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Function-valued Padé-type approximant via the formal orthogonal polynomials and its applications in solving integral equations☆

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Abstract

A kind of function-valued Padé-type approximant via the formal orthogonal polynomials (FPTAVOP) is introduced on the polynomial space and an algorithm is sketched by means of the formal orthogonal polynomials. This method can be applied to approximate characteristic values and the corresponding characteristic function of Fredholm integral equation of the second kind. Moreover, theoretical analyses show that FPTAVOP method is the most effective one for accelerating the convergence of a sequence of functions. In addition, a typical numerical example is presented to illustrate when the estimates of characteristic value and characteristic function by using this new method are more accurate than other methods.

Keywords: Padé approximant; Formal orthogonal polynomial; Neumann series; Fredholm integral equation; Characteristic value; Convergence acceleration

1. Introduction

Consider a Fredholm integral equation of the second kind

$$f(x,\lambda) = g(x) + \lambda \int_a^b K(x,y) f(y,\lambda)\mathrm{d}y, \quad (x,y) \subset [a,b]\times[a,b],$$

where $g(x) \in L^2[a,b]$ and $K(x,y)$ is an L_2 kernel which are defined in $[a,b]$ and $[a,b]\times[a,b]$, respectively.

The technique utilized for solving the integral equation is based on successive substitution, which is an iterative procedure, yielding a sequence of a approximations leading to an infinite power series solution. So we turn to consider the generating function $f(x,\lambda)$ of this kind of series of functions given by

$$f(x,\lambda) = \sum_{i=0}^{\infty} c_i(x)\lambda^i, \tag{1}$$

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in which $c_i(x) \in L_2[a, b]$ are given by the successive substitution and $[a, b]$ is the domain of definition of $c_i(x)$ in this natural sense. We suppose that $f(x, \lambda)$ is holomorphic as a function of λ at the origin $\lambda = 0$. Then $f(x, \lambda)$ in (1) converges for values of $|\lambda|$ which are sufficiently small.

Many methods estimating the characteristic values and characteristic function of (1) have been derived in the previously published papers, such as the classic Padé approximant method (CPA) [7,8], the generalized inverse, functional-valued Padé approximant method (GIPA) [7,8], the modified Padé approximant method (MPA) [10], the squared Padé approximant method (SPA) [10], the integral Padé approximant method (IPA) [10], the function-valued Padé-type approximant method (FPTA) [13], and the ε-algorithm of function-valued GIPA [11,12].

The major drawback of the method of CPA [7,8] is the use of the minimal sensitivity principle [1,2] and the presence of superfluous zeros in the denominator. Thus we have to assign a particular value of x in the Neumann series in order to obtain the estimate of the characteristic values. Graves-Morris introduced the method of function-valued GIPA [7,8] by using the generalized inverse which had been defined in the case of vector-valued GIPA [9] and a solution in which the numerator function and the denominator polynomial can be represented by determinants of the same dimension. Thukral investigated on the basis of this similar principle and Padé-type approximants introduced in [4,5], which means that the denominator polynomial of the rational approximant is arbitrarily prescribed (whereas, in the classical Padé approach the denominator is left free in order to achieve the maximal order of interpolation), and introduced three methods of Padé-type approximant, namely IPA [10], MPA [10] and SPA [10]. The construction of the denominator of the MPA [10] method is simply obtained by integrating each of the cofactors in the determinant of the denominator polynomial of the classical Padé approximant method. The denominator of the SPA [10] method was constructed by squaring each of the cofactor in the determinant of the denominator polynomial of the classical Padé approximant method and then integrating these new cofactors. The construction of the denominator of the IPA [10] method is obtained by combining the coefficients of the generating function as cofactors in the determinant of the denominator polynomial. The method of FPTA [13] was constructed by a an approach similar to that of the method of IPA [10]. The construction of the ε-algorithm of function-valued GIPA [11,12] was the same as that of ε-algorithm of scalar Padé approximants, just using the generalized inverse in order to keep the denominator polynomial of GIPA [7,8] having the only one parameter of λ.

In this paper we represent a new method, namely function-valued Padé-type approximant via the formal orthogonal polynomials (FPTAVOP), for summing the series of function (1). This method overcomes all essential difficulties encountered in the previous studies and is simpler and more effective for obtaining the characteristic values and the characteristic function than all those methods we have mentioned above.

The remainder of this paper is organized as follows. In Section 2, we mainly derive the method of FPTAVOP. We extend scalar Padé-type approximant to function-valued Padé-type approximant in Section 2.1, introduce the definition of FOPs with respect to $f(x, \lambda)$ and their three-term recurrence relationship and sketch an algorithm to compute the FOPs with respect to $f(x, \lambda)$ in Section 2.2, combine function-valued Padé-type approximant and the FOPs with respect to $f(x, \lambda)$ together to construct FPTAVOP and sketch a main algorithm to compute FPTAVOP in Section 2.3, and finally, in Section 2.4, we present a numerical example to show the effectiveness of FPTAVOP for solving integral equation. In Section 3, we first give the constructions of the methods of GIPA, ε-algorithm of GIPA, IPA, MPA, CPA, SPA and FPTA in the Sections 3.1 and 3.2, and then apply these methods to a typical numerical example in Section 3.3, and finally, in Section 3.4, we make a comparison of the estimates of the characteristic values of the integral equation derived using all these different methods. The concluding remarks are given in Section 4.

2. Function-valued Padé-type approximant via the formal orthogonal polynomials

In this section, we derive a new approach via the formal orthogonal polynomials, i.e., FPTAVOP to approximate the function-valued power series $f(x, \lambda)$ in (1).

2.1. Function-valued Padé-type approximant (FPTA)

Let $c : \mathbf{P} \to \mathbf{C}$ be a linear functional on the polynomial space $\mathbf{P}$, and define it by

$$c(t^i) = c_i(x), \quad i = 0, 1, \ldots \tag{2}$$

with the convention that $c_i(x) = 0$ for $i < 0$. From the linear functional in (2) we can obtain that

$$c((1-t\lambda)^{-1}) = c(1 + t\lambda + (t\lambda)^2 + \cdots)$$
$$= c_0(x) + c_1(x)\lambda + \cdots + c_n(x)\lambda^n + \cdots = f(x, \lambda).$$

Let $v_n \in \mathbf{P}_n$ which is a scalar polynomial of degree n,

$$v_n(\lambda) = b_0 + b_1\lambda + \cdots + b_n\lambda^n$$

and assume $b_n \neq 0$. Define the polynomial $w_n(x, \lambda)$, having function-valued coefficients, by

$$w_n(x, \lambda) = c\left(\frac{v_n(t) - v_n(\lambda)}{t - \lambda}\right).$$

Note that c acts on $\mathbf{P}$ and $w_n(x, \lambda)$ is a function-valued polynomial in λ of degree $n - 1$. Set

$$\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1}), \qquad \tilde{w}_n(x, \lambda) = \lambda^{n-1} w_n(x, \lambda^{-1}).$$

Thus, $R_{n-1,n}(x, \lambda) = \tilde{w}_n(x, \lambda)/\tilde{v}_n(\lambda)$ is defined as a *function-valued Padé-type approximant* (FPTA) of type $(n-1, n)$ and holds

$$\tilde{v}_n(\lambda) f(x, \lambda) - \tilde{w}_n(x, \lambda) = \mathcal{O}(\lambda^n). \tag{3}$$

The polynomial v_n, called the *generating polynomial* of the Padé-type approximation $(n - 1/n)$, can be arbitrarily chosen and so we have n degrees of freedom.

