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Periodic orbits and chaos in fast–slow systems with Bogdanov–Takens type fold points

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ABSTRACT

The existence of stable periodic orbits and chaotic invariant sets of singularly perturbed problems of fast–slow type having Bogdanov–Takens bifurcation points in its fast subsystem is proved by means of the geometric singular perturbation method and the blow-up method. In particular, the blow-up method is effectively used for analyzing the flow near the Bogdanov–Takens type fold point in order to show that a slow manifold near the fold point is extended along the Boutroux's tritronquée solution of the first Painlevé equation in the blow-up space.

1. Introduction

Let $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbf{R}^{n+m}$ be the Cartesian coordinates. A system of singularly perturbed ordinary differential equations of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n, y_1, \ldots, y_m, \varepsilon), \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n, y_1, \ldots, y_m, \varepsilon), \\ \dot{y}_1 = \varepsilon g_1(x_1, \ldots, x_n, y_1, \ldots, y_m, \varepsilon), \\ \quad \vdots \\ \dot{y}_m = \varepsilon g_m(x_1, \ldots, x_n, y_1, \ldots, y_m, \varepsilon), \end{cases} \tag{1.1}$$

is called a *fast–slow system*, where the dot ($\dot{}$) denotes the derivative with respect to time t, and where $\varepsilon > 0$ is a small parameter. Fast–slow systems are characterized by two different time scales, fast and

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slow time. In other words, the dynamics consists of fast motions ($(x_1, \dots, x_n)$ direction in the above system) and slow motions ($(y_1, \dots, y_m)$ direction). This structure yields nonlinear phenomena such as a relaxation oscillation, which is observed in many physical, chemical and biological problems. See Grasman [13], Hoppensteadt and Izhikevich [16] and references therein for applications of fast–slow systems. To analyze the fast–slow system, the unperturbed system (*fast system*) of Eq. (1.1) is defined to be

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n, y_1, \dots, y_m, 0), \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, \dots, x_n, y_1, \dots, y_m, 0), \\ \dot{y}_1 = 0, \\ \quad \vdots \\ \dot{y}_m = 0. \end{cases} \tag{1.2}$$

The set of fixed points of the unperturbed system is called a *critical manifold*, which is defined by

$$\mathcal{M} = \left\{ (x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbf{R}^{n+m} \mid f_i(x_1, \dots, x_n, y_1, \dots, y_m, 0) = 0,\ i = 1, \dots, n \right\}. \tag{1.3}$$

Typically $\mathcal{M}$ is an m-dimensional manifold. Fenichel [11] proved that if $\mathcal{M}$ is normally hyperbolic, then the original system (1.1) with sufficiently small $\varepsilon > 0$ has a locally invariant manifold $\mathcal{M}_\varepsilon$ near $\mathcal{M}$, and that dynamics on $\mathcal{M}_\varepsilon$ is approximately given by the m-dimensional system

$$\begin{cases} \dot{y}_1 = \varepsilon g_1(x_1, \dots, x_n, y_1, \dots, y_m, 0), \\ \quad \vdots \\ \dot{y}_m = \varepsilon g_m(x_1, \dots, x_n, y_1, \dots, y_m, 0), \end{cases} \tag{1.4}$$

where $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbf{R}^{n+m}$ is restricted to the critical manifold $\mathcal{M}$. The $\mathcal{M}_\varepsilon$ is diffeomorphic to $\mathcal{M}$ and called the *slow manifold*. The dynamics of (1.1) approximately consists of the fast motion governed by (1.2) and the slow motion governed by (1.4). His method for constructing an approximate flow is called the *geometric singular perturbation method*.

However, if the critical manifold $\mathcal{M}$ has degenerate points $\boldsymbol{x}_0 \in \mathcal{M}$ in the sense that the Jacobian matrix $\partial \boldsymbol{f}/\partial \boldsymbol{x}$, $\boldsymbol{f} = (f_1, \dots, f_n)$, $\boldsymbol{x} = (x_1, \dots, x_n)$ at $\boldsymbol{x}_0$ has eigenvalues on the imaginary axis, then $\mathcal{M}$ is not normally hyperbolic near the $\boldsymbol{x}_0$ and Fenichel's theory is no longer applicable. The most common case is that $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ has one zero-eigenvalue at $\boldsymbol{x}_0$ and the critical manifold $\mathcal{M}$ is folded at the point (*fold point*). In this case, orbits on the slow manifold $\mathcal{M}_\varepsilon$ may jump and get away from $\mathcal{M}_\varepsilon$ in the vicinity of $\boldsymbol{x}_0$. As a result, the orbit repeatedly switches between fast motions and slow motions, and complex dynamics such as a relaxation oscillation can occur. See Mishchenko and Rozov [25] and Jones [18] for treatments of jump points and the existence of relaxation oscillations based on the boundary layer technique and the geometric singular perturbation method.

The blow-up method was developed by Dumortier [6] to investigate local flows near non-hyperbolic fixed points and it was applied to singular perturbed problems by Dumortier and Roussarie [7]. The most typical example is the system of the form

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = \varepsilon g(x, y), \end{cases} \tag{1.5}$$

where $(x, y) \in \mathbf{R}^2$. The critical manifold is a graph of $y = x^2$ and the origin is the fold point, at which the Jacobian matrix of the fast system has a zero-eigenvalue. Indeed, the fast system $\dot{x} = -y + x^2$ undergoes a saddle-node bifurcation as y varies. To analyze this family of vector fields, the trivial equation $\dot{\varepsilon} = 0$ is attached as

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = \varepsilon g(x, y), \\ \dot{\varepsilon} = 0. \end{cases} \tag{1.6}$$

Then, the Jacobian matrix at the origin $(0, 0, 0)$ degenerates as

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & g(0,0) \\ 0 & 0 & 0 \end{pmatrix} \tag{1.7}$$

with the Jordan block. The blow-up method is used to desingularize such singularities based on certain coordinate transformations. The most simple case $g(0,0) \neq 0$ is deeply investigated by Krupa and Szmolyan et al. [20,12] with the aid of a geometric view point. Straightforward extensions to higher dimensional cases are done by Szmolyan and Wechselberger [33] for $n = 1$, $m = 2$ and by Mishchenko and Rozov [25] for any n and m. Under the assumptions that $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ has only one zero-eigenvalue at a fold point and that the slow dynamics (1.4) has no fixed points near the fold point, they show that in the blow-up space, the system is reduced to the Riccati equation $dx/dy = y - x^2$ for any $n \geqslant 1$ and $m \geqslant 1$, and a certain special solution of the Riccati equation plays an important role to extend a slow manifold $\mathcal{M}_\varepsilon$ to a neighborhood of the fold point, which guides jumping orbits. It is to be noted that the classical work of Mishchenko and Rozov [25] is essentially equivalent to the blow-up method.

