = \sup_{x_i \in E_i} [x_i,\ ] - f(x_i)$. Let dom f be *the domain* of f that is $\operatorname{dom} f = \{x \in E_i \mid f(x) < +\infty\}$. Then we obviously have $f^* = \sup_{x_i \in \operatorname{dom} f} [x_i,\ ] - f(x_i)$. For $(x_j, z) \in E_j \times H$ we also have $f^*(x_j) \leqslant z \Leftrightarrow [\ , x_j] - z \leqslant f$.

PROPOSITION I.3.1. *For every map f: $E_i \to \bar{H}$, f^* belongs to $\Gamma(E_j, E_i)$.*

Proof. If f does not take the value $-\infty$, we conclude with the relation $f^* = \sup_{x \in \operatorname{dom} f} [x, \ ] - f(x)$. If f takes this value, then $f^* = (+\infty)^{E_j}$ and, by Proposition I.2.1, we know that $(+\infty)^{E_j}$ belongs to $\Gamma(E_j, E_i)$. The next theorem is of Fenchel–Moreau type ([2–4, 10–13]).

THEOREM I.3.2. *For every map f: $E_i \to \bar{H}$, the $\gamma(E_i, E_j)$-regularized map $f^{\gamma(E_i,E_j)}$ of f is exactly the biconjugate f^{**} of f.*

Proof. For all $x \in E_i$ we have

$$f^{**}(x) = \sup_{x_j \in E_j} [x, x_j] - f^*(x_j).$$

Let us majorize $f^*(x_j)$ by $-[x, x_j] + f(x)$; then $f^{**}(x) \leqslant f(x)$. By Proposition I.3.1, f^{**} is an element of $\Gamma(E_i, E_j)$ majorized by f. Therefore $f^{**} \leqslant f^{\gamma(E_i,E_j)}$. On the other hand,

$$\begin{aligned} f^{\gamma(E_i,E_j)} &= \sup\{[\ , x_j] - z \mid (x_j, z) \in E_j \times H \text{ and } [\ , x_j] - z \leqslant f\} \\ &= \sup\{[\ , x_j] - z \mid (x_j, z) \in E_j \times H \text{ and } f^*(x_j) \leqslant z\} \\ &\leqslant \sup\{[\ , x_j] - f^*(x_j) \mid x_j \in \operatorname{dom} f^*\} = f^{**}. \end{aligned}$$

COROLLARY I.3.3. *The following maps are one to one, onto and mutually reciprocal:*

$$\begin{array}{cc} \Gamma(E_1, E_2) \to \Gamma(E_2, E_1), & \Gamma(E_2, E_1) \to \Gamma(E_1, E_2), \\ f \to f^*, & f \to f^*. \end{array}$$

Proof. It is sufficient to remark that

$$f \in \Gamma(E_i, E_j) \Leftrightarrow f = f^{\gamma(E_i,E_j)} \Leftrightarrow f = f^{**}.$$

COROLLARY I.3.4. *For every map f from E_i to $\bar{H}$, $f^{\gamma(E_i,E_j)}$ is the smallest map from E_i to $\bar{H}$ such that*

$$f^* = (f^{\gamma(E_i,E_j)})^*.$$

Proof. We have $f^{**} = f^{\gamma(E_i,E_j)} = f^{(\gamma(E_i,E_j))^{**}}$. On the other hand, f^* and $(f^{\gamma(E_i,E_j)})^*$ belong to $\Gamma(E_j, E_i)$. From Corollary I.3.3 we deduce that $f^* = (f^{\gamma(E_i,E_j)})^*$. Conversely, let g: $E_i \to \bar{H}$ be a map such that $f^* = g^*$. Then, let $(x_j, z) \in E_j \times H$ such that $[\ , x_j] - z \leqslant f$. We have

$$g^*(x_j) = f^*(x_j) \leqslant z.$$

Therefore $[\ , x_j] - z \leqslant g$ and finally $f^{\gamma(E_i,E_j)} \leqslant g$.