In order to construct rational approximants to $f(x, \lambda)$ with various degrees in the numerator and in the denominator, we have the following relation, which can be deduced from [4]:

$$(m/n) f(x, \lambda) = c_0(x) + c_1(x)\lambda + \cdots + c_{m-n}(x)\lambda^{m-n} + \lambda^{m-n+1}(n - 1/n)_{f_{m-n}}(x, \lambda)$$

with

$$f_{m-n}(x, \lambda) = c_{m-n+1}(x) + c_{m-n+2}(x)\lambda + \cdots.$$

So an FPTA of type (m, n) such as $\tilde{p}_{mn}(x, \lambda)/\tilde{v}_{mn}(\lambda)$ can be defined by

$$\tilde{v}_{mn}(\lambda) f(x, \lambda) - \tilde{p}_{mn}(x, \lambda) = O(\lambda^{m+1}),$$

where

$$\tilde{p}_{mn}(x, \lambda) = \tilde{v}_{mn}(\lambda) \sum_{i=0}^{m-n} c_i(x)\lambda^i + \lambda^{m-n+1}\tilde{w}_{mn}(x, \lambda), \quad m \geq n,$$

and $\tilde{v}_{mn}(\lambda)$, $\tilde{w}_{mn}(x, \lambda)$ satisfy

$$(n - 1/n)_{f_{m-n}}(x, \lambda) = \tilde{w}_{mn}(x, \lambda)/\tilde{v}_{mn}(\lambda)$$

and

$$w_{mn}(x, \lambda) = c^{(m-n+1)}\left(\frac{v_{mn}(t) - v_{mn}(\lambda)}{t - \lambda}\right).$$

Because the first coefficient of $f_{m-n}(x, \lambda)$ is c_{m-n+1}, so it is natural to define a new functional $c^{(m-n+1)}$ by

$$c^{(m-n+1)}(t^i) = c(t^{m-n+1+i}) = c_{m-n+1+i}(x), \quad i = 0, 1, \ldots. \tag{4}$$

The methods we have discussed in the introduction, such as GIPA [7,8], ε-algorithm of function-valued GIPA [11,12], CPA [7,8], MPA [10], SPA [10] and IPA [10] all belong to the family of Padé-type approximants. The denominator polynomials of these rational approximants are arbitrarily prescribed, and the corresponding orders of the approximation are the order of the numerator polynomial of the rational approximant plus one. So the precision of approximation of the characteristic function of those approaches is limited.

2.2. Formal orthogonal polynomials with respect to $f(x,\lambda)$ and their three-term recurrence relationship

In this subsection, we give many notions of formal orthogonal polynomials associated with $f(x,\lambda)$. Brezinski [4,5] defined the scalar formal orthogonal polynomials (FOPs) associated with c which was defined as a linear functional on the space of complex polynomials by $c(t^i) = c_i$ for $i > 0$ with $c_0, c_1, \ldots$ is prescribed nonzero complex numbers. We will analogously give the definition of generalized orthogonal polynomials (GOPs) and formal orthogonal polynomials (FOPs) associated with $f(x,\lambda)$ respectively.

Definition 1. For the linear functional $c^{(m-n+1)}(t^i) = c_{m-n+1+i}(x)$, $i = 0, 1, \ldots$, $\{q_k^{(m-n+1)}\}$ is said to be a family of *generalized orthogonal polynomials* associated with $c^{(m-n+1)}$ if, $\forall k \geq 0$, $q_k^{(m-n+1)}$ has degree k at most if we could arrange that

$$c^{(m-n+1)}(t^i q_k^{(m-n+1)}(t)) = 0 \quad \text{for } i = 0, 1, \ldots, k-1. \tag{5}$$

Now let

$$q_k^{(m-n+1)}(t) = a_0 + a_1 t + \cdots + a_k t^k.$$

Then in terms of k equalities in (5) and the functional definition in (4), there exists a system of k equations in $k+1$ unknowns

$$a_0 c_{m-n+i+1}(x) + a_1 c_{m-n+i+2}(x) + \cdots + a_k c_{m-n+i+k+1}(x) = 0 \quad \text{for } i = 0, \ldots, k-1. \tag{6}$$

We can simply denote (6) by

$$\sum_{j=0}^{k} a_j c_{m-n+i+j+1}(x) = 0 \quad \text{for } i = 0, \ldots, k-1. \tag{7}$$

We notice that the solution of (7) for a_j normally depends on x. So the system of equations represented by (7) normally has no solution for the a_j for $j = 0, 1, \ldots, k$. Thus the system (5) also has no solution for $q_k^{(m-n+1)}(t)$. For this reason we call them generalized orthogonal polynomials.

Obviously, $q_k^{(m-n+1)}(t)$ has exact degree k if and only if a_k is different from zero. In this case, we say that the functional $c^{(m-n+1)}$ is definite. In the sequel, we will assume that the GOPs of $q_k^{(m-n+1)}(t)$ has been normalized to be monic, that is, $a_k = 1$ for all k.

Now let us construct a function-valued *Hankel-like* matrix, whose elements consist of the function-valued coefficients of the power series $f(x,\lambda)$, i.e., $\{c_i(x)\}_{i=m-n+1}^{m-n+2k}$,

$$H_{k,k+1}^{(m-n+1)} = \begin{bmatrix} c_{m-n+1}(x) & c_{m-n+2}(x) & \cdots & c_{m-n+k}(x) & c_{m-n+k+1}(x) \\ c_{m-n+2}(x) & c_{m-n+3}(x) & \cdots & c_{m-n+k+1}(x) & c_{m-n+k+2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m-n+k}(x) & c_{m-n+k+1}(x) & \cdots & c_{m-n+2kn-1}(x) & c_{m-n+2k}(x) \end{bmatrix}.$$

Assuming that $\psi(t) \in P_{k-1}$, $q_k^{(m-n+1)}(t) \in P_k$, by means of the matrix $H_{k,k+1}^{(m-n+1)}$, we define a unsymmetrical bilinear form by

$$\langle \psi(t), q_k^{(m-n+1)}(t) \rangle := v^T H_{k,k+1}^{(m-n+1)} u,$$

for any two polynomials

$$q_k^{(m-n+1)}(t) \equiv [1, t, \ldots, t^k]u, \qquad \psi(t) \equiv [1, t, \ldots, t^{k-1}]v, \quad u \in C^{k+1}, v \in C^k,$$

of degree at most k and $k-1$, $k = 0, 1, \ldots$. Note that

$$\langle t^i, 1 \rangle = e_{i+1}^T H_{k,k+1}^{(m-n+1)} e_1 = c_{m-n+i+1}(x) \quad \text{for } i = 0, 1, \ldots, k-1,$$

where e_i is the ith column of identity matrix of dimension k and e_1 is the first column of identity matrix of dimension $k+1$, hence it is clear that the preceding functional $c^{(m-n+1)}$ is equivalent to this unsymmetrical bilinear form when $k = 0, 1, \ldots$. We define formal orthogonal polynomials as follows.

Definition 2. A polynomial $q_k(t) \in P_k$, $q_k(t) \neq 0$, which satisfies

$$\langle \psi(t), q_k^{(m-n+1)}(t) \rangle = 0 \tag{8}$$

for all $\psi(t) \in P_{k-1}$ is called a *formal orthogonal polynomial* (associated with $f(x, \lambda)$) of degree k.

Actually, the equality (8) can also be rewritten as

$$\langle t^i, q_k^{(m-n+1)}(t) \rangle = e_{i+1}^T H_{k,k+1}^{(m-n+1)} u = 0 \quad \text{for } i = 0, 1, \ldots, k-1. \tag{9}$$

Therefore, when we say a scalar polynomial $q_k^{(m-n+1)}$ belonging to a family of formal orthogonal polynomials associated with $f(x, \lambda)$, we mean that, for $\forall k \geq 0$, $q_k^{(m-n+1)}$ has degree k at most and it satisfies the relationship represented in (9).

One of the most important properties of a family of FOPs (at least for our purpose in this paper) is their three-term recurrence relationship.