On the other hand, if the dynamics (1.4) has fixed points on (a set of) fold points, for example, if $g(0,0) = 0$ in Eq. (1.5), then more complex phenomena such as canard explosion can occur. Such situations are investigated by [7,20,32,22,24] by using the blow-up method. For example, for Eq. (1.5) with $g(0,0) = 0$, the original system is reduced to the system $\dot{x} = -y + x^2$, $\dot{y} = x$ in the blow-up space. If the dimension m of slow direction is larger than 1, there are many types of fixed points of (1.4) and thus we need more hard analysis as is done in [22].

The fast system for Eq. (1.5) undergoes a saddle-node bifurcation at the fold point. Thus we call the fold point the *saddle-node type fold point*. The cases that fast systems undergo a transcritical bifurcation and a pitchfork bifurcation are studied in [21]. It is shown that in the blow-up space, systems are reduced to the equations $dx/dy = x^2 - y^2 + \lambda$ and $dx/dy = xy - x^3$, respectively, whose special solutions are used to construct slow manifolds near fold points.

Despite many works, behavior of flows near fold points at which the Jacobian matrix $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ of the fast system has more than one zero-eigenvalues is not understood well. The purpose of this article is to investigate a three dimensional fast–slow system of the form

$$\begin{cases} \dot{x} = f_1(x, y, z, \varepsilon, \delta), \\ \dot{y} = f_2(x, y, z, \varepsilon, \delta), \\ \dot{z} = \varepsilon g(x, y, z, \varepsilon, \delta), \end{cases} \tag{1.8}$$

whose fast system has fold points with two zero-eigenvalues, where f_1, f_2, g are C^∞ functions, $\varepsilon > 0$ is a small parameter, and where $\delta > 0$ is a small parameter which controls the strength of the stability of the critical manifold (see the assumption (C5) in Section 2). Note that the critical manifold

$$\mathcal{M}(\delta) = \left\{ (x, y, z) \in \mathbf{R}^3 \;\middle|\; f_1(x, y, z, 0, \delta) = f_2(x, y, z, 0, \delta) = 0 \right\} \tag{1.9}$$

gives curves on $\mathbf{R}^3$ in general. We consider the situation that at a fold point $(x_0, y_0, z_0) \in \mathbf{R}^3$ on $\mathcal{M}$, the Jacobian matrix $\partial(f_1, f_2)/\partial(x, y)$ has two zero-eigenvalues with the Jordan block, and the two dimensional unperturbed system (fast system) undergoes a Bogdanov–Takens bifurcation. We call such a fold point the *Bogdanov–Takens type fold point*. For this system, we will show that the first Painlevé equation

$$\frac{d^2 y}{dz^2} = y^2 - z$$

appears in the blow-up space and plays an important role in the analysis of a local flow near the Bogdanov–Takens type fold points. This is in contrast with the fact that the Riccati equation appears in the case of saddle-node type fold points. It is shown that in the blow-up space, the slow manifold is extended along one of the special solutions, the Boutroux's tritronquée solution [1,19], of the first Painlevé equation. One of the main results in this article is that a transition map of Eq. (1.8) near the Bogdanov–Takens type fold point is constructed, in which an asymptotic expansion and a pole of the Boutroux's tritronquée solution are essentially used. This result shows that the distance between a solution of (1.8) near the Bogdanov–Takens type fold point and a solution of its unperturbed system is of order $O(\varepsilon^{4/5})$ as $\varepsilon \to 0$ (see Theorem 1 and Theorem 3.2), while it is of $O(\varepsilon^{2/3})$ for a saddle-node type fold point (see Mishchenko and Rozov [25]).

It is remarkable that all equations appeared in the blow-up space are related to the Painlevé theory. For example, the equation $dx/dy = y - x^2$ obtained from the saddle-node type fold point is transformed into the Airy equation $du/dy = uy$ by putting $x = (du/dy)/u$, which gives classical solutions of the second Painlevé equation. The equation $dx/dy = x^2 - y^2 + \lambda$ obtained from the transcritical type fold point is transformed into the Hermite equation

$$\frac{d^2u}{dy^2} + 2y\frac{du}{dy} + (\lambda + 1)u = 0$$

by putting $x + y = -(du/dy)/u$, which gives classical solutions of the fourth Painlevé equation. For other cases listed above, we also see that equations appeared in the blow-up space have the Painlevé property [5,17]; that is, all movable singularities (in the sense of the theory of ODEs on the complex plane) are poles, not branch points and essential singularities. This seems to be common for a wide class of fast–slow systems. Painlevé equations have many good properties [5]. For example, poles of solutions of Painlevé equations can be transformed into zeros of solutions of certain analytic systems by analytic transformations, which allow us to prove that the dominant part of the transition map near the Bogdanov–Takens type fold point is given by an analytic function describing a position of poles of the first Painlevé equation.

