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Differential coefficients of orthogonal matrix polynomials

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Dedicated to Rod Wong on his 60th birthday

Abstract

We find explicit formulas for raising and lowering first order differential operators for orthogonal matrix polynomials. We derive recurrence relations for the coefficients in the raising and lowering operators. Some examples are given.

MSC: primary 33E30; secondary 42C05

Keywords: Orthogonal matrix polynomials; Creation operators; Annihilation operators

1. Introduction

A systematic study of second-order differential equations satisfied by polynomials orthogonal with respect to a weight function was the subject of the recent papers [4–6,21,22]. See also [23,24]. This recent approach is not only different from the method used by Shohat in his classic [25], or the later approach used by Atkinson and Everitt [3] but it also applies to more general polynomials. The work [6] also studied the Lie algebra generated by the creation and annihilation operators of the orthogonal polynomials under consideration. The annihilation operators and the second-order differential equations

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played a crucial role in the evaluation of discriminants of general orthogonal polynomials [19] and in the electrostatic interpretation of their zeros [20,18].

Orthogonal matrix polynomials are a more recent addition to the very rich subjects of orthogonal polynomials [27] and moment problems [1,26]. The theory of matrix orthogonal polynomial is still far from being complete although, many of the landmarks of the theory of scalar orthogonal polynomials, like the spectral theorem for orthogonal polynomials (aka Favard's theorem), quadrature formulae and asymptotics properties (Markov's theorem, ratio, weak and zero asymptotics), have been extended to orthogonal matrix polynomials [2,7–16]. The theory of second-order differential equations, analogues of discriminants and electrostatics of orthogonal polynomials have yet to be generalized to orthogonal matrix polynomials.

This work is a contribution to the theory of first-order raising and lowering operators as well as the second-order differential equations for orthogonal matrix polynomials. In Section 2, we record preliminary results which will be used in later sections. Section 3 contains several examples of creation and annihilation operators for orthogonal matrix polynomials. Closed form expressions for the differential coefficients in the creation and annihilation operators are given in Section 4. In Section 4, we also combine these operators to give a second-order differential equation which the polynomials satisfy.

2. Preliminaries

All the matrices considered in this work are square matrices and are of a fixed size, say $N \times N$. Let W be a positive definite matrix of $\mathscr{C}^1$ weight functions defined on a set E and assume that the moments $\int_E t^n W(t)\,\mathrm{d}t$ and $\int_E t^n W'(t)\,\mathrm{d}t$ exist for all orders, $n=0,1,\ldots$. The matrix W induces an inner product on matrix polynomials by $\int_E P(t)W(t)Q^*(t)\,\mathrm{d}t$, for matrix polynomials P and Q. This inner product gives rise to matrix polynomials $\{P_n, n \geqslant 0\}$ orthonormal with respect to the inner product. Let the orthogonality relation be

$$\int_E P_m(t)W(t)P_n^*(t)\,\mathrm{d}t = \delta_{m,n}\mathrm{Id},$$

Id being the identity matrix. These polynomials satisfy a three term recurrence relation

$$zP_n(z) = A_{n+1}P_{n+1}(z) + B_nP_n(z) + A_n^*P_{n-1}(z), \tag{2.1}$$

where for all n, $n=0,1,2,\ldots$, A_n are invertible matrices and B_n are Hermitian matrices, and we shall assume that $P_0(z)$ is the identity matrix and $P_{-1}(z)=\theta$, the zero matrix. The polynomials of the second kind are defined by

$$Q_n(z) = \int_E \frac{P_n(t)-P_n(z)}{t-z}W(t)\,\mathrm{d}t, \quad n \geqslant 0,$$

also satisfy the three term recurrence relation (2.1), $n \geqslant 1$, but with initial conditions $Q_0(z)=\theta$, $Q_1(z)=A_1^{-1}$. The Christoffel–Darboux formula for orthogonal matrix polynomials is

$$P_{n-1}^*(z)A_nP_n(w) - P_n^*(z)A_n^*P_{n-1}(w) = (w-z)\sum_{k=0}^{n-1}P_k^*(z)P_k(w); \tag{2.2}$$

[9, (2.1)], while the Liouville–Ostrogadski formula is

$$Q_n(z)P^*_{n-1} - P_n(z)Q^*_{n-1}(z) = A_n^{-1}; \tag{2.3}$$

[9, (2.6)]. Moreover the hermitian property [9, p. 1188],

$$Q_n(z)P^*_n(z) = P_n(z)Q^*_n(z), \tag{2.4}$$

holds.