Theorem 3. *Let us assume that, for all k, $q_k^{(m-n+1)}(t)$ which is a monic FOP having degree k exactly. Then*

$$q_{k+1}^{(m-n+1)}(t) = (t + B_{k+1}^{(m-n+1)}) q_k^{(m-n+1)}(t) - C_{k+1}^{(m-n+1)} q_{k-1}^{(m-n+1)}(t) \tag{10}$$

with $q_{-1}^{(m-n+1)}(t) = 0$ and $q_0^{(m-n+1)}(t) = 1$. The coefficients can be computed by the following relations:

$$c^{(m-n+1)}(t^k q_k^{(m-n+1)}(t)) = C_{k+1}^{(m-n+1)} C_k^{(m-n+1)} \cdots C_2^{(m-n+1)} c^{(m-n+1)}(q_0^{(m-n+1)}(t)), \tag{11}$$

$$\begin{aligned} c^{(m-n+1)}(t^{k+1} q_k^{(m-n+1)}(t)) &= -C_{k+1}^{(m-n+1)} C_k^{(m-n+1)} \cdots C_2^{(m-n+1)} (B_{k+1}^{(m-n+1)} + \cdots + B_2^{(m-n+1)}) \\ &\quad \times c^{(m-n+1)}(q_0^{(m-n+1)}(t)), \end{aligned} \tag{12}$$

where

$$\hat{c}_j^{(m-n+1)}(t^i) = \int_a^b c_j(x) c_{m-n+i+1}(x) \mathrm{d}x \quad \textit{for } i = 0, 1, \ldots \tag{13}$$

and $j = m - n + 1, \ldots, m - n + 2k$.

The relations (12) and (13) are deduced from [4, pp. 43–49]. It is clear that we have $2k$ choices for the functional of (13). In fact, different choices lead to different precisions. The functional

$$\hat{c}_{m-n+2k}^{(m-n+1)}(t^i) = \int_a^b c_{m-n+2k}(x) c_{m-n+i+1}(x) \mathrm{d}x \quad \text{for } i = 0, 1, \ldots \tag{14}$$

may be the best choice if there is no breakdown in the process of the following Algorithm 4, although it demands higher order coefficients. We will empirically illustrate this point in Remark 9 in Section 3.4.

This theorem can be derived from [4,5], where FOP is related with functional $c^{(n)}(t^i) = c_{n+i}$ for $i = 0, 1, \ldots$. What should be noted is that the functional in (13) is different from $c^{(m-n+1)}$. And we will illuminate why we can make the change with the functional in (13) and what advantages we will get by this way.

The reason we make the change in (13) is based on two points: first, the difficulty we are facing makes us do it this way. The FOPs $q_k^{(m-n+1)}(t)$ are scalar polynomials, but there are no function-valued items in the relations of (11) and (12). Thus we multiply both sides of the equations in (6) with $c_j(x)$ for $j = m-n+1, \ldots, m-n+2k$, respectively, and then take the integral to both sides of them with regard to x from a to b, and so we have k orthogonal relations associated with functional $\hat{c}_j^{(m-n+1)}$ as follows

$$\hat{c}_j^{(m-n+1)}(t^i q_k^{(m-n+1)}(t)) = 0 \quad \text{for } i = 0, 1, \ldots, k-1. \tag{15}$$

That is, the functionals presented in (14) is right in principle, and the $q_k^{(m-n+1)}(t)$ which satisfies the relations in (15) belongs to FOPs. For the convenience, we will replace $c_j^{(m-n+1)}$ by $c^{(m-n+1)}$ in the following text.

Secondly, we know from [8] that when $k = n$, the elements in the determinant of the denominator polynomial of the function-valued Padé approximation have many different forms, such as the following five forms

$$c_{m-n+i+1}(x), \quad \int_a^b c_{m-n+i+1}(x)\mathrm{d}x, \quad \int_a^b c^2_{m-n+i+1}(x)\mathrm{d}x,$$
$$\int_a^b c_{m+n-1}(x)c_{m-n+i+1}(x)\mathrm{d}x, \quad \int_a^b c_{m-n+1}(x)c_{m-n+i+1}(x)\mathrm{d}x.$$

These different forms of elements correspond to different kinds of function-valued Padé-type approximant, namely CPA [10], MPA [10], SPA [10], IPA [10] and FPTA [13]. Referred from [10], the IPA whose determinant's elements are $\int_b^a c_{m+n-1}(x)c_{m-n+i+1}(x)\mathrm{d}x$ for $i = 0, 1, \dots, 2n-2$, may be the best choice in all these kinds of function-valued Padé-type approximations, because this choice is more effective in accelerating the convergence of the sequence of the function $f(x, \lambda)$, and estimates of characteristic value and characteristic function derived using this functional are also found to be more accurate than the others.

For convenience, we will use l_i to substitute $\hat{c}^{(m-n+1)}(t^i)$ hereafter.

Now we sketch the process for gaining a monic FOP for the function-valued series $f(x, \lambda)$ in (1) as follows.

Algorithm 4. (For gaining a monic FOP of degree n associated with the $f(x, \lambda)$)

(0) Set $q^{(m-n+1)}_{-1}(t) = 0$, $q^{(m-n+1)}_0(t) = 1$, and $q^{(m-n+1)}_1(t) = t - \frac{l_1}{l_0}$.
For $k = 1, 2, \dots, n$, do
(1) Use (12) to compute $C^{(m-n+1)}_{k+1}$.
(2) Substitute $C^{(m-n+1)}_{k+1}$ into (13) to compute $B^{(m-n+1)}_{k+1}$.
(3) Use three-term relationship (10) to compute $q^{(m-n+1)}_{k+1}(t)$.

Remark 5. If $k = 0$ in (11), we have

$$q^{(m-n+1)}_1(t) = t + B^{(m-n+1)}_1.$$

By the definition of FOP, $q^{(m-n+1)}_1(t)$ satisfies that

$$\hat{c}^{(m-n+1)}(q^{(m-n+1)}_1(t)) = 0,$$

that is,

$$l_1 + l_0 B^{(m-n+1)}_1 = 0.$$

So we have $B^{(m-n+1)}_1 = -l_1/l_0$.

Thus, we obtain the formal orthogonal polynomial with the first degree in step (0) as follows

$$q^{(m-n+1)}_1(t) = t - \frac{l_1}{l_0}.$$

Another fact to be noticed is that the algorithm does not break down unless

$$\hat{c}^{(m-n+1)}(q^{(m-n+1)}_0(t)) = l_0 = 0.$$

In other words, whether the algorithm is having a breakdown or not is easy to judge.

2.3. Function-valued Padé-type approximation via the FOPs

In terms of the error formula of Padé-type approximation [5], it holds that

$$\begin{aligned} f(x,\lambda) - \frac{\tilde{p}_{mn}(x,\lambda)}{\tilde{v}_{mn}(\lambda)} &= \frac{\lambda^{m+1}}{\tilde{v}_{mn}(\lambda)} c^{(m-n+1)}\left(\frac{v_{mn}(t)}{1-t\lambda}\right) \\ &= \frac{\lambda^{m+1}}{\tilde{v}_{mn}(\lambda)} c^{(m-n+1)}\left(v_{mn}(t)\left(1 + t\lambda + \cdots + t^{n-1}\lambda^{n-1} + \frac{t^n\lambda^n}{1-t\lambda}\right)\right). \end{aligned} \tag{16}$$

It is easy to see from (16) that if we want to improve the order of approximation, we can let $c^{(m-n+1)}(v_{mn}(t)) = 0$, then we will have $(m/n)_f(x,\lambda) - f(x,\lambda) = \mathcal{O}(\lambda^{m+2})$. If $v_{mn}(t)$ satisfies $c^{(m-n+1)}(v_{mn}(t)) = 0$ and $c^{(m-n+1)}(tv_{mn}(t)) = 0$, then we will obtain an error in $\mathcal{O}(\lambda^{m+3})$, and so on. Such approximation, with an improved order of approximation, are called *high-order Padé approximants*.

Since we have n degrees of freedom, let us take v_{mn} such that

$$c^{(m-n+1)}(t^i v_{mn}(t)) = 0, \quad i = 0, \ldots, n-1. \tag{17}$$

In that case we have

$$\tilde{v}_{mn}(\lambda) f(x,\lambda) - \tilde{p}_{mn}(x,\lambda) = \mathcal{O}(\lambda^{m+n+1}). \tag{18}$$

Thus $\tilde{p}_{mn}(x,\lambda)/\tilde{v}_{mn}(\lambda)$ is called *Padé approximant* for $f(x,\lambda)$ and is denoted by $[m/n]_f(x,\lambda)$.