II. A Unificatory Frame

In this section E_1 is a non-empty set and E_2 a non-empty subset of the group of maps from E_1 to H with the property

$$x_2 \in E_2 \quad \text{and} \quad n \in \mathbb{N} \Rightarrow nx_2 = x_2 + \overset{n \text{ times}}{\cdots} + x_2 \in E_2.$$

We note that the constant map 0^{E_1} always belongs to E_2. The map $\langle\ ,\ \rangle$: $E_1 \times E_2 \to H$ will be here defined by $\langle x_1, x_2 \rangle = x_2(x_1)$ for every $(x_1, x_2) \in E_1 \times E_2$. Then, for every $n \in \mathbb{N}$ and $(x_1, x_2) \in E_1 \times E_2$ we have

$$\langle x_1, nx_2 \rangle = n \langle x_1, x_2 \rangle.$$

II.1. *Characterization of Support Functions*

Let $A \subset E_1$; the conjugate of the A-indicator function will be noted α_A. Then, for all $x_2 \in E_2$ we have

$$\alpha_A(x_2) = (\psi_A)^* (x_2) = \sup_{\operatorname{dom} \psi_A} \langle\ , x_2 \rangle - \psi_A = \sup_A \langle\ , x_2 \rangle.$$

α_A is called *the support function of A*. Let us note that α_A verifies the property $(\mathscr{C})$. $(\mathscr{C})$: $\alpha_A(0^{E_1}) = 0$ and $\forall n \in \mathbb{N}$, $\forall x_2 \in E_2$: $\alpha_A(nx_2) \leqslant n\alpha_A(x_2)$. In fact, it is obvious that $\alpha_A(0^{E_1}) = \sup_A \langle\ , 0^{E_1} \rangle = 0$. On the other hand,

$$\alpha_A(nx_2) = \sup_A \langle\ , nx_2 \rangle = \sup_A n \langle\ , x_2 \rangle \leqslant n \sup_A \langle\ , x_2 \rangle.$$

Lemma II.1.1. *Let ϕ: $E_2 \to \bar{H}$ be a map having property $(\mathscr{C})$. Then the conjugate ϕ^* of ϕ is exactly the indicator function of the set $\{x_1 \in E_1 \mid \langle x_1,\ \rangle \leqslant \phi\}$.*

Proof. Let $x_1 \in E_1$ such that $\langle x_1,\ \rangle \leqslant \phi$. We have $\phi^*(x_1) = \sup_{E_2} \langle x_1\ \ \rangle - \phi$; therefore $\phi^*(x_1) \leqslant 0$ and finally $\phi^*(x_1) = 0$ because the supremum is reached with 0^{E_1}. Inversely let $x_1 \in E_1$ such that $\langle x_1,\ \rangle \nleqslant \phi$. We have to prove that $\phi(x_1) = +\infty$. By hypothesis there is $x_2 \in E_2$ such that $\langle x_1, x_2 \rangle \nleqslant \phi(x_2)$. Therefore we cannot have $\phi(x_2) = +\infty$. On the other hand, $\phi^*(x_1) \geqslant \langle x_1, x_2 \rangle - \phi(x_2)$. If $\phi(x_2) = -\infty$ we have obviously $\phi^*(x_1) = +\infty$; if not, $\phi(x_2)$ belongs to H and, for every $n \in \mathbb{N}$, we have $\phi^*(x_1) \geqslant \langle x_1, nx_2 \rangle - \phi(nx_2)$ whence $\phi^*(x_1) \geqslant n\langle x_1, x_2 \rangle - n\phi(x_2)$, that is, $\phi^*(x_1) \geqslant n(\langle x_1, x_2 \rangle - \phi(x_2))$. Then, necessarily, $\phi(x_1) = +\infty$ for otherwise we would have $\langle x_1, x_2 \rangle - \phi(x_2) \leqslant 0$ which is absurd.