We also investigate global behavior of the system. Under some assumptions, we will prove that there exists a stable periodic orbit (relaxation oscillation) if $\varepsilon > 0$ is sufficiently small for fixed δ, and further that there exists a chaotic invariant set if $\delta > 0$ is also small in comparison with small ε. Roughly speaking, δ controls the strength of the stability of stable branches of the critical manifolds. While chaotic attractors on 3-dimensional fast–slow systems are reported by Guckenheimer, Wechselberger and Young [14] in the case of $n = 1$, $m = 2$, our system is of $n = 2$, $m = 1$. In the situation of [14], the chaotic attractor arises according to the theory of Hénon-like maps. On the other hand, in our system, the mechanism of the onset of a chaotic invariant set is similar to that in Silnikov's works [28–30], in which the existence of a hyperbolic horseshoe is shown for a 3-dimensional system which have a saddle-focus fixed point with a homoclinic orbit. See also Wiggins [34]. Indeed, in our situation, the critical manifold $\mathcal{M}(\delta)$ plays a similar role to a saddle-focus fixed point in the Silnikov's system. Thus the proof of the existence of a relaxation oscillation in our system will be done in usual way: the Poincaré return map proves to be contractive, while the proof of the existence of chaos is done in a similar way to that of the Silnikov's system: as δ decreases, the Poincaré return map becomes non-contractive, undergoes a cascade of bifurcations, and horseshoes are created. When one want to prove the existence of a stable periodic orbit, it is sufficient to show that the image of the return map is exponentially small. However, to prove the existence of a horseshoe, one has to show that the image of a rectangle under the return map becomes a horseshoe-shaped (ring-shaped). Thus our analysis for constructing the return map involves hard calculations, which can be avoided when proving only a periodic orbit.

Our chaotic invariant set seems to be attracting as that in [14], however, it remains unsolved. See Homburg [15] for the proof of the existence of chaotic attractors in the Silnikov's system.

The results in the present article are used in [3] to investigate chaotic invariant sets on the Kuramoto model, which is one of the most famous models to explain synchronization phenomena. In [3], it is shown that the Kuramoto model with appropriate assumptions can be reduced to a three dimensional fast–slow system by using the renormalization group method [2].

This paper is organized as follows. In Section 2, we give statements of our theorems on the existence of a periodic orbit and a chaotic invariant set. An intuitive explanation of the theorems is also shown with an example. In Section 3, local analysis near the Bogdanov–Takens type fold point is given by means of the blow-up method. Section 4 is devoted to global analysis, and proofs of main theorems are given. Concluding remarks are included in Section 5.

2. Main results

To obtain a local result and the existence of relaxation oscillations, the parameter δ in Eq. (1.8) does not play a role. Thus we consider the system of the form

$$\begin{cases} \dot{x} = f_1(x, y, z, \varepsilon), \\ \dot{y} = f_2(x, y, z, \varepsilon), \\ \dot{z} = \varepsilon g(x, y, z, \varepsilon), \end{cases} \tag{2.1}$$

with C^∞ functions $f_1, f_2, g : U \times I \to \mathbf{R}$, where $U \subset \mathbf{R}^3$ is an open domain in $\mathbf{R}^3$ and $I \subset \mathbf{R}$ is a small interval containing zero. The unperturbed system is given as

$$\begin{cases} \dot{x} = f_1(x, y, z, 0), \\ \dot{y} = f_2(x, y, z, 0), \\ \dot{z} = 0. \end{cases} \tag{2.2}$$

Since z is a constant, this system is regarded as a family of 2-dimensional systems. The critical manifold is the set of fixed point of (2.2) defined to be

$$\mathcal{M} = \big\{(x, y, z) \in U \;\big|\; f_1(x, y, z, 0) = f_2(x, y, z, 0) = 0\big\}. \tag{2.3}$$

The reduced flow on the critical manifold is defined as

$$\dot{z} = \varepsilon g(x, y, z, 0)|_{(x,y,z)\in\mathcal{M}}. \tag{2.4}$$

To investigate a Bogdanov–Takens type fold point, we make the following assumptions.

(A1) The critical manifold $\mathcal{M}$ has a smooth component $S^+ = S_a^+ \cup \{L^+\} \cup S_r^+$, where S_a^+ consists of stable focus fixed points, S_r^+ consists of saddle fixed points, and where L^+ is a fold point.
(A2) The L^+ is a Bogdanov–Takens type fold point; that is, L^+ is a Bogdanov–Takens bifurcation point of the vector field $(f_1(x, y, z, 0), f_2(x, y, z, 0))$. In particular, Eq. (2.2) has a cusp at L^+.
(A3) The reduced flow (2.4) on S_a^+ is directed toward the fold point L^+ and $g(L^+, 0) \neq 0$.

A few remarks are in order. It is easy to see from (A1) that the Jacobian matrix $\partial(f_1, f_2)/\partial(x, y)$ has two zero eigenvalues at L^+ since S_r^+ and S_a^+ are saddles and focuses, respectively. Thus there exists a coordinate transformation $(x, y, z) \mapsto (X, Y, Z)$ defined near L^+ such that L^+ is placed at the origin and Eq. (2.2) takes the following normal form

$$\begin{cases} \dot{X} = a_1(Z) + a_2(Z)Y^2 + a_3(Z)XY + O\big(X^3, X^2Y, XY^2, Y^3\big), \\ \dot{Y} = b_1(Z) + b_2(Z)X + O\big(X^3, X^2Y, XY^2, Y^3\big), \\ \dot{Z} = 0, \end{cases} \tag{2.5}$$

where $a_1(0) = b_1(0) = 0$ so that the origin is a fixed point (for the normal form theory, see Chow, Li and Wang [4]). Then the assumption (A2) means that $a_2(0) \neq 0$, $a_3(0) \neq 0$, $b_2(0) \neq 0$. In this case, it is well known that the flow of Eq. (2.5) has a cusp at the origin (see also Lemma 3.1). Since Eq. (2.5) has

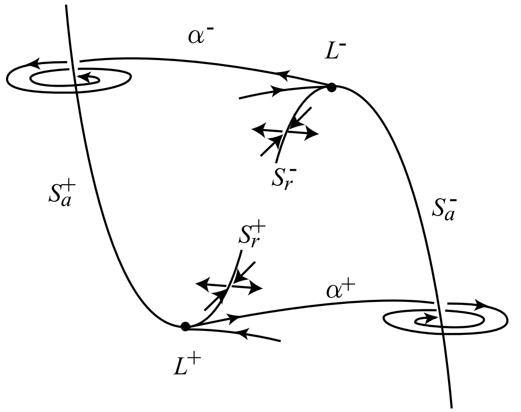

Fig. 1. Critical manifold and the flow of Eq. (2.1) with the assumptions (B1) to (B4).

a cusp at L^+, there exists exactly one orbit α^+ emerging from L^+. The assumption (A3) means that if the critical manifold is locally convex downward (resp. convex upward), then $g(x, y, z, 0) < 0$ (resp. $g(x, y, z, 0) > 0$) on $S_a^+ \cup \{L^+\}$. Thus an orbit of the reduced flow on S_a^+ reaches L^+ in finite time. As a result, an orbit of (2.1) may jump in the vicinity of L^+. The next theorem describes an asymptotic behavior of such a jumping orbit.