In the case of the Padé approximant, v_{mn} is not arbitrarily chosen any more, but is determined by (17). The relations $c^{(m-n+1)}(t^i v_{mn}(t)) = 0$ for $i = 0, \ldots, n-1$ show that v_{mn} is the polynomial of degree n belonging to the family of generalized orthogonal polynomials associated with the functional $c^{(m-n+1)}$. In the sequel, $\tilde{p}_{mn}(x,\lambda)$ and $\tilde{v}_{mn}(\lambda)$ will be denoted by $\tilde{p}_m^{(m-n+1)}(x,\lambda)$ and $\tilde{q}_n^{(m-n+1)}(\lambda)$ respectively.

For convenience, we give a new definition about quasi-orthogonal polynomial associated with $f(x,\lambda)$.

Definition 6. Assume a scalar polynomial

$$v_n(t) = b_0 + b_1 t + \cdots + b_n t^n = \sum_{i=0}^{n} b_i t^i$$

belonging to a family of formal orthogonal polynomials associated with $f(x,\lambda)$, then another scalar polynomial

$$\tilde{v}_n(t) = t^n v_n(t^{-1}) = b_n + b_{n-1} t + \cdots + b_0 t^n = \sum_{i=0}^{n} b_i t^{n-i}$$

is called a *quasi-orthogonal polynomial* associated with $f(x,\lambda)$ for $v_n(t)$.

With all these preliminaries, we present the main theorem as follows for computing function-valued Padé-type approximation via the formal orthogonal polynomials.

Theorem 7. *Let $q_n^{(m-n+1)}(t)$ be a scalar polynomial which belongs to a family of formal orthogonal polynomials associated with $f(x,\lambda)$, that is,*

$$q_n^{(m-n+1)}(t) = a_0 + a_1 t + \cdots + a_n t^n = \sum_{i=0}^{n} a_i t^i$$

satisfies

$$\hat{c}^{(m-n+1)}(t^i q_n^{(m-n+1)}(t)) = 0 \quad \textit{for } i = 0, 1, \ldots, n-1,$$

and its quasi-orthogonal polynomial is

$$\tilde{q}_n^{(m-n+1)}(t) = t^n q_n^{(m-n+1)}(t^{-1}) = \sum_{i=0}^{n} a_i t^{n-i}.$$

Define the polynomial $\tilde{p}_m^{(m-n+1)}(x,\lambda)$ with the function-valued coefficient by

$$\tilde{p}_m^{(m-n+1)}(x,\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i} f_{m-n+i}(x,\lambda),$$

where

$$f_k(x,\lambda) = \sum_{j=0}^{k} c_j(x)\lambda^j \quad \textit{for } k \geq 0,$$
$$= 0 \quad \textit{for } k < 0.$$

Then, it holds

$$(m/n)_f(x,\lambda) = \frac{\tilde{p}_m^{(m-n+1)}(x,\lambda)}{\tilde{q}_n^{(m-n+1)}(\lambda)} = \frac{\sum_{i=0}^{n} a_i \lambda^{n-i} f_{m-n+i}(x,\lambda)}{\sum_{i=0}^{n} a_i \lambda^{n-i}},$$

that is,

$$\tilde{q}_n^{(m-n+1)}(\lambda) f(x,\lambda) - \tilde{p}_m^{(m-n+1)}(\lambda) = \mathcal{O}(\lambda^{m+1}).$$

Proof. We deduce that

$$\begin{aligned}
&\tilde{q}_n^{(m-n+1)}(\lambda) f(x,\lambda) - \tilde{p}_m^{(m-n+1)}(x,\lambda) \\
&= \sum_{i=0}^{n} a_i \lambda^{n-i} \sum_{j=0}^{\infty} c_j(x)\lambda^j - \sum_{i=0}^{n} a_i \lambda^{n-i} \sum_{j=0}^{m-n+i} c_j(x)\lambda^j \\
&= \sum_{i=0}^{n} a_i \lambda^{n-i} \sum_{j=m-n+i+1}^{\infty} c_j(x)\lambda^j \\
&\quad \text{(to be expanded in terms of } i \text{ from 0 to } n) \\
&= a_0\lambda^n \sum_{j=m-n+1}^{\infty} c_j(x)\lambda^j + a_1\lambda^{n-1} \sum_{j=m-n+2}^{\infty} c_j(x)\lambda^j + \cdots + a_n \sum_{j=m+1}^{\infty} c_j(x)\lambda^j \\
&\quad \text{(to be considered as a polynomial of } \lambda \text{ and the terms with the same order will be lumped together)} \\
&= \sum_{i=0}^{n} a_i c_{m-n+i+1}(x)\lambda^{m+1} + \sum_{i=0}^{n} a_i c_{m-n+i+2}(x)\lambda^{m+2} + \cdots + \sum_{i=0}^{n} a_i c_{m+i}(x)\lambda^{m+n} \\
&\quad + \sum_{i=0}^{n} a_i c_{m+i+1}(x)\lambda^{m+n+1} + \cdots \\
&= \mathcal{O}(\lambda^{m+1}). \quad \square
\end{aligned}$$

From the proof we can see that if we could let the system of

$$\sum_{i=0}^{n} a_i c_{m-n+i+j+1}(x) = 0 \quad \text{for } j = 0, 1, \ldots, n-1$$

has solution for a_j, the coefficients of $\lambda^{m+1}, \lambda^{m+2}, \ldots, \lambda^{m+n+1}$ in the second line from bottom of our proof will be equal to zero. Then the approximation order will be improved to $\mathcal{O}(\lambda^{m+n+1})$, which is higher than the order we have obtained. Meanwhile, according to the equality (18), the approximants we obtained will belong to function-valued Padé approximants. But unfortunately, we have not been able to guarantee that the solution of the system comes into existence, so far. We hope we could solve this problem by some certain ways in the future.

We now give the complete algorithm to compute function-valued Padé-type approximants via the formal orthogonal polynomials.

Algorithm 8. Do step (0)–(3) in Algorithm 4 to get $q_n^{(m-n+1)}(t)$.

(4) Rewrite $q_n^{(m-n+1)}(t)$ in the form of $\sum_{i=0}^{n} a_i t^i$ and compute the denominator of FPTAVOP by

$$\tilde{q}_n^{(m-n+1)}(\lambda) = \lambda^n q_n^{(m-n+1)}(\lambda^{-1}) = \sum_{i=0}^{n} a_i \lambda^{n-i}.$$

(5) Compute the numerator of the FPTAVOP by

$$\tilde{p}_m^{(m-n+1)}(x,\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i} f_{m-n+i}(x,\lambda).$$

(6) Obtain FPTAVOP for $f(x,\lambda)$

$$(m/n)_f(x,\lambda) = \frac{\sum_{i=0}^{n} a_i \lambda^{n-i} f_{m-n+i}(x,\lambda)}{\sum_{i=0}^{n} a_i \lambda^{n-i}}.$$

2.4. A numerical example

Let us consider Fredholm integral equation of the second kind [6,13]

$$\varphi(x) = f(x,\lambda) = 1 + \lambda \int_0^{\pi} \sin(x+y)\varphi(y)\mathrm{d}y.$$

The first few terms of the power series for $f(x,\lambda)$ are given by

$$f(x,\lambda) = 1 + 2\cos x\lambda + \pi \sin x\lambda^2 + \frac{\pi^2}{2}\cos x\lambda^3 + \frac{\pi^3}{4}\cos x\lambda^4 + \cdots.$$

Suppose that we want to find a function-valued Padé approximation $(2/2)_f(x,\lambda)$ for the given integral equation, here, $m = n = 2$ and $m - n + 1 = 1$. First we compute the denominator, $\tilde{q}_n^{(m-n+1)}(\lambda)$, namely $\tilde{q}_2^{(1)}(\lambda)$. So we shall compute the FOP of $q_2^{(1)}(t)$ for $f(x,\lambda)$. Here we let j in (14) be equal to $m - n + 1 = 1$, and so

$$\hat{c}_j^{(m-n+1)}(t^i) = \hat{c}_1^{(1)}(t^i) = \int_a^b c_1(x)c_{i+1}(x)\mathrm{d}x, \quad i = 0, 1, \ldots.$$

Now we apply Algorithm 8 to compute FPTAVOP of type (2/2) for this integral equation.