Theorem II.1.2. *Let ϕ be a map from E_2 to $\bar{H}$. The following assertions are equivalent*

(i) *ϕ is a $\gamma(E_2, E_1)$-map having property $(\mathscr{C})$,*

(ii) *ϕ is a support-function of a subset of E_1,*

(iii) *ϕ is a support-function of a $\gamma(E_1, E_2)$-set.*

Proof. (i) $\Rightarrow$ (ii). By Lemma II.1.1, $\phi^* = \psi_{\{x \in E_1 | \langle x, \rangle \leqslant \phi\}}$. On the other hand, $\phi \in \Gamma(E_2, E_1)$ and by Corollary I.3.3 we have $\phi = \phi^{**}$. Therefore $\phi = \alpha_{\{x \in E_1 | \langle x, \rangle \leqslant \phi\}}$.

(ii) $\Rightarrow$ (iii). Let $\phi = \alpha_A$. For every $x_2 \in E_2$ we have by Corollary I.1.3

$$\alpha_A(x_2) = \sup_A \langle \ , x_2 \rangle = \sup_{A^{\gamma(E_1, E_2)}} \langle \ , x_2 \rangle.$$

Then $\phi = \alpha_{A^{\gamma(E_1, E_2)}}$.

(iii) $\Rightarrow$ (i). For all $A \subset E_i$ we know by Proposition I.3.1 that $\alpha_A = (\psi_A)^*$ belongs to $\Gamma(E_2, E_1)$. On the other hand, we have already noted that every support function verifies property ($\mathscr{C}$).

THEOREM II.1.3. *Let $(A_k)_{k \in K}$ be a family of subsets of E_1 and $A = \bigcup_{k \in K} A_k$, then*

$$\alpha_{A^{\gamma(E_1, E_2)}} = \sup_{k \in K} \alpha_{A_k}.$$

Proof. $\alpha_A = (\psi_{\bigcup_{k \in K} A_k})^* = (\inf_{k \in K} \psi_{A_k})^*$. Now, if $(f_k)_{k \in K}$ is a family of maps from E_1 to H, it is easy to see that

$$(\operatorname*{Inf}_{k \in K} f_k)^* = \sup_{k \in K} f_k^*$$

and here $\alpha_A = \sup_{k \in K} \psi_{A_k}^* = \sup_{k \in K} \alpha_{A_k}$. We conclude with the already known equality $\alpha_A = \alpha_{A^{\gamma(E_1, E_2)}}$.

II.2. *Unification of $\gamma(E_1, E_2)$-Envelopes of Sets and Maps Notions*

THEOREM II.2.1. *For every part A of E_1, the $\gamma(E_1, E_2)$-envelope of the indicator function of A is exactly the indicator function of the $\gamma(E_1, E_2)$-envelope of A, that is, $(\psi_A)^{\gamma(E_1, E_2)} = \psi_{A^{\gamma(E_1, E_2)}}$.*

Proof. By Theorem I.3.2, $(\psi_A)^{\gamma(E_1, E_2)} = \psi_A^{**} = \alpha_A^*$. Now, from Lemma II.1.1

$$\alpha_A^* = \psi_{\{x \in E_1 | \langle x, \rangle \leqslant \alpha_A\}}.$$

But $\{x \in E_1 \mid \langle x, \ \rangle \leqslant \alpha_A\} = \{x \in E_1 \mid x_2 \in E_2 \Rightarrow \langle x, x_2 \rangle \leqslant \sup_A \langle \ , x_2 \rangle\}$. By Theorem I.1.2 these two sets coincide with $A^{\gamma(E_1, E_2)}$. Finally $(\psi_A)^{\gamma(E_1, E_2)} = \psi_{A^{\gamma(E_1, E_2)}}$.