Theorem 1. *Suppose that the system* (2.1) *satisfies assumptions* (A1) *to* (A3)*. Consider a solution* $\mathbf{x}(t)$ *whose initial point is in the vicinity of* S_a^+*. Then, there exist* $t_0, t_1 > 0$ *such that the distance between* $\mathbf{x}(t)$, $t_0 < t < t_1$ *and the orbit* α^+ *of the unperturbed system emerging from* L^+ *is of* $O(\varepsilon^{4/5})$ *as* $\varepsilon \to 0$.

Note that for a saddle-node type fold points, the distance between $\mathbf{x}(t)$ and an orbit emerging from a fold point is of $O(\varepsilon^{2/3})$. To prove the existence of relaxation oscillations, we need global assumptions for the system (2.1).

(B1) The critical manifold $\mathcal{M}$ has two smooth components $S^+ = S_a^+ \cup \{L^+\} \cup S_r^+$ and $S^- = S_a^- \cup \{L^-\} \cup S_r^-$, where $S_a^\pm$ consist of stable focus fixed points, $S_r^\pm$ consist of saddle fixed points, and where $L^\pm$ are fold points (see Fig. 1).
(B2) The $L^\pm$ are Bogdanov–Takens type fold points; that is, $L^\pm$ are Bogdanov–Takens bifurcation points of the vector field $(f_1(x, y, z, 0), f_2(x, y, z, 0))$. In particular, Eq. (2.2) has cusps at $L^\pm$.
(B3) Eq. (2.2) has two heteroclinic orbits α^+ and α^- which connect L^+, L^- with points on S_a^-, S_a^+, respectively.
(B4) The reduced flow (2.4) on $S_a^\pm$ is directed toward the fold points $L^\pm$ and $g(L^\pm, 0) \neq 0$, respectively.

Assumptions (B1) and (B2) assure that $S^\pm$ are locally expressed as parabolas, and thus they are of "J-shaped". Components S^+ and S^- are allowed to be connected. In this case, $S^+ \cup S^-$ is of "S-shaped". As was mentioned above, since (2.2) has cusps at $L^\pm$, there exist two orbits α^+ and α^- of Eq. (2.2) emerging from L^+ and L^-. The assumption (B3) means that these orbits are connected to S_a^- and S_a^+, respectively. If $S^+ \cup S^-$ is of "S-shaped", the assumption (B3) is typically satisfied because at least near the fold points, the unperturbed system (2.2) has heteroclinic orbits connecting each point on $S_r^\pm$ to $S_a^\pm$, respectively, due to the basic bifurcation theory. Note that the assumption (B3) also determines a positional relationship between S^+ and S^-. For example, if S^+ is convex downward, S^- should be convex upward. By applying Theorem 1 combined with the geometric singular perturbation (boundary layer technique), we can obtain the following result.

Theorem 2. *Suppose that the system* (2.1) *satisfies assumptions* (B1) *to* (B4)*. Then there exists a positive number ε_0 such that Eq.* (2.1) *has a hyperbolically stable periodic orbit near $S_a^+ \cup \alpha^+ \cup S_a^- \cup \alpha^-$ if $0 < \varepsilon < \varepsilon_0$.*

To prove the existence of a periodic orbit, the local assumptions are not so important, though a positional relationship between components of the critical manifold and the existence of heteroclinic orbits $\alpha^\pm$ are essential. Indeed, similar results for fast–slow systems having saddle-node type fold points are obtained by many authors.

To prove the existence of chaos, we have to control the strength of the stability of $S_a^\pm$. Let us consider the system (1.8) with C^∞ functions $f_1, f_2, g : U \times I \times I' \to \mathbf{R}$, where U and I as above and $I' \subset \mathbf{R}$ is a small interval containing zero. The unperturbed system of Eq. (1.8) is given by

$$\begin{cases} \dot{x} = f_1(x, y, z, 0, \delta), \\ \dot{y} = f_2(x, y, z, 0, \delta), \\ \dot{z} = 0. \end{cases} \tag{2.6}$$

The critical manifold $\mathcal{M}(\delta)$ defined by (1.9) is parameterized by δ. At first, we suppose that the assumptions (B1) to (B4) are satisfied uniformly in δ. More exactly, we assume following.

(C1) There exists δ_0 such that for every $\delta \in [0, \delta_0)$, the critical manifold $\mathcal{M}(\delta)$ has two smooth components $S^+(\delta) = S_a^+(\delta) \cup \{L^+(\delta)\} \cup S_r^+(\delta)$ and $S^-(\delta) = S_a^-(\delta) \cup \{L^-(\delta)\} \cup S_r^-(\delta)$. When $\delta > 0$, $S_a^\pm(\delta)$ consist of stable focus fixed points, $S_r^\pm(\delta)$ consist of saddle fixed points, and $L^\pm(\delta)$ are fold points (see Fig. 1). Further, the δ family $\mathcal{M}(\delta)$ is smooth with respect to $\delta \in [0, \delta_0)$.

(C2) For every $\delta \in [0, \delta_0)$, $L^\pm(\delta)$ are Bogdanov–Takens type fold points; that is, $L^\pm(\delta)$ are Bogdanov–Takens bifurcation points of the vector field $(f_1(x, y, z, 0, \delta), f_2(x, y, z, 0, \delta))$. In particular, Eq. (2.6) has cusps at $L^\pm(\delta)$.