(0) $q_0^{(1)}(t) = 1$, and

$$q_1^{(1)}(t) = ta - \frac{l_1}{l_0} = t - \frac{\int_0^{\pi} 2\cos(x)\cdot\pi\sin(x)\mathrm{d}x}{\int_0^{\pi} 4\cos^2(x)\mathrm{d}x} = t - \frac{0}{-2\pi} = t;$$

(1) Use (12) to compute $C_2^{(1)}$:

$$\hat{c}_1^{(1)}(tq_1^{(1)}(t)) = C_2^{(1)}\hat{c}_1^{(1)}(q_0^{(1)}(t)) = C_2^{(1)}\hat{c}_1^{(1)}(1) = C_2^{(1)}l_0.$$

From this equality and $q_1^{(1)}$ which has been obtained in step (0), we can easily obtain that

$$C_2^{(1)} = \frac{\hat{c}_1^{(1)}(t^2)}{l_0} = \frac{l_2}{l_0} = \frac{\int_0^{\pi} 2\cos(x)\cdot\frac{\pi^2}{2}\cos(x)\mathrm{d}x}{\int_1^{\pi} 4\cos^2(x)\mathrm{d}x} = \frac{-\pi^3/2}{-2\pi} = \frac{\pi^2}{4}.$$

(2) Substitute $C_2^{(1)}$ into (13) to compute $B_2^{(1)}$:

$$\hat{c}_1^{(1)}(t^2 q_1^{(1)}(t)) = -C_2^{(1)}B_2^{(1)}l_1,$$

$$B_2^{(1)} = \frac{\hat{c}_1^{(1)}(t^3)}{-C_2^{(1)}l_0} = \frac{l_3}{-(\pi^2/4)\cdot(-2\pi)} = \frac{0}{\pi^3/2} = 0.$$

(3) Use three-term relationship (11) and coefficients above to obtain $q_2^{(1)}$:

$$\begin{aligned} q_2^{(1)}(t) &= (t + B_2^{(1)})q_1^{(1)}(t) - C_2^{(1)}q_0^{(1)}(t) \\ &= t^2 - \frac{\pi^2}{4}. \end{aligned}$$

(4) Rewrite $q_2^{(1)}(t)$ in the form of $\sum_{i=0}^{2} a_i t^i$. Obviously, $a_0 = -\pi^2/4$, $a_1 = 0$, $a_2 = 1$, and

$$\tilde{q}_2^{(1)}(\lambda) = \lambda^2 q_2^{(1)}(\lambda^{-1}) = -\frac{\pi^2}{4}t^2 + 1.$$

(5) Compute the numerator of FPTAVOP and get

$$\tilde{p}_2^{(1)}(\lambda) = \sum_{i=0}^{2} a_i \lambda^{2-i} f_i(x,\lambda) = -\frac{\pi^2}{4}\lambda^2 \cdot 1 + (1 + 2\cos x\lambda + \pi \sin x\lambda^2).$$

(6) Finally, we obtain

$$\begin{aligned}(2/2)_f(x,\lambda) &= \frac{1 - \frac{\pi^2}{4}\lambda^2 \cdot 1 + 2\cos x\lambda + \pi \sin x\lambda^2}{1 - \frac{\pi^2}{4}\lambda^2} \\ &= 1 + \frac{4(2\cos x\lambda + \pi \sin x\lambda^2)}{4 - \pi^2\lambda^2}.\end{aligned}$$

In fact, the solution obtained from FPTAVOP is the exact solution of the original equation.

If we choose j in (14) as $m - n + 1 + 1$, which is equal to 2 here, then we have

$$\hat{c}_j^{(m-n+1)}(t^i) = \hat{c}_2^{(1)}(t^i) = \int_a^b c_2(x)c_{i+1}(x)\mathrm{d}x, \quad i = 0, 1, \ldots.$$

Unfortunately, it follows that

$$l_0 = \hat{c}_2^{(1)}(t^0) = \int_a^b c_2(x)c_1(x)\mathrm{d}x = \int_0^\pi 2\cos(x)\cdot\sin(x)\pi\,\mathrm{d}x = 0.$$

So we could not use it in the denominator, and then $q_1^{(1)}(t)$ which is equal to $t - l_1/l_0$ could not be computed. The process is stopped.

3. Comparisons by a typical numerical example with the previous methods

To determine the consistency of the new method, we actually tested it on the previous investigations [1,2,7, 8,10–13]. These findings are generalized by illustrating the effectiveness of the new method for determining the characteristic values and the characteristic function of a familiar linear integral equation.

We first introduce the construction of GIPA [7,8] and ε-algorithm of function-valued GIPA [12,13] in Section 3.1, the construction of IPA [10], CPA [7,8], MPA [10], SPA [10] and FPTA [13] in Section 3.2, and then apply them including FPTAVOP on a typical example in Sections 3.3 and 3.4.

3.1. The methods of function-valued GIPA [7,8] and ε-algorithm of function-valued GIPA [12,13]

We define a rational function $r(x,\lambda)$ to be a function-valued GIPA of type $(m, 2n)$ [7,8] for $f(x,\lambda)$ if

$$r(x,\lambda) = p(x,\lambda)/q(\lambda),$$

where $p(x,\lambda)$, $q(\lambda)$ are polynomials in x, $p(x,\lambda) \in L_2[a,b]$ as a function of λ and

$$\partial\{p(x,\lambda)\} \le n - \alpha, \qquad \partial\{q(\lambda)\} \le 2n - 2\alpha, \quad \text{for } \alpha \ge 0, \tag{19a}$$

$$q(\lambda) \left| \int_a^b |p(x,\lambda)|^2 \mathrm{d}x \right., \tag{19b}$$

$$q(\lambda) = q^*(\lambda) \tag{19c}$$

$$q(0) \ne 0, \tag{19d}$$

$$p(x,\lambda) - q(\lambda)f(x,\lambda) = \mathcal{O}(\lambda^{m+1}). \tag{19e}$$

The asterisk in (19c) denotes the functional complex conjugate.

If $p(x,\lambda)$, $q(\lambda)$, satisfy (19a)–(19e), then $r(x,\lambda)$ is unique. The explicit formula for the denominator polynomial is given by

$$q(\lambda)=\begin{vmatrix} 0 & M_{01} & \cdots & M_{0,2n-1} & M_{0,2n} \\ -M_{01} & 0 & \cdots & M_{1,2n-1} & M_{1,2n} \\ \vdots & \vdots & \cdots & \vdots & \\ -M_{0,2n-1} & -M_{1,2n-1} & \cdots & 0 & M_{2n-1,2n} \\ \lambda^{2n} & \lambda^{2n-1} & \cdots & \lambda & 1 \end{vmatrix}. \tag{20}$$

The elements of (20) are defined by

$$M_{ij}:=\sum_{l=0}^{j-i-1}\int_a^b c_{l+i+m-2n+1}(x)[c_{j-i+m-2n}(x)]^*\mathrm{d}x,$$

for $i=0,1,\ldots,2n$ and $j=i+1,i+2,\ldots,2n$, and taking $c_j:=0$ if $j<0$.

It is well-known that the polynomials produced by (20) are strictly positive for $\lambda\in R$ and their zeros occur in complex conjugate pairs close to the real axis [7]. Ideally, we take the real parts of the zeros of $q(\lambda)$ as the estimates of the characteristic values λ_c.

The numerator polynomial $P(x,\lambda)$ follows from (19e) as

$$p(x,\lambda)=[q(\lambda)f(x,\lambda)]_0^n. \tag{21}$$

If in the determinant (20), $q(0)\neq 0$, then $r(x,\lambda)$ defined by (18), (20) and (21) is the GIPA of type $(m,2n)$ for $f(x,\lambda)$. The case of hybrid method of GIPA requires much more numerical computation than GIPA and so the reader is referred to [7,8].