Let

$$P_n(z) = z^n \Gamma_n + \text{lower order terms}. \tag{2.5}$$

By equating coefficients of z^{n+1} in (2.1) we get

$$\Gamma_n = A_{n+1}\Gamma_{n+1}. \tag{2.6}$$

Hence

$$\Gamma_n = A_n^{-1}A_{n-1}^{-1}\cdots A_1^{-1}, \quad n>0, \quad \Gamma_0 = I. \tag{2.7}$$

Finally, we include the following basic formula on integrating by parts which will be used later.

Lemma 2.1. *Let all the weight functions be supported on E, a union of finitely many disjoint intervals (finite or infinite intervals), and write Δ for the boundary of E, then for any polynomials P and any matrix function F with $\mathscr{C}^1$ entries, we have*

$$\int_E F(t)W'(t)P^*(t)\,\mathrm{d}t = F(t)W(t)P^*(t)|_\Delta - \int_E F'(t)W(t)P^*(t)\,\mathrm{d}t - \int_E F(t)W(t)(P^*)'(t)\,\mathrm{d}t.$$

Proof. We first expand P as $\int_E F(t)W'(t)P^*(t)\,\mathrm{d}t$ as a linear combination of power of t with matrix coefficients and joint the power of t to the function F to the left-hand side of W'. It is now enough to apply an integration by parts and then to recover the polynomial P and its derivative P' by moving the powers of t to the right-hand side of W. □

3. Differential equations

Our first result provides lowering (annihilation) and raising (creation) operators for general weights.

Theorem 3.1. *Let all the weight functions be supported on E, a union of finitely many disjoint intervals (finite or infinite intervals), and write Δ for the boundary of E. Define $\mathscr{A}_n(z)$ and $\mathscr{B}_n(z)$ by*

$$\mathscr{A}_n(z) = \left.\frac{P_n(t)W(t)P^*_n(t)}{t-z}\right|_\Delta - \int_E P_n(t)W'(t)P^*_n(t)\frac{\mathrm{d}t}{t-z} \tag{3.1}$$

and

$$\mathscr{B}_n(z) = \left.\frac{P_n(t)W(t)P^*_{n-1}(t)}{t-z}\right|_\Delta - \int_E P_n(t)W'(t)P^*_{n-1}(t)\frac{\mathrm{d}t}{t-z}, \tag{3.2}$$

respectively, provided that all the functions on the right-hand sides of (3.1) *and* (3.2) *exist for* $z \notin E$. *Then* $P_n(z)$ *satisfies the following differential recurrence relations* (*lowering and raising operators, respectively*).

$$P_n'(z) = \mathscr{A}_n(z)A_n^* P_{n-1}(z) - \mathscr{B}_n(z)A_n P_n(z), \tag{3.3}$$

$$P_n'(z) = [\mathscr{A}_n(z)(z - B_n) - \mathscr{B}_n(z)A_n]P_n(z) - \mathscr{A}_n(z)A_{n+1}P_{n+1}(z). \tag{3.4}$$

Proof. We first prove (3.3). Since $\{P_k(z), k \geqslant 0\}$ is a basis for the space of matrix polynomials we set

$$P_n'(z) = \sum_{k=0}^{n-1} C_k P_k(z), \quad C_k = \int_E P_n'(t)W(t)P_k^*(t)\,\mathrm{d}t.$$

Thus,

$$P_n'(z) = \int_E P_n'(t)W(t)\left[\sum_{k=0}^{n-1} P_k^*(t)P_k(z)\right]\mathrm{d}t. \tag{3.5}$$

Integration by parts gives

$$P_n'(z) = P_n(t)W(t)\sum_{k=0}^{n-1} P_k^*(t)P_k(z)\Bigg|_{\Delta} - \sum_{k=0}^{n-1}\int_E P_n(t)[W'(t)P_k^*(t) + W(t)(P_k^*(t))']P_k(z)\,\mathrm{d}t.$$

The integral $\int_E P_n(t)W(t)(P_k^*(t))'\,\mathrm{d}t$ vanishes due to the orthogonality of the P_k's. The Christoffel–Darboux formula (2.2) simplifies the above expression to (3.3).