COROLLARY II.2.2. *For every subset A of E_1 the following properties are equivalent*

(i) *A is a $\gamma(E_1, E_2)$-set,*

(ii) *ψ_A is a $\gamma(E_1, E_2)$-map.*

Proof. From Corollary I.2.3 we already know that (ii) $\Rightarrow$ (i). Conversely let A be a $\gamma(E_1, E_2)$-set. Then we have $(\psi_A)^{\gamma(E_1,E_2)} = \psi_{A^{\gamma(E_1,E_2)}} = \psi_A$.

THEOREM II.2.3. *Let A and B be two $\gamma(E_1, E_2)$-sets, then*

$$A \subset B \Leftrightarrow \alpha_A \leqslant \alpha_B.$$

Proof. It is obvious that $A \subset B \Rightarrow \alpha_A \leqslant \alpha_B$. Conversely, we easily see that

$$\alpha_A \leqslant \alpha_B \Rightarrow \alpha_B^* \leqslant \alpha_A^*.$$

Therefore $\psi_B^{**} \leqslant \psi_A^{**}$ or $(\psi_B)^{\gamma(E_1,E_2)} \leqslant (\psi_A)^{\gamma(E_1,E_2)}$ and, by Theorem II.2.1, $\psi_{B^{\gamma(E_1,E_2)}} \leqslant \psi_{A^{\gamma(E_1,E_2)}}$. We conclude with the fact that A and B are $\gamma(E_1, E_2)$-sets.

II.3. *Monotone H-Valued Maps*

Let E_1 be a non-empty partially ordered set. We set respectively E_2', E_2'', E_2 for increasing, decreasing, monotone maps from E_1 to H.

PROPOSITION II.3.1. *For every subset A of E_1, $A^{\gamma(E_1,E_2')}$, $A^{\gamma(E_1,E_2'')}$, $A^{\gamma(E_1,E_2)}$ coincide respectively with $d(A) = \{x \in E_1 \mid \exists\, y \in A \mid x \leqslant y\}$, $i(A) = \{x \in E_1 \mid \exists\, y \in A \mid y \leqslant x\}$, $[A] = i(A) \cap d(A)$.*

Proof. $d(A)$ is contained in $A^{\gamma(E_1,E_2')}$ for if $x \in d(A)$ and if $\langle\ , x_2\rangle$ is an increasing map from E_1 to H, then $\exists\, y \in A \mid x \leqslant y$ and $\langle x, x_2\rangle \leqslant \langle y, x_2\rangle \leqslant \sup_A\langle\ , x_2\rangle$. Inversely let $x_0 \notin d(A)$. We take $z \in H$, $z > 0$ and define the map $\langle\ , x_2\rangle$: $E_1 \to H$

$$\begin{aligned} x &\to z \qquad \text{if} \quad x_0 \leqslant x \\ &\to 0 \qquad \text{if not.} \end{aligned}$$

We see easily that $\langle\ , x_2\rangle$ is an increasing map which is 0 on A and z at x_0. Therfore we do not have $\langle x_0, x_2\rangle \leqslant \sup_A\langle\ , x_2\rangle$ and by this $x_0 \notin A^{\gamma(E_1,E_2')}$. Hence $d(A) = A^{\gamma(E_1,E_2')}$ and taking the opposite order on E_1 we also have $i(A) = A^{\gamma(E_1,E_2'')}$. At last, if f: $E_1 \to \bar{H}$ is a map, it is easy to see that $f^{\gamma(E_1,E_2)} = \sup\{f^{\gamma(E_1,E_2')}, f^{\gamma(E_1,E_2'')}\}$. In particular with $f = \psi_A$ we obtain

$$A^{\gamma(E_1,E_2)} = A^{\gamma(E_1,E_2')} \cap A^{\gamma(E_1,E_2'')} = d(A) \cap i(A) = [A].$$

COROLLARY II.3.2. *For every part A of E_1, the upper envelope of monotone maps from E_1 to H majorized by the indicator function of A coincide with the indicator function of the order convex hull of A.*