We define the generalized inverse of the arbitrary $g(x,\lambda)\in L_2(a,b)$ as follows

$$g(x,\lambda)^{(-1)}=\frac{1}{g(x,\lambda)}=\frac{g(x,\lambda)}{\int_a^b\mid g(x,\lambda)\mid^2\mathrm{d}x}. \tag{22}$$

Then ε-algorithm of function-valued GIPA [12] can be defined by the generalized inverse defined by (22):

$$\begin{aligned} &\varepsilon_{-1}^{(j)}=0,\quad j=0,1,2,\ldots,\\ &\varepsilon_0^{(j)}=\sum_{i=0}^{j}c_i(x)\lambda^i,\\ &\varepsilon_{k+1}^{(j)}=\varepsilon_{k-1}^{(j+1)}+(\varepsilon_k^{(j+1)}-\varepsilon_k^{(j)})^{-1},\quad j,k\geq 0. \end{aligned} \tag{23}$$

If in the process of constructing $\varepsilon_{2k}^{(j)}$ there is no breakdown, that is, the denominator in the third equality in (23) is not equal to zero, we can obtain a rational approximant $R(x,\lambda)=P_{j+2k}(x,\lambda)/Q_{2k}(\lambda)$, which was proved to be just a GIPA of type $(j+2k,2k)$ in [12], i.e.,

$$\varepsilon_{2k}^{(j)}=\frac{P_{j+2k}(x,\lambda)}{Q_{2k}(\lambda)}\quad\text{as } j,k\geq 0$$

is a GIPA of type $(j+2k,2k)$ for $f(x,\lambda)$. In this case, we take the real parts of the zeros of $Q_{2k}(\lambda)$ as the estimates of the characteristic values λ_c.

3.2. The methods of IPA [10], CPA [7,8], MPA [10], SPA [10] and FPTA [13]

We define a rational function $r(x,\lambda)$ to be an IPA [10] of type (m,n) for $f(x,\lambda)$ if

$$r(x,\lambda)=N(x,\lambda)/D(\lambda), \tag{24}$$

where $N(x,\lambda)$, $D(\lambda)$ are polynomials in λ, $N(x,\lambda)\in L_2[a,b]$ as a function of x and they satisfy

$$N(x,\lambda)-D(\lambda)f(x,\lambda)=\mathcal{O}(\lambda^{m+1}). \tag{25}$$

The denominator polynomial of the integral Padé approximant was defined by

$$D(\lambda)=\begin{vmatrix}\int_a^b c_{m+n-1}(x)c_{m-n}(x)\mathrm{d}x & \int_a^b c_{m+n-1}(x)c_{m-n+1}(x)\mathrm{d}x & \cdots & \int_a^b c_{m+n-1}(x)c_{m}(x)\mathrm{d}x\\ \int_a^b c_{m+n-1}(x)c_{m-n+1}(x)\mathrm{d}x & \int_a^b c_{m+n-1}(x)c_{m-n+2}(x)\mathrm{d}x & \cdots & \int_a^b c_{m+n-1}(x)c_{m+1}(x)\mathrm{d}x\\ \vdots & \vdots & \cdots & \vdots\\ \int_a^b c_{m+n-1}(x)c_{m-1}(x)\mathrm{d}x & \int_a^b c_{m+n-1}(x)c_{m}(x)\mathrm{d}x & \cdots & \int_a^b c_{m+n-1}(x)c_{m+n-1}(x)\mathrm{d}x\\ \lambda^n & \lambda^{n-1} & \cdots & 1\end{vmatrix} \tag{26}$$

provided $D(0)\neq 0$ and $c_i(x)$ are the coefficients of (1), and naturally the numerator polynomial $N(x,\lambda)$ follows from (25) as

$$N(x,\lambda)=[D(\lambda)f(x,\lambda)]_0^m,$$

where this notation indicates that truncation at degree m in λ has been effected.

The purpose of integrating the elements in (26) is to make the estimates of the characteristic values independent of the variable x. This is similar to the approach for the method of GIPA [7,8] in which we overcome the series problem, using [1,2], with CPA [7,8]. We take the appropriate roots of denominator polynomial, given by (26), as the estimates of the characteristic value for the IPA method.

We can analogously define the methods of CPA [7,8], MPA [10] and SPA [10]. First we give the general determinant of the denominator of the approximants as follows

$$D(\lambda)=\begin{vmatrix}\mu_{m-n+1} & \mu_{m-n+2} & \cdots & \mu_{m+1}\\ \mu_{m-n+2} & \mu_{m-n+3} & \cdots & \mu_{m+2}\\ \vdots & \vdots & \cdots & \vdots\\ \mu_m & \mu_{m+1} & \cdots & \mu_{m+n}\\ \lambda^n & \lambda^{n-1} & \cdots & 1\end{vmatrix}, \tag{27}$$

where $D(0)\neq 0$ and by the means of [10] we know that if

$$\mu_i=c_i(x)\quad \text{for } i=m-n+1,\ldots,m+n, \tag{28}$$

the $D(\lambda)$ is the determinant of the denominator polynomial of the method of CPA [7,8], if

$$\mu_i=\int_a^b c_i(x)\mathrm{d}x\quad \text{for } i=m-n+1,\ldots,m+n, \tag{29}$$

the $D(\lambda)$ is the determinant of the denominator polynomial of the method of MPA [10], and if

$$\mu_i=\int_a^b c_i^2(x)\mathrm{d}x\quad \text{for } i=m-n+1,\ldots,m+n, \tag{30}$$

the $D(\lambda)$ is the determinant of the denominator polynomial of the method of SPA [10]. Their definitions are given by (24), (25) and (27) combined with (28)–(30) respectively.

If we evaluate μ_i with

$$\int_a^b c_{m-n+1}(x)c_i(x)\mathrm{d}x\quad \text{for } i=m-n+1,\ldots,m+n, \tag{31}$$

and $D(0)\neq 0$, then $r(x,\lambda)$ defined by (24), (25), (27) and (31) is an FPTA [13] of type (m,n) for $f(x,\lambda)$.

3.3. Application to an integral equation

We investigate the convergence of sequences of these eight methods presented for the Neumann series solution of the linear integral equation

$$f(x,\lambda) = 1 + \lambda \int_0^1 (1 + |x-y|) f(y,\lambda) \mathrm{d}y. \tag{32}$$

This integral equation is a Fredholm of the second kind with a nondegenerate kernel and has been previously considered in [7,10].

The characteristic functions of this equation can be found by converting it to a second-order ordinary differential equation [3]. The explicit solution of (32) is

$$f(x,\lambda) = \frac{2\cosh \nu(x-1/2)}{2\cosh \nu/2 - 3\nu \sinh \nu/2}, \tag{33}$$

where $\nu = \sqrt{2\lambda}$. The denominator of (33) is analytic as a function of λ and has just one simple zero at

$$\nu_c = 1.22290658\ldots,$$

corresponding to a single characteristic value

$$\lambda_c = 0.7477502556\ldots.$$

It is familiar that the Neumann series of (32) converges for $|\lambda| < \lambda_c$. The first few terms of this series are

$$f(x,\lambda) = \sum_{i=0}^{\infty} c_i(x)\lambda^i = 1 + \left[\frac{5}{4} + \left(x - \frac{1}{2}\right)^2\right]\lambda + \left[\frac{161}{96} + \frac{5}{4}\left(x-\frac{1}{2}\right)^2 + \frac{1}{6}\left(x - \frac{1}{2}\right)^4\right]\lambda^2 + \cdots \tag{34}$$

as may be found by iteration of (32).

Before we apply these eight methods on the integral equation (32), we simply enumerate each process for computing the methods of GIPA [7,8], ε-algorithm of GIPA [12], IPA [10], MPA [10], SPA [10], CPA [7,8], FPTA [13] and FPTAVOP respectively.