To establish the raising differential relation, eliminate $A_n^* P_{n-1}(z)$ between (3.3) and (2.1). This completes the proof. □

It is clear that the lowering and raising operators are not unique. For example for a given function $\mathscr{A}_n$, one can define

$$\mathscr{B}_n(z) = (\mathscr{A}_n(z)A_n^* P_{n-1}(z) - P_n'(z))P_n^{-1}(z)A_n^{-1}$$

and then (3.3) will trivially hold.

We next derive a different pair of differential recurrence relations using the Christoffel–Darboux formula (2.2).

Example 1. By using the Christoffel–Darboux formula in (3.5) above, we get again the lowering differential operator (3.3)

$$P_n'(z) = \mathscr{C}_n(z)A_n^* P_{n-1}(z) - \mathscr{D}_n(z)A_n P_n(z) \tag{3.6}$$

but now the differential coefficients are given by

$$\mathscr{C}_n(z) = \int_E P_n'(t)W(t)P_n^*(t)\frac{\mathrm{d}t}{t-z} \tag{3.7}$$

and

$$\mathscr{D}_n(z) = \int_E P_n'(t)W(t)P_{n-1}^*(t)\frac{\mathrm{d}t}{t-z}. \tag{3.8}$$

Example 2. This example is the case of scalar polynomials and is taken from [6]. One writes the weight function $w(t)=\mathrm{e}^{-v(t)}$, where v is a twice continuously differentiable function on an interval (a, b) (finite or infinite). The lowering operator is [6, Theorem 2.1].

$$p_n'(z) = \mathscr{E}_n(z)p_{n-1}(z) - \mathscr{F}_n(z)p_n(z) \tag{3.9}$$

and the coefficients are given by

$$\mathscr{E}_n(z) = a_n \frac{w(t)}{t-z}p_n^2(t)\bigg|_{\Delta} + a_n\int_E \frac{v'(t)-v'(z)}{t-z}p_n^2(t)w(t)\,\mathrm{d}t \tag{3.10}$$

and

$$\mathscr{F}_n(z) = a_n \frac{w(t)}{t-z}p_n(t)p_{n-1}(t)\bigg|_{\Delta} + a_n\int_E \frac{v'(t)-v'(z)}{t-z}p_n(t)p_{n-1}(t)w(t)\,\mathrm{d}t. \tag{3.11}$$

A straightforward calculation gives the following relationships among $\mathscr{A}_n$, $\mathscr{B}_n$ and $\mathscr{E}_n$, $\mathscr{F}_n$:

$$\begin{aligned}
\mathscr{E}_n(z) &= a_n\left(\mathscr{A}_n(z) - v'(z)\int_E \frac{p_n^2(t)}{t-z}w(t)\,\mathrm{d}t\right),\\
\mathscr{F}_n(z) &= a_n\left(\mathscr{B}_n(z) - v'(z)\int_E \frac{p_n(t)p_{n-1}(t)}{t-z}w(t)\,\mathrm{d}t\right).
\end{aligned} \tag{3.12}$$

The representations (3.12) of the coefficients $\mathscr{E}_n$ and $\mathscr{F}_n$ do not extend to the matrix case since, in general $V'(t)\mathrm{e}^{-V(t)} \neq \mathrm{e}^{-V(t)}V'(t)$. The reason is that the derivative of the matrix exponential $\mathrm{e}^{-V(t)}$ is $-V'(t)\mathrm{e}^{-V(t)}$ when $V(t)V(s) = V(s)V(t)$, which is not true in general. Indeed when $W = U^*DU$ with U unitary, and D diagonal, then $W = \mathrm{e}^{-V}$ and $[V(s), V(t)] = 0$, but this is essentially the scalar case.

In Section 4, we will establish closed expressions for all these differential coefficients (that for Example 2 being new).

Both lowering and raising differential operators can be combined to give a second-order differential equation which the polynomials satisfy. Just differentiate (3.3), then substitute P_{n-1}' from (3.4), with $n \to n-1$, and P_{n-1} again from (3.3).

Observe that (3.1) implies

$$(\mathscr{A}_n(z))^* = \mathscr{A}_n(\overline{z}).$$

We now derive two recursion relations involving $\mathscr{A}_n$ and $\mathscr{B}_n$ (the second is an extension of a result due to Ismail and Wimp [21]).