II.4. *Complements*

PROPOSITION II.4.1. *Let E_2 be a non-empty part of the group of maps from E_1 to H with the property $k \in \mathbb{Z}$ and $x_2 \in E_2 \Rightarrow kx_2 \in E_2$. Then, the following assertions are equivalent*

(i) *points of E_1 are $\gamma(E_1, E_2)$-sets,*

(ii) *E_2 separate points of E_1.*

Proof. (i) $\Rightarrow$ (ii). Let $x \neq x'$ be in E_1. By hypothesis, $x \notin \{x'\}^{\gamma(E_1,E_2)}$. Then, we have $x_2 \in E_2$ suh that $\langle x, x_2 \rangle \nleqslant \langle x', x_2 \rangle$ and therefore $\langle x, x_2 \rangle \neq \langle x', x_2 \rangle$.

(ii) $\Rightarrow$ (i). Let $x \in E_1$ and $x' \in \{x\}^{\gamma(E_1,E_2)}$. For every $x_2 \in E_2$, $\langle x', x_2 \rangle \leqslant \langle x, x_2 \rangle$ and $\langle x', -x_2 \rangle \leqslant \langle x, -x_2 \rangle$, that is, $-\langle x', x_2 \rangle \leqslant -\langle x, x_2 \rangle$. Therefore, for every $x_2 \in E_2$ we have $\langle x', x_2 \rangle = \langle x, x_2 \rangle$ and from the fact that E_2 separate points of E_1 we conclude that $x = x'$ and that $\{x\}$ is a $\gamma(E_1, E_2)$-set.

The next theorem is motivated by Proposition 3.3 of [8].

THEOREM II.4.2. *Let E_2 be a non-empty part of the group of maps from E_1 to H such that $0^{E_1} \in E_2$ and $x_2, x_2' \in E_2 \Rightarrow x_2 + x_2' \in E_2$. Let f: $E_1 \to \bar{H}$ be a map such that $f \geqslant \langle\ , x_2 \rangle - z$ with $(x_2, z) \in E_2 \times H$. Then*: $\operatorname{dom} f^{\gamma(E_1,E_2)} \subset (\operatorname{dom} f)^{\gamma(E_1,E_2)}$.

Proof. If $f = (+\infty)^{E_1}$ we have by Propositions I.1.1 and I.2.1, $(+\infty)^{E_1} = f^{\gamma(E_1,E_2)}$ and $(\operatorname{dom} f)^{\gamma(E_1,E_2)} = \phi^{\gamma(E_1,E_2)} = \phi$. Therefore in this case, $\operatorname{dom} f^{\gamma(E_1,E_2)} = (\operatorname{dom} f)^{\gamma(E_1,E_2)} = \phi$. Let us suppose $f \neq (+\infty)^{E_1}$, that is, $\operatorname{dom} f \neq \phi$. Now, let $x \notin (\operatorname{dom} f)^{\gamma(E_1,E_2)}$. We have to show that $x_0 \notin \operatorname{dom} f^{\gamma(E_1,E_2)}$. By hypothesis there is $x_2' \in E_2$ such that

$$\langle x_0, x_2' \rangle \nleqslant \sup_{\operatorname{dom} f} \langle\ , x_2' \rangle. \tag{1}$$

Let us set $\alpha_n = \sup_{\operatorname{dom} f}(\langle\ , x_2 + nx_2' \rangle - f)$ $(n \in \mathbb{N})$. It is easy to see that $f \geqslant \langle\ , x_2 + nx_2' \rangle - \alpha_n$. If $\alpha_n < +\infty$ we then have

$$f^{\gamma(E_1,E_2)} \geqslant \langle\ , x_2 + nx_2' \rangle - \alpha_n. \tag{2}$$

If $\alpha_n = +\infty$, (2) obviously holds. Now

$$\alpha_n = \sup_{\operatorname{dom} f} \langle\ , x_2 \rangle + n\langle\ , x_2' \rangle - f.$$