(C3) For every $\delta \in (0, \delta_0)$, Eq. (2.6) has two heteroclinic orbits $\alpha^+(\delta)$ and $\alpha^-(\delta)$ which connect $L^+(\delta)$, $L^-(\delta)$ with points on $S_a^-(\delta)$, $S_a^+(\delta)$, respectively.

(C4) For every $\delta \in [0, \delta_0)$, the reduced flow on $S_a^\pm(\delta)$ is directed toward the fold points $L^\pm(\delta)$ and $g(L^\pm, 0, \delta) \neq 0$, respectively.

In addition to the assumptions above, we make the assumptions for the strength of the stability of $S_a^\pm$ as follows:

(C5) For every $\delta \in [0, \delta_0)$, eigenvalues of the Jacobian matrix $\partial(f_1, f_2)/\partial(x, y)$ of Eq. (2.6) at $(x, y, z) \in S_a^+(\delta)$ and at $(x, y, z) \in S_a^-(\delta)$ are expressed by $-\delta \cdot \mu^+(z, \delta) \pm \sqrt{-1}\omega^+(z, \delta)$ and $-\delta \cdot \mu^-(z, \delta) \pm \sqrt{-1}\omega^-(z, \delta)$, respectively, where $\mu^\pm$ and $\omega^\pm$ are real-valued functions satisfying

$$\mu^\pm(z, 0) > 0, \qquad \omega^\pm(z, 0) \neq 0. \tag{2.7}$$

The assumption (C5) means that the parameter δ controls the strength of the stability of stable focus fixed points on $S_a^\pm(\delta)$.

Finally, we suppose that the basin of $S_a^\pm(\delta)$ of the unperturbed system can be taken uniformly in $\delta \in (0, \delta_0)$: By the assumption (C5), there exist open sets $V^\pm \supset S_a^\pm(\delta)$ such that real parts of eigenvalues of the Jacobian matrix $\partial(f_1, f_2)/\partial(x, y)$ on $V^\pm$ is of order $O(\delta)$. In general, the "size" of $V^\pm$ depend on δ and they may tend to zero as $\delta \to 0$. To prove Theorem 3 below, we assume following.

(C6) There exist open sets $V^\pm \supset S_a^\pm(\delta)$, which is independent of δ, such that real parts of eigenvalues of the Jacobian matrix $\partial(f_1, f_2)/\partial(x, y)$ on $V^\pm$ are negative and of order $O(\delta)$ as $\delta \to 0$. The orbits $\alpha^\pm(\delta)$ emerging from $L^\pm$ enter the set $V^\mp$, respectively, in finite time for any $\delta \in [0, \delta_0]$.

The first sentence of this assumption also assures that the attraction basin of $S_a^\pm(\delta)$ of the unperturbed system can be taken uniformly in $\delta \in (0, \delta_0)$, see an example below. For the second sentence, note that there exist orbits $\alpha^\pm(\delta)$ emerging from $L^\pm$ even for $\delta = 0$ because of (C2), although they

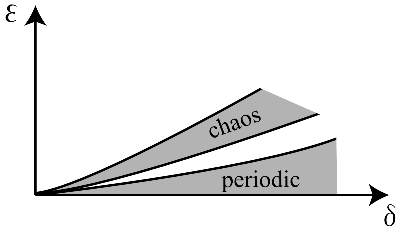

Fig. 2. Typical bifurcation diagram of (1.8) with assumptions (C1) to (C6).

may not be connected to $S_a^{\mp}$ at $\delta = 0$ because (C3) is assumed for an open interval $(0, \delta_0)$. The second sentence of (C6) implies that the transition map from the section near $L^{\pm}$ to the section in $V^{\mp}$ is well defined as $\delta \to 0$.

Theorem 3. *Suppose that the system* (1.8) *satisfies assumptions* (C1) *to* (C6)*. Then, there exist a positive number ε_0 and positive valued functions $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ such that if* $0 < \varepsilon < \varepsilon_0$ *and* $\delta_1(\varepsilon) < \delta < \delta_2(\varepsilon)$*, then Eq.* (1.8) *has a chaotic invariant set near $S_a^+(\delta) \cup \alpha^+(\delta) \cup S_a^-(\delta) \cup \alpha^-(\delta)$, where $\delta_{1,2}(\varepsilon) \to 0$ as $\varepsilon \to 0$. More exactly, the Poincaré return map Π along the flow of* (1.8) *near $S_a^+(\delta) \cup \alpha^+(\delta) \cup S_a^-(\delta) \cup \alpha^-(\delta)$ is well defined, and Π has a hyperbolic horseshoe* (*an invariant Cantor set, on which Π is topologically conjugate to the full shift on two symbols*)*.*

Theorems 2 and 3 mean that if $\varepsilon > 0$ is sufficiently small for a fixed δ, then there exists a stable periodic orbit. However, as δ decreases, the periodic orbit undergoes a succession of bifurcations and if δ gets sufficiently small in comparison with ε, then a chaotic invariant set appears. In our proof in Section 4, δ will be assumed to be of $O(\varepsilon(-\log\varepsilon)^{1/2})$. We conjecture that this chaotic invariant set is attracting, although the proof is not given in this paper. In general, given fast–slow systems do not have the parameter δ explicitly. However, Theorem 3 suggests that as ε increases for fixed δ, a periodic orbit undergoes bifurcations and a chaotic invariant set may appears, see Fig. 2. Obviously the assumptions (C1) to (C4) include assumptions (A1) to (A3) and (B1) to (B4). In what follows, we consider the system (1.8) with the parameter δ. When proving Theorems 1 and 2, δ is assumed to be constant, and when proving Theorem 3, δ is assumed to be of $\delta \sim O(\varepsilon(-\log\varepsilon)^{1/2})$ as $\varepsilon \to 0$. Note that $\varepsilon \ll \varepsilon(-\log\varepsilon)^{1/2} \ll 1$ as $\varepsilon \to 0$. Although $\delta > 0$ is also small, uniformity assumptions on δ and the fact $\varepsilon \ll \delta$ allow us to use the perturbation techniques with respect to only on ε.