(1) *GIPA* [7,8]

We begin with expanding the denominator polynomial of the GIPA of type $(m, 2)$,

$$q(\lambda) = \begin{vmatrix} 0 & \int_0^1 c_{m-1}^2(x)\mathrm{d}x & 2\int_0^1 c_{m-1}(x)c_m(x)\mathrm{d}x \\ -\int_0^1 c_{m-1}^2(x)\mathrm{d}x & 0 & \int_0^1 c_m^2(x)\mathrm{d}x \\ \lambda^2 & \lambda & 1 \end{vmatrix},$$

which gives

$$q(\lambda) = \int_0^1 c_{m-1}^2(x)\mathrm{d}x \left[\int_0^1 c_{m-1}^2(x)\mathrm{d}x + \lambda^2 \int_0^1 c_m^2(x)\mathrm{d}x\right] - 2\lambda \int_0^1 c_{m-1}^2(x)\mathrm{d}x \left[\int_0^1 c_{m-1}(x)c_m(x)\mathrm{d}x\right]. \tag{35}$$

Solving (35) by a standard quadratic formula and simplifying gives rise to

$$\lambda = \frac{\int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x \pm \sqrt{[\{\int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x\}^2] - \int_0^1 c_m^2(x)\mathrm{d}x \int_0^1 c_{m-1}^2(x)\mathrm{d}x}}{\int_0^1 c_m^2(x)\mathrm{d}x}.$$

We know from previous studies [7,8] that the above formula produces a pair of complex conjugates which are close to the real axis, that is

$$\left[\int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x\right]^2 < \int_0^1 C_m^2(x)\mathrm{d}x \int_0^1 C_{m-1}^2(x)\mathrm{d}x.$$

Therefore, the estimate of the characteristic value of the method of GIPA of type $(m, 2)$ is given by

$$\lambda = \int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x \bigg/ \int_0^1 c_m^2(x)\mathrm{d}x.$$

(2) *ε-algorithm of GIPA* [12]

From the iterative formulas in (23) and the generalized inverse, we have

$$\begin{aligned}
\varepsilon_0^{(0)} &= c_0(x),\\
\varepsilon_0^{(1)} &= c_0(x) + c_1(x)\lambda,\\
\varepsilon_0^{(2)} &= c_0(x) + c_1(x)\lambda + c_2(x)\lambda^2,\\
\varepsilon_1^{(0)} &= \frac{1}{c_1(x)\lambda} = \frac{c_1(x)}{\lambda \int_0^1 (c_1(x))^2\mathrm{d}x},\\
\varepsilon_1^{(1)} &= \frac{1}{c_2(x)\lambda^2} = \frac{c_2(x)}{\lambda^2 \int_0^1 (c_2(x))^2\mathrm{d}x},\\
\varepsilon_2^{(0)} &= c_0(x) + c_1(x)\lambda + 1 \bigg/ \left(\frac{c_2(x)}{\lambda^2 \int_0^1 (c_2(x))^2\mathrm{d}x} - \frac{c_1(x)}{\lambda \int_0^1 (c_1(x))^2\mathrm{d}x} \right)\\
&= P_2(x,\lambda)/Q_2(\lambda) = R(x,\lambda),\\
&\vdots
\end{aligned}$$

After iteratively computing out the $\varepsilon_{2n}^{(j)}$ we obtain the approximant of type of $(j+2n, 2n)$ for $f(x,\lambda)$, and we take the real parts of the zeros of the denominator of the estimate of the characteristic value of the ε-algorithm of GIPA.

(3) *IPA* [10]

The denominator polynomial of the method of IPA of type $(m, 1)$ is given by

$$D(\lambda) = \begin{vmatrix} \int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x & \int_0^1 c_m^2(x)\mathrm{d}x \\ \lambda & 1 \end{vmatrix},$$

which gives

$$D(\lambda) = \int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x - \lambda \int_0^1 c_m^2(x)\mathrm{d}x. \tag{36}$$

The characteristic value of the method IPA of type $(m, 1)$ is calculated by solving (36) and thus we have

$$\lambda = \int_0^1 c_m(x)c_{m-1}(x)\mathrm{d}x \bigg/ \int_0^1 c_m^2(x)\mathrm{d}x.$$

(4) *CPA* [7,8]

From (27), the denominator polynomial of the methods of CPA [7,8], MPA [10], SPA [10] and FPTA [13] of type $(m, 1)$ can be expressed as

$$D(\lambda) = \begin{vmatrix} \mu_m & \mu_{m+1} \\ \lambda & 1 \end{vmatrix},$$

which gives

$$D(\lambda) = \mu_m - \lambda\mu_{m+1}. \tag{37}$$

The characteristic value is calculated by solving (37) and we have

$$\lambda = \mu_m/\mu_{m+1}.$$

For CPA, $\mu_i = c_i(x)$, and

$$\lambda = c_m(x)/c_{m+1}(x).$$

Thus we must assign a particular value to x, and this is usually done using the principle of minimal sensitivity [1]. Because $x = 1/2$ is a turning point of each coefficient in (34), it is also a turning point of each CPA, and we take $x = 1/2$ as the most natural value to take in this example, that is, the estimates of characteristic value of the method CPA of type $(m, 1)$ can be obtained by

$$\lambda = c_m(x)/c_{m+1}(x),$$

where $x = 1/2$.

(5) *MPA* [12]

For the method of MPA of type $(m, 1)$, the elements in the denominator polynomial $D(\lambda)$ of (37) μ_i are equal to $\int_0^1 c_i(x)\mathrm{d}x$ for $i = m, m+1$, and thus we have

$$\lambda = \int_0^1 c_m(x)\mathrm{d}x \bigg/ \int_0^1 c_{m+1}(x)\mathrm{d}x.$$

(6) *SPA* [10]

In the case of SPA, the elements in the denominator polynomial $D(\lambda)$ of (37) μ_i are equal to $\int_0^1 c_i^2(x)\mathrm{d}x$ for $i = m, m+1$. We take the square root of (37) as the estimate of the characteristic value

$$\lambda = \sqrt{|\mu_m/\mu_{m+1}|} = \sqrt{\left|\int_0^1 c_m^2(x)\mathrm{d}x \bigg/ \int_0^1 c_{m+1}^2(x)(x)\mathrm{d}x\right|}.$$

We refer the reader to [10] for a more detailed discussion of SPA.

(7) *FPTA* [13]

For the method of FPTA of type $(m, 1)$, the elements in the denominator polynomial $D(\lambda)$ of (37) μ_i are equal to $\int_0^1 c_m c_i(x)\mathrm{d}x$, and thus we have

$$\lambda = \int_0^1 c_m^2(x)\mathrm{d}x \bigg/ \int_0^1 c_{m+1}c_m(x)(x)\mathrm{d}x.$$

(8) *FPTAVOP*

First we calculate the denominator of the FPTAVOP method of type $(m, 1)$, i.e., formal orthogonal polynomials associated with the function-valued power series (1) by Algorithm 4

$$q_1^{(m)}(\lambda) = \lambda - \frac{\int_0^1 c_{m+1}^2(x)\mathrm{d}x}{\int_0^1 c_m(x)c_{m+1}(x)\mathrm{d}x},$$

and then we obtain $\tilde{q}_1^{(m)}(\lambda)$ by Algorithm 8:

$$\tilde{q}_1^{(m)}(\lambda) = -\frac{\int_0^1 c_{m+1}^2(x)\mathrm{d}x}{\int_0^1 c_m(x)c_{m+1}(x)\mathrm{d}x}\lambda + 1. \tag{38}$$

The characteristic value of the FPTAVOP method of type $(m, 1)$ is calculated by solving (38) and thus we have

$$\lambda = \int_0^1 c_m(x)c_{m+1}(x)\mathrm{d}x \bigg/ \int_0^1 c_{m+1}^2(x)\mathrm{d}x.$$

We make a comparison of the estimates formed using the row sequence of the method of FPTAVOP of type $(m, 1)$ with corresponding estimates derived from the method of IPA of type $(m, 1)$, the method of MPA of type $(m, 1)$, the method of CPA of type $(m, 1)$, the SPA method of type $(m, 1)$, the method of FPAT of type $(m, 1)$, the method of GIPA

Table 1
Estimates of the characteristic value λ_c; The exact value is $\lambda_c = 0.747750255563845043\ldots$

m	$\lambda_c^{\text{FPTAVOP}}$	$\lambda_c^I = \lambda_c^G = \lambda_c^\varepsilon$	λ_c^{FPTA}	λ_c^{SPA}	λ_c^{MPA}	λ_c^{CPA}
1	0.74775011	0.74766	0.747753	0.7488	0.75	0.8
2	0.7477502553	0.74775011	0.747750261	0.747752	0.74766	0.745
3	0.7477502555635	0.7477502553	0.747750259	0.7477502583	0.747754	0.7485
4	0.7477502555638444	0.7477502555635	0.747750256	0.747750255568	0.74775011	0.747746

of type $(m, 2)$, and ε-algorithm of GIPA of type $(m, 2)$. The results in Table 1 are the estimates of the characteristic value for each of these eight iterative methods. We find that the estimates from FPTAVOP give the best approximation among all these methods. In each case, the comparisons with other methods were made using a similar amount of data, which is, using a similar number of terms of (34).