Theorem 3.2. *The recurrence relations*

$$A_{n+1}\mathscr{B}_{n+1}(z) + A_n^*\mathscr{B}_n^*(z) = (z - B_n)\mathscr{A}_n(z) \tag{3.13}$$

and

$$A_{n+1}\mathscr{A}_{n+1}(z)A_{n+1}^* - A_n^*\mathscr{A}_n(z)A_n = 1 + (z - B_n)\mathscr{B}_{n+1}(z)A_{n+1}^* - A_n^*\mathscr{B}_n^*(z)(z - B_n) \tag{3.14}$$

hold for $n = 0, 1, \ldots$.

Proof. The left-hand side of (3.13) is

$$\begin{aligned}
&[A_{n+1}P_{n+1}(t) + A_n^*P_{n-1}(t)]\,\frac{W(t)P_n^*(t)}{t-z}\Big|_\Delta \\
&\quad + \int_E [A_{n+1}P_{n+1}(t) + A_n^*P_{n-1}(t)]\frac{W'(t)}{z-t}P_n^*(t)\,\mathrm{d}t \\
&\quad = \frac{(t-B_n)}{t-z}P_n(t)W(t)P_n^*(t)\Big|_\Delta - (z-B_n)\,\frac{P_n(t)W(t)P_n^*(t)}{t-z}\Big|_\Delta \\
&\qquad + (z-B_n)\mathscr{A}_n(z) - \int_E P_n(t)W'(t)P_n^*(t)\,\mathrm{d}t.
\end{aligned}$$

After applying integration by parts to the last integral and using the orthogonality relation one realizes that the boundary terms cancel and the above expression reduces to the right-hand side of (3.13).

We next prove that $\mathscr{A}_n$'s and $\mathscr{B}_n$'s satisfy the recurrence relation (3.14).

Denote the left-hand side of (3.14) by $Q(z)$. Eliminate $A_{n+1}P_{n+1}(t)$ from the terms involving $\mathscr{A}_{n+1}$ in Q using (2.1) then replace $t - B_n$ by $(t-z) + (z - B_n)$. The result is

$$\begin{aligned}
Q(z) &= (z-B_n)\mathscr{B}_{n+1}(z)A_{n+1}^* - \int_E P_n(t)W'(t)P_{n+1}^*(t)A_{n+1}^*\,\mathrm{d}t \\
&\quad + A_n^*\int_E P_{n-1}(t)\frac{W'(t)}{t-z}[P_{n+1}^*(t)A_{n+1}^* + P_{n-1}^*(t)A_n]\,\mathrm{d}t \\
&\quad + \left\{P_n(t) - \frac{A_n^*P_{n-1}(t)}{t-z}\right\}W(t)P_{n+1}^*(t)A_{n+1}^*\Big|_\Delta \\
&\quad - A_n^*\,\frac{P_{n-1}(t)W(t)P_{n-1}^*(t)}{t-z}\,A_n\Big|_\Delta .
\end{aligned}$$

Now we have

$$\int_E P_n(t)W'(t)P_{n+1}^*(t)A_{n+1}^*\,\mathrm{d}t = P_n(t)W(t)P_{n+1}^*(t)A_{n+1}^*|_\Delta - \int_E P_n(t)W(t)(P_{n+1}^*(t))'A_{n+1}^*\,\mathrm{d}t,$$

due to the orthogonality of the P_n's. In the notation of (2.5)–(2.7), we find

$$\begin{aligned}
(P_{n+1}^*(t))' &= (n+1)t^n\Gamma_{n+1}^* + \cdots = (n+1)P_n^*(t)(\Gamma_n^*)^{-1}\Gamma_{n+1}^* + \cdots \\
&= (n+1)P_n^*(t)(A_{n+1}^*)^{-1} + \cdots .
\end{aligned}$$