In the rest of this section, we give an intuitive explanation of the theorems with an example. Consider the system

$$\begin{cases} \dot{x} = z + 3(y^3 - y) + \delta x\left(\dfrac{1}{3} - y^2\right), \\ \dot{y} = -x, \\ \dot{z} = \varepsilon \sin\left(\dfrac{5}{2}y\right). \end{cases} \tag{2.8}$$

The critical manifold $\mathcal{M} = \mathcal{M}(\delta)$ is given by the curve $z = 3(y - y^3), x = 0$, and the fold points are given by $L^{\pm} = (0, \mp\frac{1}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}})$, see Fig. 3.

It is easy to verify that the assumptions (C1), (C2), (C4) and (C5) are satisfied for (2.8). The assumption (C3) of existence of heteroclinic orbits are verified numerically (we do not give a proof here).

The assumption (C6) is also verified by a straightforward calculation. Now we show that (C6) implies that the attraction basin of $S_a^{\pm}(\delta)$ of the unperturbed system can be taken uniformly in $\delta \in (0, \delta_0)$. We change the coordinates by an affine transformation $(x, y, z) \mapsto (X, Y, Z)$ so that the point

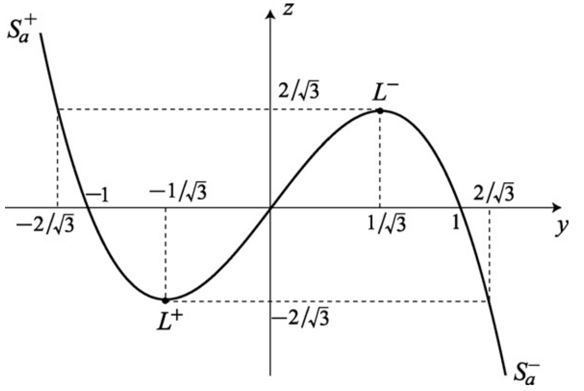

Fig. 3. Critical manifold of the system (2.8).

$(0, -2/\sqrt{3}, 2/\sqrt{3})$ is placed at the origin and the linear part of Eq. (2.8) is diagonalized. Then the unperturbed system of Eq. (2.8) is rewritten as

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\sqrt{-1}}{2}\sqrt{36-\delta^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} - \frac{\delta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + h(X, Y, \delta), \tag{2.9}$$

where the explicit form of the polynomial h, whose degree is greater than one, is too complicated to be written here. However, one can verify that h is of the form

$$h(X, Y, \delta) = \sqrt{-1}h_1(X, Y, \delta) + \delta h_2(X, Y, \delta), \tag{2.10}$$

where h_1 and h_2 are polynomials with respect to X and Y such that all coefficients of h_1 are real. Note that $\sqrt{36-\delta^2}/2$ and $\delta/2$ correspond to $\omega^+(z,\delta)$ and $\delta\mu^+(z,\delta)$, respectively, in the assumption (C5).

Now we bring Eq. (2.9) into the normal form with respect to the first term of the right-hand side. There exist a neighborhood W of the origin, which is independent of δ, and a coordinate transformation $(X, Y) \mapsto (r, \theta)$ defined on W such that Eq. (2.9) is put in the form

$$\begin{cases} \dot{r} = -\dfrac{\delta}{2}r + a_3 r^3 + a_5 r^5 + \cdots, \\ \dot{\theta} = \sqrt{36-\delta^2}/2 + O(r^2). \end{cases} \tag{2.11}$$

Note that the equation of the radius r is independent of θ (see Chow, Li and Wang [4]). In our case, a_3 is given by

$$a_3 = \delta\frac{-180 + 29\delta^2}{6(36-\delta^2)^2}. \tag{2.12}$$

Further, we can prove that $a_i \sim O(\delta)$, $i = 3, 5, \ldots$ as $\delta \to 0$ by using the induction together with the property that $h(X, Y, 0)$ takes purely imaginary values if $(X, Y) \in \mathbf{R}^2$ (see Eq. (2.10)). See Chiba [2] for explicit formulas of normal forms which are convenient for induction. Thus the derivative of the right-hand side of Eq. (2.11) is calculated as

$$\frac{d}{dr}\left(-\frac{\delta}{2}r + a_3 r^3 + \cdots\right) = -\frac{\delta}{2}\left(1 + b_3 r^2 + b_5 r^4 + \cdots\right) + O\left(\delta^2\right), \tag{2.13}$$