3.4. Precision of the approximate solution

In Fig. 1 we display the exact (analytic) solution and its approximations obtained using the method of FPTAVOP of type $(m, 1)$, the method of IPA of type $(m, 1)$, the method of FPTA of type $(m, 1)$, the method of GIPA of type $(m, 2)$, the ε-algorithm of GIPA of type $(m, 2)$ and the method of MPA of type $(m, 1)$. Also in Fig. 1 we see a remarkable precision of the methods of FPTAVOP, IPA and FPTA, where graphically there is no significant difference between the exact and these three methods. Accordingly, in Table 2, we show the errors incurred by the methods of FPTAVOP, IPA, MPA, SPA, GIPA, ε-algorithm of GIPA and FPTA for $x = 0(0.1)0.5$ in the solution of (33). And it is clear to see that the method of FPTAVOP gives the closest result to the exact solution. We list the appropriate rational functions displayed.

Solution of (33) based on the method of IPA of type (2, 1) is

$$r(x,\lambda) = \frac{1 + (x^2 - x + \frac{4399}{27045})\lambda + (\frac{1}{6}x^4 - \frac{1}{3}x^3 + \frac{4399}{27045}x^2 + \frac{217}{54090}x - \frac{217}{36060})\lambda^2}{1 - \frac{72337}{54090}\lambda}.$$

Solution of (33) based on the method of FPTAVOP of type (2, 1) is

$$r(x,\lambda) = \frac{1 + [\frac{5}{4} + (x-\frac{1}{2})^2]\lambda + [\frac{161}{96} + \frac{5}{4}(x-\frac{1}{2})^2 + \frac{1}{6}(x-\frac{1}{2})^4]\lambda^2 - \frac{2220059599}{1660050132}(\lambda + [\frac{5}{4} + (x-\frac{1}{2})^2]\lambda^2)}{1 - \frac{2220059599}{1660050132}\lambda}.$$

Solution of (33) based on the method of GIPA of type (2, 2) is

$$r(x,\lambda) = \frac{1 + [-1.425 + (x-\frac{1}{2})^2]\lambda + [0.122 - 1.425(x-\frac{1}{2})^2 + 0.167(x-\frac{1}{2})^4]\lambda^2}{1 - 2.675\lambda + 1.788\lambda^2}.$$

Solution of (33) based on the method of SPA of type (2, 1) is

$$r(x,\lambda) = \frac{1 + (x^2 - x + \frac{2917}{101115})\lambda + (\frac{1}{6}x^4 - \frac{1}{3}x^3 - \frac{2917}{101115}x^2 + \frac{18409}{40446}x - \frac{18409}{36964})\lambda^2}{1 - \frac{72337}{40446}\lambda}.$$

Solution of (33) based on the method of MPA of type (2, 1) is

$$r(x,\lambda) = \frac{1 + (x^2 - x + \frac{13}{80})\lambda + (\frac{1}{6}x^4 - \frac{1}{3}x^3 + \frac{13}{80}x^2 + \frac{1}{15}x - \frac{1}{10})\lambda^2}{1 - \frac{107}{80}\lambda}.$$

Solution of (33) based on the method of FPTA of type (2, 1) is

$$r(x,\lambda) = \frac{1 + [\frac{5}{4} + (x-\frac{1}{2})^2 - \frac{10641347}{7957070}]\lambda + [\frac{161}{96} + \frac{5}{4}(x-\frac{1}{2})^2 + \frac{1}{6}(x-\frac{1}{2})^4 - \frac{10641347}{7957070}(\frac{5}{4} + (x-\frac{1}{2})^2)]\lambda^2}{1 - \frac{10641347}{7957070}\lambda}.$$

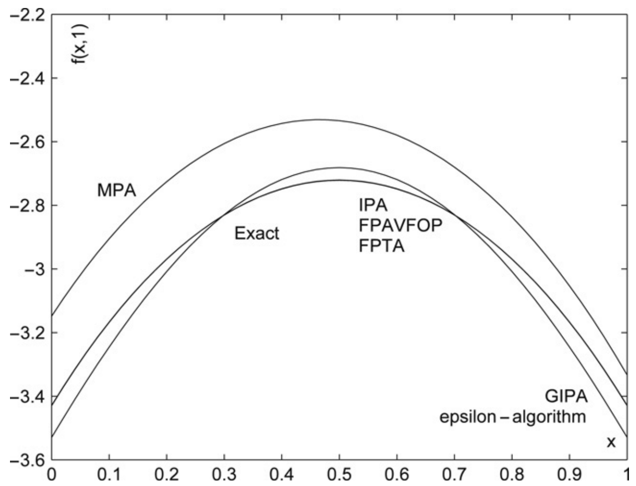

Fig. 1. The analytic solution (exact) of (33) for λ. The curves of FPTAVOP, IPA, FPTA, are indistinguishable from the exact curve, whereas the curves of GIPA, ε-algorithm, and MPA are perceptibly different and SPA is beyond the exact curve.

Table 2
Errors occurring in the solution of (33) using the different methods for the case of $\lambda = 1$

x	FPAVFOP	IPA	MPA	SPA	FPTA	GIPA and ε-algorithm
0	−0.000872	−0.000877	−0.281	−2.661	−0.000873	0.0997
0.1	−0.000655	−0.000659	−0.264	−2.459	−0.000656	0.0777
0.2	−0.000202	−0.000206	−0.243	−2.306	−0.000205	0.0388
0.3	0.000307	0.000311	−0.224	−2.197	0.000311	−0.000621

Solution of (33) based on the method of ε-algorithm of type (2, 2) is

$$r(x,\lambda) = \frac{1 + [-1.425 + (x-\frac{1}{2})^2]\lambda + [0.122 - 1.425(x-\frac{1}{2})^2 + 0.167(x-\frac{1}{2})^4]\lambda^2}{1 - 2.675\lambda + 1.788\lambda^2}.$$

Remark 9. In this example, the functional defined in (14) used in the method of FPTAVOP

$$\hat{c}_j^{(m-n+1)}(\lambda^i) = \int_b^a c_j(x) c_{m-n+i+1}(x)\mathrm{d}x, \quad i = 0, 1, \ldots$$

has two choices: $\hat{c}_2^{(2)}$ and $\hat{c}_3^{(2)}$ because here $m = 2$, $n = 1$ and $m - n + 1 = 2$. The solution we have obtained was computed by using the functional of $\hat{c}_3^{(2)}$. If we choose $\hat{c}_2^{(2)}$, we obtain

$$q_1^{(2)}(\lambda) = \lambda - \frac{l_1}{l_0} = \lambda - \frac{\int_b^a c_2(x)\cdot c_2(x)\mathrm{d}x}{\int_b^a c_2(x)\cdot c_3(x)\mathrm{d}x} = \lambda - \frac{72337/22680}{10641347/2494800}$$

and the estimate of the characteristic value is

$$\frac{10641347/2494800}{72337/22680} = 0.747753,$$

which is equal to the result from FPTA, but not good comparing with the estimate obtained by using the functional of $\hat{c}_3^{(2)}$.

4. Conclusion

A new method of producing a sequence of rational approximations to the solution of a linear integral equation has been described, and its effectiveness has been investigated in many examples. The method is essentially for accelerating the convergence of a sequence of functions. In the context of a familiar linear integral equation, the method of FPTAVOP is shown to be much more efficient in accelerating the characteristic value and more accurate for calculating the approximate solution than other similar techniques. The second advantage of the method of FPTAVOP is that we can obtain the denominator of the approximants without computing the determinant which is always an impractical method when the order of the denominator required is high. The third superiority of this new method is that we have given an analytic investigation, and an algorithm was obtained too. Finally, the choice of the functional in (13) is an open question, and the corresponding theoretical analysis is a subject of further research.

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