Therefore

$$\alpha_n \leqslant n \sup_{\operatorname{dom} f} \langle\ , x_2 \rangle - f + \sup_{\operatorname{dom} f} n\langle\ , x_2' \rangle \quad \text{and} \quad \alpha_n \leqslant z + n \sup_{\operatorname{dom} f} \langle\ , x_2' \rangle.$$

Then, we deduce from (2) that

$$f^{\gamma(E_1,E_2)}(x_0) - \langle x_0, x_2 \rangle + z \geqslant n(\langle x_0, x_2' \rangle - \sup_{\mathrm{dom} f} \langle \ , x_2' \rangle). \tag{3}$$

By (1) and by the fact that $\mathrm{dom} f \neq \phi$ we have $\sup_{\mathrm{dom} f} \langle \ , x_2' \rangle \in H$. Let us suppose that $f^{\gamma(E_1,E_2)}(x_0) < +\infty$. As $f^{\gamma(E_1,E_2)}(x_0) \geqslant \langle x_0, x \rangle - z$, we have $f^{\gamma(E_1,E_2)}(x_0) \in H$. Then (3), which is verified for all $n \in \mathbb{N}$, shows that $\langle x_0, x_2' \rangle - \sup_{\mathrm{dom} f} \langle \ , x_2' \rangle \leqslant 0$. By (1) this last inequality is absurd. So we have $f^{\gamma(E_1,E_2)}(x_0) = +\infty$.

We now give two examples showing that Theorem II.4.2 cannot be improved. We take $E_1 = H$ and for E_2 the set of increasing maps from H to H. Let $f: H \to \bar{H}$ defined by $f(x) = -\infty$ if $x \leqslant 0$ and $f(x) = +\infty$ if not. From Proposition II.3.1, $(\mathrm{dom} f)^{\gamma(E_1,E_2)} = \{x \in H \mid x \leqslant 0\}$. On the other hand, $f^{\gamma(E_1,E_2)} = (-\infty)^H$ and $\mathrm{dom} f^{\gamma(E_1,E_2)} = H$. So, we do not have $\mathrm{dom} f^{\gamma(E_1,E_2)} \subset (\mathrm{dom} f)^{\gamma(E_1,E_2)}$.

Now we take $E_1 = H = \mathbb{R}$ and for E_2 the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We see easily that $\gamma(E_1, E_2)$-sets coincide here with the closed subsets of $\mathbb{R}$. For every $n \in \mathbb{N}$, $n \geqslant 1$, let us set $f_n(x) = n$ if $x \leqslant 1/n$ and $f_n(x) = 1/x$ if $x \geqslant 1/n$. Since f_n is continuous, $f = \sup_{n \geqslant 1} f_n$ is a $\gamma(E_1, E_2)$-function and we have $f(x) = +\infty$ if $x \leqslant 0$ and $f(x) = 1/x$ if $x > 0$. Then, $\mathrm{dom} f^{\gamma(E_1,E_2)} = \mathrm{dom} f =]0, +\infty[$ and $(\mathrm{dom} f)^{\gamma(E_1,E_2)} = [0, +\infty[$. So, we do not have $(\mathrm{dom} f)^{\gamma(E_1,E_2)} \subset \mathrm{dom} f^{\gamma(E_1,E_2)}$.

THEOREM II.4.3. *Let E_2 be a subgroup of the group of maps from E_1 to H. For every map f from E_1 to H and for every $x_2 \in E_2$ we have*

$$(f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)} = f^{\gamma(E_1,E_2)} - \langle \ , x_2 \rangle.$$