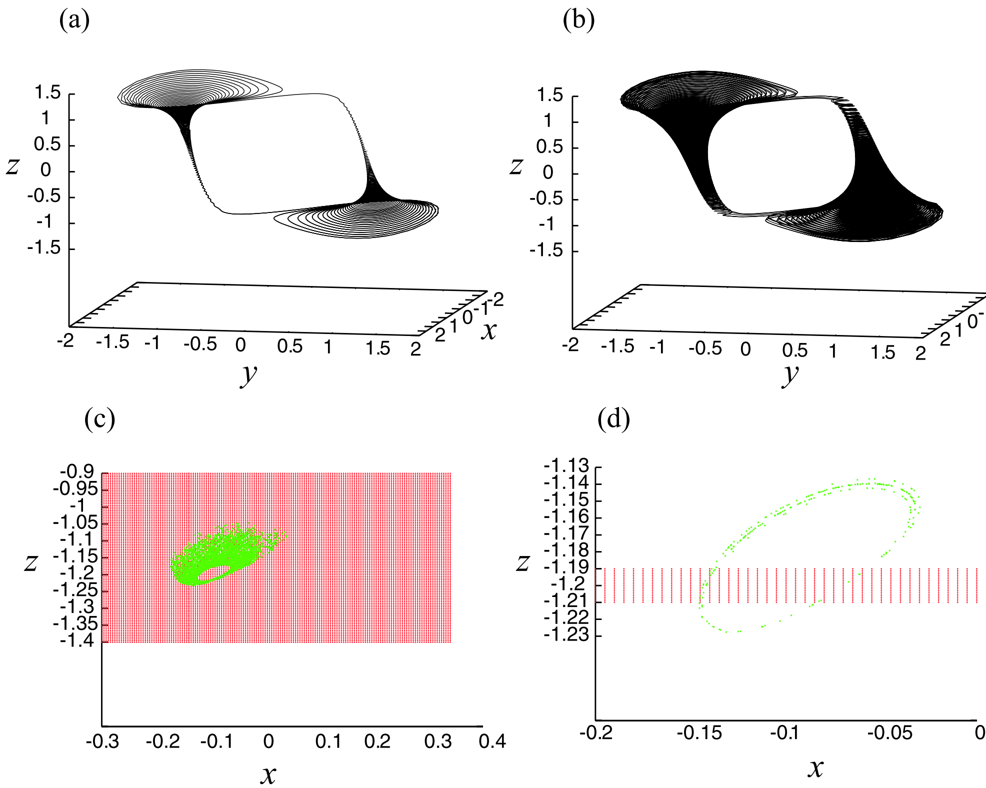

Fig. 4. Numerical results for Eq. (2.8). If (a) $\varepsilon = 0.02$ and $\delta = 0.06$, there exists a stable periodic orbit and if (b) $\varepsilon = 0.02$ and $\delta = 0.03$, there exists a chaotic attractor. In (c) and (d), the green points denote the image of the red points under the Poincaré map from Σ_1 to Σ_2 for $\varepsilon = 0.02$ and $\delta = 0.03$. They show that the Poincaré map has a horseshoe and it is attracting. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $b_3, b_5, \ldots$ are δ-independent constants. It proves that there exists a δ-independent positive constant C such that if $|r(0)| < C$, then $r(t)$ decays as $|r(t)| \sim O(e^{-\delta t/2})$ for small $\delta > 0$. The same property can be verified for any system with the assumption (C6) by means of the normal form.

Fig. 4 shows numerical results for Eq. (2.8). If $\varepsilon = 0.02$ and $\delta = 0.06$, there exists a stable periodic orbit (Fig. 4 (a)) while a chaotic behavior occurs if $\varepsilon = 0.02$ and $\delta = 0.03$ (Fig. 4 (b)). This verifies Theorems 2 and 3 for Eq. (2.8). As was mentioned, chaos may occurs when ε increases for fixed δ. For example, numerical simulations show that chaos also appears for $\varepsilon = 0.04$ and $\delta = 0.06$.

Although Theorem 3 does not state that a chaotic invariant set mentioned is attracting, Fig. 4(c) corroborates numerically that the chaotic invariant set for our example is actually a chaotic attractor. Take the Poincaré section $\Sigma_1 = \{(x, y, z) \mid y = 0.5,\ z > 0\}$ and $\Sigma_2 = \{(x, y, z) \mid y = -0.5,\ z < 0\}$, like as Σ^-_{out} and Σ^+_{out} in Fig. 5, respectively. Since Eq. (2.8) admits the symmetry $(x, y, z) \mapsto (-x, -y, -z)$ and Σ_1 corresponds to Σ_2 under the symmetry, we identify them and calculate the Poincaré map from Σ_1 to Σ_2. The results are represented in Fig. 4(c) and (d). The red points on Σ_1, identified with Σ_2, are mapped to the green points on Σ_2 by the Poincaré map. Fig. 4(c) shows that the Poincaré map is attracting, and Fig. 4(d) shows that it has a horseshoe.

To ascertain the reason why the periodic orbit or the chaotic attractor occur, we take Poincaré sections Σ^+_{out}, Σ^-_{II}, Σ^-_{in}, Σ^-_{out}, Σ^+_{II} and Σ^+_{in} as in Fig. 5.

The section Σ^+_{out} is parallel to the xz plane and located at the right of L^+. Take a rectangle R on Σ^+_{out} and consider how it behaves when it runs along solutions of Eq. (2.8). Since the unperturbed system of Eq. (2.8) has the heteroclinic orbit α^+ connecting L^+ and S^-_a, the rectangle R also approaches

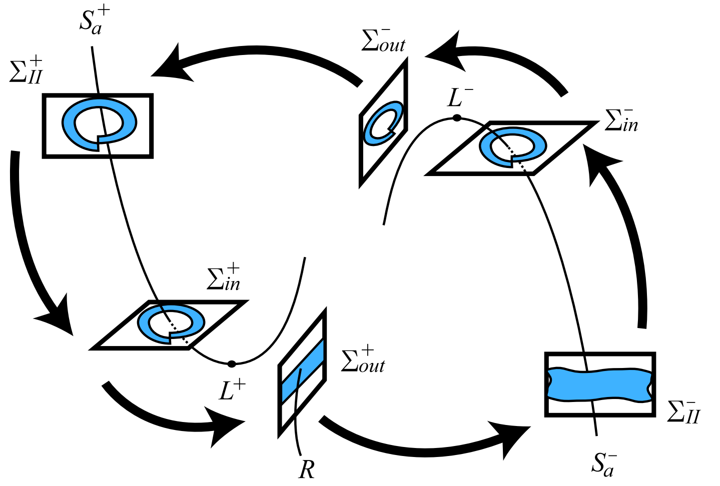

Fig. 5. Poincaré sections and a schematic view of the images of the rectangle R under a succession of the transition maps.

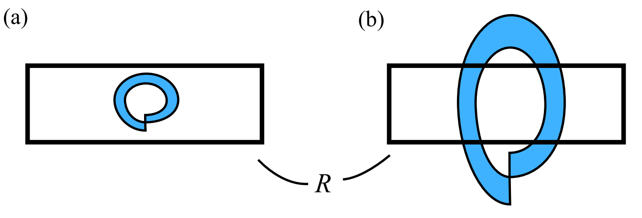

Fig. 6. Positional relationship of the rectangle R with the returned ring-shaped area.

to S_a^- along α^+ and intersects the section Σ_{II}^-, as is shown in Fig. 5. Since the velocity $\varepsilon \sin(5y/2)$ in the direction z is positive in the vicinity of S_a^- and since S_a^- consists of stable focus fixed points, the intersection area on Σ_{II}^- moves upward, rotating around S_a^-. As a result, the flow of R intersects the section Σ_{in}^-, which is parallel to the xy plane, to form a ring-shaped area as is shown in Fig. 5. Further, we can show that the ring-shaped area on Σ_{in}^- moves to Σ_{out}^- along solutions of Eq. (2.8) due to Theorem 1. The area on Σ_{out}^- goes back to the section Σ_{out}^+ in a similar manner because Eq. (2.8) has the symmetry $(x, y, z) \mapsto (-x, -y, -z)$. Thus the Poincaré return map Π from Σ_{out}^+ into itself is well defined and it turns out that $\Pi(R)$ is ring-shaped.