Therefore

$$
\begin{aligned}
&Q(z) - (z - B_n)\mathscr{B}_{n+1}(z)A_{n+1}^* \\
&\quad = (n+1)\mathrm{Id} + A_n^* \int_E P_{n-1}(t)\frac{W'(t)}{t-z}P_n^*(t)(t - B_n^*)\,\mathrm{d}t \\
&\qquad - A_n^* P_{n-1}(t)\frac{W(t)}{t-z}\{P_{n-1}^*(t)A_n + P_{n-1}^*(t)A_{n+1}^*\}\Big|_\Delta \\
&\quad = (n+1)\mathrm{Id} + A_n^* \int_E P_{n-1}(t)W'(t)P_n^*(t)\,\mathrm{d}t \\
&\qquad + A_n^* \int_E P_{n-1}(t)\frac{W'(t)}{t-z}P_n^*(t)(z - B_n^*)\,\mathrm{d}t - A_n^* P_{n-1}(t)\frac{W(t)}{t-z}P_n^*(t)(t - B_n^*)\Big|_\Delta \\
&\quad = (n+1)\mathrm{Id} + A_n^* P_{n-1}(t)W'(t)P_n^*(t)|_\Delta \\
&\qquad - A_n^* \int_E P_{n-1}(t)W(t)nP_{n-1}^*(t)(A_n^*)^{-1}\,\mathrm{d}t \\
&\qquad + A_n^* \int_E P_{n-1}(t)\frac{W'(t)}{t-z}P_n^*(t)(z - B_n^*)\,\mathrm{d}t \\
&\qquad - A_n^* P_{n-1}(t)\frac{W(t)}{t-z}P_n^*(t)(t - B_n^*)\Big|_\Delta .
\end{aligned}
$$

It is easy now to simplify the above equality to (3.14). □

4. Differential coefficients of orthogonal matrix polynomials

We now give closed expressions for the differential coefficients $\mathscr{A}_n$'s and $\mathscr{B}_n$'s. Note that these closed expressions are also new for the scalar case.

Theorem 4.1. *Define a matrix function F_w by*

$$F_W(z) = \int_E \frac{\mathrm{d}W(t)}{t-z}, \quad z \notin E.$$

Then we have

$$\mathscr{A}_n(z) = -P_n(z)\,[F_W(z) + P_n^{-1}(z)Q_n(z)]'P_n^*(z), \tag{4.1}$$

$$
\begin{aligned}
\mathscr{B}_n(z) = &- P_n(z)[F_W(z) + P_{n-1}^{-1}(z)Q_{n-1}(z)]'P_{n-1}^*(z) \\
&+ P_n(z)[P_n^{-1}(z)Q_n(z) - P_{n-1}^{-1}(z)Q_{n-1}(z)](P_{n-1}^*)'(z).
\end{aligned} \tag{4.2}
$$

Proof. Performing integrating by parts in (3.1) and in view of Lemma 2.1, we find that

$$\mathscr{A}_n(z) = \int_E \frac{P_n(t)}{t-z}W(t)(P_n^*(t))'\,\mathrm{d}t + \int_E \left(\frac{P_n(t)}{t-z}\right)'W(t)P_n^*(t)\,\mathrm{d}t. \tag{4.3}$$

From the orthogonality of the polynomials $(P_n)_n$, it follows that the first integral in the right-hand side of (4.3) is given by

$$\begin{aligned}\int_E \frac{P_n(t)}{t-z}W(t)(P_n^*(t))'\,\mathrm{d}t &= \int_E P_n(t)W(t)\frac{(P_n^*)'(t)-(P_n^*)'(z)}{t-z}\,\mathrm{d}t + \int_E P_n(t)W(t)\frac{(P_n^*)'(z)}{t-z}\,\mathrm{d}t \\ &= \int_E P_n(t)W(t)\frac{(P_n^*)'(z)}{t-z}\,\mathrm{d}t \\ &= \int_E \frac{P_n(t)-P_n(z)}{t-z}W(t)\mathrm{d}t\,(P_n^*)'(z) + \int_E \frac{P_n(z)}{t-z}W(t)\,\mathrm{d}t\,(P_n^*)'(z) \\ &= Q_n(z)(P_n^*)'(z) + P_n(z)F_W(z)(P_n^*)'(z).\end{aligned}$$

Similarly we see that the second integral in the right-hand side of (4.3) is equal to

$$\begin{aligned}&\int_E \left(\frac{P_n(t)}{t-z}\right)' W P_n^*(t)\,\mathrm{d}t \\ &\quad= \int_E \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n(t)-P_n(z)}{t-z}\right) W(t)P_n^*(t)\,\mathrm{d}t + \int_E \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n(z)}{t-z}\right) W(t)P_n^*(t)\,\mathrm{d}t \\ &\quad= \int_E \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n(z)}{t-z}\right) W(t)P_n^*(t)\,\mathrm{d}t \\ &\quad= -P_n(z)\int_E \frac{1}{(t-z)^2}W(t)P_n^*(t)\,\mathrm{d}t \\ &\quad= -P_n(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_E W(t)\frac{P_n^*(t)}{t-z}\,\mathrm{d}t\right) \\ &\quad= -P_n(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_E W(t)\frac{P_n^*(t)-P_n^*(z)}{t-z}\,\mathrm{d}t + \int_E \frac{W(t)}{t-z}P_n^*(z)\,\mathrm{d}t\right) \\ &\quad= -P_n(z)[Q_n^*)'(z) - P_n(z)(F_W(z)P_n^*(z)]'.\end{aligned}$$