Proof. Let us suppose that $\langle \ , x_2' \rangle - z \leqslant f - \langle \ , x_2 \rangle$ then $\langle \ , x_2 + x_2' \rangle - z \leqslant f^{\gamma(E_1,E_2)}$, that is, $\langle \ , x_2' \rangle - z \leqslant f^{\gamma(E_1,E_2)} - \langle \ , x_2 \rangle$. Taking the supremum we obtain $(f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)} \leqslant f^{\gamma(E_1,E_2)} - \langle \ , x_2 \rangle$. Conversely, let us suppose that $\langle \ , x_2' \rangle - z \leqslant f$. Then $\langle \ , x_2' - x_2 \rangle - z \leqslant f - \langle \ , x_2 \rangle$ and by this $\langle \ , x_2' - x_2 \rangle - z \leqslant (f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)}$. Therefore $\langle \ , x_2' \rangle - z \leqslant (f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)} + \langle \ , x_2 \rangle$ and taking the supremum we obtain

$$f^{\gamma(E_1,E_2)} \leqslant (f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)} + \langle \ , x_2 \rangle.$$

Finally, $f^{\gamma(E_1,E_2)} - \langle \ , x_2 \rangle \leqslant (f - \langle \ , x_2 \rangle)^{\gamma(E_1,E_2)}$.

LEMMA II.4.4. *Let G be a group and ϕ an homomorphism from G to H. If there is $z \in H$ such that $\forall x \in G$: $\phi(x) \leqslant z$ then $\phi = 0^G$.*

Proof. For every $(n, x) \in \mathbb{N} \times G$ we have $n\phi(x) = \phi(nx) \leqslant z$. Therefore, for all $x \in G$, $\phi(x) \leqslant 0$ and obviously $\phi(-x) \leqslant 0$. Now, $\phi(-x) = -\phi(x)$; hence $-\phi(x) \leqslant 0$, that is, $\phi(x) \geqslant 0$ and finally $\phi(x) = 0$.

THEOREM II.4.5. *Let E_1 be a group, E_2 a subgroup of the group of homomorphisms from E_1 to H, f and g two $\gamma(E_1, E_2)$-maps which do not take the value $-\infty$. Let us suppose that there is $(x_2, z) \in E_2 \times H$ such that $f + g \leqslant \langle\ , x_2\rangle - z$. Then we have $f = \langle\ , x_2'\rangle - z'$ and $g = \langle\ , x_2''\rangle - z''$ with (x_2', z') and $(x_2'', z'') \in E_2 \times H$.*

Proof. Since f is a $\gamma(E_1, E_2)$-map which does not take the value $-\infty$, there is $(y_2, t) \in E_2 \times H$ such that $f \geqslant \langle\ , y_2\rangle - t$. Then, for all $(y_2', t') \in E_2 \times H$ with $g \geqslant \langle\ , y_2'\rangle - t'$ we have $\langle\ , y_2\rangle - t + \langle\ , y_2'\rangle - t' \leqslant f + g \leqslant \langle\ , x_2\rangle - z$; hence $\langle\ , y_2 + y_2' - x_2\rangle \leqslant -z + t + t'$. From Lemma II.4.4 we deduce that $\langle\ , y_2 + y_2' - x_2\rangle = 0^{E_1}$, that is, $\langle\ , y_2'\rangle = \langle\ , x_2 - y_2\rangle$. Then

$$\begin{aligned} g &= \sup(\langle\ , x_2 - y_2\rangle - t' \mid \langle\ , x_2 - y_2\rangle - t' \leqslant g) \\ &= \langle\ , x_2 - y_2\rangle + \sup(-t' \mid \langle\ , x_2 - y_2\rangle - t' \leqslant g) \\ &= \langle\ , x_2 - y_2\rangle + \sup_{g^*(x_2 - y_2) \leqslant t'} -t' = \langle\ , x_2 - y_2\rangle - g^*(x_2 - y_2). \end{aligned}$$

Moreover, g does not take the value $-\infty$; g does not take the value $+\infty$ because $f + g \leqslant \langle\ , x_2\rangle - z$. We have then $g^*(x_2 - y_2) \in H$. Interchanging the parts played by f and g we obtain $y_2' \in E_2$ such that $f = \langle\ , x_2 - y_2'\rangle - f^*(x_2 - y_2')$ and this completes the proof.

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