The above calculations show that $\mathscr{A}_n(z)$ is of the form

$$\begin{aligned}\mathscr{A}_n(z) &= Q_n(z)(P_n^*)'(z) + P_n(z)F_W(z)(P_n^*(z))' \\ &\quad - P_n(z)(Q_n^*)'(z) - P_n(z)(F_W(z)P_n^*(z))' \\ &= -P_n(z)F_W'(z)P_n^*(z) - P_n(z)(Q_n^*)'(z) + Q_n(z)(P_n^*)'(z).\end{aligned} \tag{4.4}$$

Taking into account the fact that $P_n(z)Q_n^*(z) = Q_n(z)P_n^*(z)$, see (2.4), we find that

$$(Q_n^*)'(z) = (P_n^{-1}(z)Q_n(z))'P_n^*(z) + P_n^{-1}(z)Q_n(z)(P_n^*)'(z).$$

Using the above expression, we simplify (4.4) and establish (4.1).

The proof of the formula (4.2) is similar. First observe that the integral

$$\int_E \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n(t)-P_n(z)}{t-z}\right) W(t) P_{n-1}^*(t)\,\mathrm{d}t$$

again vanishes because the degree of the polynomial

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n(t)-P_n(z)}{t-z}\right)$$

is only $n-2$. □

Note that proceeding in the same way we can find the following closed expression for the differential coefficients $\mathscr{C}_n, \mathscr{D}_n, \mathscr{E}_n$ and $\mathscr{F}_n$ given by (3.7), (3.8), (3.10) and (3.11). Indeed the first two have the form

$$\begin{aligned}\mathscr{C}_n(z) &= P_n'(z)Q_n^*(z) + P_n'(z)F_W(z)P_n^*(z),\\ \mathscr{D}_n(z) &= P_n'(z)Q_{n-1}^*(z) + P_n'(z)F_W(z)P_{n-1}^*(z);\end{aligned} \tag{4.5}$$

while the remaining two functions take the from (3.12):

$$\begin{aligned}\mathscr{E}_n(z) &= -a_n p_n^2(z)\left[\left(F_W(z)+\frac{q_n(z)}{p_n(z)}\right)' + v'(z)\left(F_W(z)+\frac{q_n(z)}{p_n(z)}\right)\right],\\ \mathscr{F}_n(z) &= -a_n p_n(z)p_{n-1}(z)\left[\left(F_W(z)+\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)' + v'(z)\left(F_W(z)+\frac{q_n(z)}{p_n(z)}\right)\right]\\ &\quad + a_n p_n(z)p_{n-1}'(z)\left(\frac{q_n(z)}{p_n(z)} - \frac{q_{n-1}(z)}{p_{n-1}(z)}\right).\end{aligned} \tag{4.6}$$

In deriving (4.6) we used (3.12).

The formula (4.1) and Markov's theorem for orthogonal matrix polynomials [9, Theorem 1.1] provide information about the size of the differential coefficient $\mathscr{A}_n$. To see this write Δ_n for the set of zeros of P_n and put

$$\Gamma = \bigcap_{N\geqslant 0} M_N, \quad \text{where } M_N = \overline{\bigcup_{n\geqslant N} \Delta_n}.$$

The information alluded to above is contained in the following corollary.

Corollary 4.2. *Assume that the matrix weight W is determinate, i.e., there is no other matrix weight with the same moments. Then we have the limiting relation*

$$\lim_{n\to\infty} P_n^{-1}(z)\mathscr{A}_n(z)(P_n^*)^{-1}(z) = -2F_W'(z) = 0$$

holds uniformly on compact subsets of $\mathbb{C}\backslash\Gamma$.

The analogous result for $\mathscr{B}_n$ is more difficult to derive. To motivate the result we first consider the scalar case. The expression for $\mathscr{B}_n$ as given by (4.2) can be simplified as follows:

$$\begin{aligned}\mathscr{B}_n(z) &= -p_n(z)p_{n-1}(z)\left(F_W(z)+\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)' + p_n(z)p'_{n-1}(z)\left(\frac{q_n(z)}{p_n(z)}-\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)\\ &= -p_n(z)p_{n-1}(z)\left(F_W(z)+\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)' + p_n(z)p'_{n-1}(z)\frac{q_n(z)p_{n-1}(z)-q_{n-1}p_n(z)}{p_n(z)p_{n-1}(z)}\\ &= -p_n(z)p_{n-1}(z)\left(F_W(z)+\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)' + p'_{n-1}(z)\frac{1}{a_n p_{n-1}(z)}.\end{aligned}$$

From this we obtain

$$\frac{\mathscr{B}_n(z)}{p_n(z)p_{n-1}(z)} = -\left(F_W(z)+\frac{q_{n-1}(z)}{p_{n-1}(z)}\right)' + \frac{p'_{n-1}(z)}{np_{n-1}(z)}\frac{n}{a_n p_n(z)p_{n-1}(z)}.$$

As before, Markov's theorem implies that the first term on the right-hand side of that formula converges to 0 uniformly in compact subsets of $\mathbb{C}\setminus\Gamma$. As for the second term

$$\frac{p'_{n-1}(z)}{np_{n-1}(z)}\frac{n}{a_n p_n(z)p_{n-1}(z)}$$

we expand the first factor into partial fractions

$$\frac{p'_{n-1}(z)}{np_{n-1}(z)} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{z-x_k}$$

and it easily follows that the above expression is uniformly bounded on compact subsets of $\mathbb{C}\setminus\Gamma$. The second factor however may not be to be bounded. Indeed, we have

$$\frac{n}{a_n p_n(z)p_{n-1}(z)} = n\left(\frac{q_n(z)}{p_n(z)}-\frac{q_{n-1}(z)}{p_{n-1}(z)}\right). \tag{4.7}$$

Markov's theorem guarantees that

$$\frac{q_n(z)}{p_n(z)}-\frac{q_{n-1}(z)}{p_{n-1}(z)}$$

converges to zero uniformly on compact subsets of $\mathbb{C}\setminus\Gamma$, but with the only assumption of the determinacy of the measure the convergence can be as slow as one wants (slower, for instance, than $1/n$). But assuming further hypothesis, we can prove that this second factor (4.7) also tends to zero uniformly in compact subsets of $\mathbb{C}\setminus\Gamma$. For instance, if the polynomials $\{p_n : n\geqslant 0\}$ are in the Nevai class, that is, $\{a_n : n\geqslant 0\}$ and $\{b_n : n\geqslant 0\}$ are convergent sequences. Assume first that $\lim_{n\to\infty} a_n = a > 0$, in which case we can put $a=\frac{1}{2}$. Then, the sequence of orthonormal polynomials $(p_n)_n$ satisfies the following n-root asymptotic behavior

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{|p_n(z)|}} = \left|z-\sqrt{z^2-1}\right| = \rho(z)$$

uniformly in compact subsets of $\mathbb{C}\backslash\Gamma$. This means that $1/|p_n(z)| \sim \rho^n(z)$; but $\rho(z)<1$, which implies that (4.7) tends to zero as n tends to infinity. If $\lim_n a_n = 0$, we can proceed as follows:

$$\frac{n}{a_n p_n(z) p_{n-1}(z)} = \frac{n}{a_n \gamma_n \gamma_{n-1} \hat{p}_n(z) \hat{p}_{n-1}(z)},$$

where γ_n is the leading coefficient of p_n and $\hat{p}_n$ stands for the monic orthogonal polynomials. Taking into account (2.6) and (2.7), we can write:

$$\frac{n}{a_n p_n(z) p_{n-1}(z)} = n \frac{a_{n-1}^2 a_{n-2}^2 \cdots a_1^2}{(z-x_{n,1})\cdots(z-x_{n,n})(z-x_{n-1,1})\cdots(z-x_{n-1,n-1})}, \tag{4.8}$$

where $x_{m,k}$, $k=1,\dots,m$, are the zeros of p_k. Given a compact set K, $K\subset \mathbb{C}\backslash\Gamma$, we write $c=\mathrm{d}(K,\Gamma)$. Then (4.8) gives that

$$\left|\frac{n}{a_n p_n(z) p_{n-1}(z)}\right| \leqslant n \frac{|a_{n-1}a_{n-2}\cdots a_1|^2}{c^{2n-1}}; \tag{4.9}$$

Since $\lim_n a_n = 0$, we can take $N \geqslant 0$ such that if $k \geqslant N$ then $|a_k| \leqslant c/2$; from (4.9), we get that

$$\left|\frac{n}{a_n p_n(z) p_{n-1}(z)}\right| \leqslant C\frac{n}{4^n},$$

where $C>0$ is a constant which does not depend on n. This shows that also for $a=0$, (4.7) tends to zero as n tends to infinity uniformly in compact subsets of $\mathbb{C}\backslash\Gamma$.

Actually, this result (for any compact set K, $K\subset \mathbb{C}\backslash\Gamma$, there exist constants $\alpha>0$, $0<c<1$ for which

$$\left|\int \frac{\mathrm{d}w(t)}{t-z} - \frac{q_n(z)}{p_n(z)}\right| \leqslant \alpha c^n,$$

is also true for measures with compact support (see [17]); hence we have proved that

Corollary 4.3. *If we assume that the function w has compact support, then we have that*

$$\lim_n \frac{\mathscr{B}_n(z)}{p_n(z)p_{n-1}(z)} = -2F_W'(z) = 0$$

uniformly in compact subset of $\mathbb{C}\backslash\Gamma$.

In the matrix case, to estimate the size of $\mathscr{B}_n$ is more involved, mainly because then the sequence $P_n'(z)P_n^{-1}(z)/n$ does not need to be bounded. Indeed, if we expand the rational matrix function $P_n'(z)P_n^{-1}(z)$ in simple fractions (it is always possible: see [9, p. 1186]), we find

$$P_n'(z)P_n^{-1}(z) = \sum_{k=1}^{m} \frac{C_k}{z-x_k},$$

where x_k, $k=1,\dots,m$, are the zeros of the matrix polynomial P_n, x_k with multiplicity $l_k \leqslant N$ and

$$C_k = \frac{1}{(\det(P_n(t)))^{(l_k)}} P_n'(x_k)(\mathrm{Adj}(P_n(t)))^{(l_k-1)}(x_k).$$

In the scalar case $l_k = 1$ and then $C_k = 1$, but in the matrix case this is, in general, no longer true. In other words, in the scalar case the rational function $p_n'(z)/np_n(z)$ is the Hilbert transform of the counting measure $\mu_n = (1/n)\sum_{k=1}^n \delta_{x_n}$ for the zeros of p_n: since μ_n is a probability measure its Hilbert transform is bounded on compact subsets of $\mathbb{C}\setminus\Gamma$. In the matrix case neither $(1/n)P_n'(z)P_n^{-1}(z)$ nor $(1/n)P_n^{-1}(z)P_n'(z)$ is the Hilbert transform of the corresponding counting measure for the set of zeros of P_n (see [14, p. 40]).

Using Markov's theorem we can also estimate the size for the differential coefficients $\mathscr{C}_n$ and $\mathscr{D}_n$ (see (3.7) and (3.8)); indeed from (4.5), we get

$$(P_n'(z))^{-1}\mathscr{C}_n(z)(P_n^*)^{-1}(z) = Q_n^*(z)(P_n^*)^{-1}(z) + F_W(z),$$
$$(P_n'(z))^{-1}\mathscr{D}_n(z)(P_{n-1}^*)^{-1}(z) = Q_{n-1}^*(z)(P_{n-1}^*)^{-1}(z) + F_W(z);$$

Markov's theorem then gives that if W is determinate then

$$\lim_n (P_n'(z))^{-1}\mathscr{C}_n(z)(P_n^*)^{-1}(z) = 0,$$
$$\lim_n (P_n'(z))^{-1}\mathscr{D}_n(z)(P_{n-1}^*)^{-1}(z) = 0,$$

uniformly in compact sets of $\mathbb{C}\setminus\Gamma$.

Finally, for the differential coefficients $\mathscr{E}_n$ and $\mathscr{F}_n$ (see (3.10) and (3.11)), using (4.6) and proceeding as before, we have that if w is determinate then

$$\lim_n \frac{\mathscr{E}_n(z)}{a_n p_n^2(z)} = 0$$

and if, in addition, we assume that w has compact support then

$$\lim_n \frac{\mathscr{F}_n}{a_n p_n(z) p_{n-1}(z)} = 0$$

uniformly in compact sets of $\mathbb{C}\setminus\Gamma$.

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