There are two possibilities of locations of the returned ring-shaped area. If the strength of the stability of stable fixed points on $S_a^\pm$, say δ as in the assumption (C5), is sufficiently large, then the radius of the ring-shaped area gets sufficiently small when passing around $S_a^\pm$. As a result, the returned ring-shaped area is included in the rectangle R as in Fig. 6(a). It means that the Poincaré map Π is contractive and it has a stable fixed point, which corresponds to a stable periodic orbit of Eq. (2.8). On the other hand, if the strength δ is not so large, the radius of the ring-shaped area is not so small and it intersects with the rectangle as in Fig. 6(b). In this case, the Poincaré map Π has a horseshoe.

3. Local analysis around the fold points

In this section, we give a local analysis around the fold points $L^\pm$ by using the blow-up method, and calculate a transition map to observe how orbits of Eq. (1.8) behave near the fold points. To prove the existence of chaos, we will give a detailed analysis of the transition map, which does not need for the standard proof of the existence of a periodic orbit. The main theorem in this section (Theorem 3.2) will be made in the end of Section 3.1. We will calculate only for L^+ because discussion for L^- is done in the same way.

3.1. Normal form coordinates

At first, we transform Eq. (1.8) into the normal form in the vicinity of $L^+(\delta)$. In what follows, if a (formal) power series h centered at the origin begins with n-th degree terms (*i.e.* $\partial^i h(0)/\partial \boldsymbol{x}^i = 0$ $(i = 0, 1, \ldots, n-1)$ and $\partial^n h(0)/\partial \boldsymbol{x}^n \neq 0$), we denote the fact as $h \sim O_p(n)$. The notation $O(\cdot)$ is also used to the usual Landau notation. For example if $h(x, y, z) \sim O(x^2, y^2, z^2, xy, yz, zx)$ as $x, y, z \to 0$, we simply denote it as $h \sim O_p(2)$.

Lemma 3.1. *Suppose* (C1), (C2) *and* (C4). *For every $\delta \in [0, \delta_0)$, there exists a C^∞ local coordinate transformation $(x, y, z) \mapsto (X, Y, Z)$ defined near $L^+(\delta)$ such that Eq.* (1.8) *is brought into the form*

$$\begin{cases} \dot{X} = Z - Y^2 + c_1(\delta)XY + Zh_1(X, Y, Z, \delta) + Y^2 h_2(X, Y, Z, \delta) + \varepsilon h_3(X, Y, Z, \varepsilon, \delta), \\ \dot{Y} = -X + Zh_4(X, Y, Z, \delta) + \varepsilon h_5(X, Y, Z, \varepsilon, \delta), \\ \dot{Z} = -\varepsilon + \varepsilon h_6(X, Y, Z, \varepsilon, \delta), \end{cases} \tag{3.1}$$

where $c_1(\delta)$ and h_i $(i = 1, \ldots, 6)$ are C^∞ functions such that $c_1(\delta) > 0$ for $\delta > 0$ and

$$h_1, h_2, h_4 \sim O(X, Y, Z), \qquad h_6 \sim O(X, Y, Z, \varepsilon). \tag{3.2}$$

If we assume (C5), *then $c_1(\delta) \sim O(\delta)$ as $\delta \to 0$.*

In these coordinates, $L^+(\delta)$ is placed at the origin and the branch $S^+(\delta)$ of the critical manifold is of the form $Z = Y^2 + O_p(3)$, $X = O_p(2)$.

Proof of Lemma 3.1. We start by calculating the normal form of the unperturbed system (2.6). We will use the same notation (x, y, z) as the original coordinates after a succession of coordinate transformations for simplicity. Since the Jacobian matrix of (f_1, f_2) at $L^+(\delta)$ has two zero eigenvalues due to the assumption (C1), the normal form for the equations of (x, y) is of the form (see Chow, Li and Wang [4])

$$\begin{cases} \dot{x} = a_1(\delta)z + a_2(\delta)y^2 + a_3(\delta)xy + zh_1(x, y, z, \delta) + y^2 h_2(x, y, z, \delta), \\ \dot{y} = b_1(\delta)x + b_2(\delta)z + zh_4(x, y, z, \delta), \end{cases} \tag{3.3}$$

where $a_1(\delta), a_2(\delta), a_3(\delta), b_1(\delta), b_2(\delta)$ and $h_1, h_2, h_4 \sim O(x, y, z)$ are C^∞ functions. Note that $a_2(\delta) \neq 0$, $b_1(\delta) \neq 0$ for $\delta \in [0, \delta_0)$ because of the assumption (C2). Since we can assume that $S^+(\delta)$ is locally expressed as $z \sim y^2$, $x \sim 0$ without loss of generality, by a suitable coordinate transformation, we obtain $a_2(\delta) = -a_1(\delta)$ and $b_2(\delta) = 0$. Since fixed points on $S_a^+(\delta)$ are attracting and since fixed points on $S_r^+(\delta)$ are saddles for $\delta > 0$, we obtain $a_1(\delta)b_1(\delta) < 0$ and $a_3(\delta) > 0$ for $\delta > 0$. If we assume (C5), then $a_3(\delta) \sim O(\delta)$. We can assume that $a_1(\delta) > 0$ because we are allowed to change the coordinates as $x \mapsto -x$, $y \mapsto -y$ if necessary. Thus, the normal form of Eq. (2.6) is written as

$$\begin{cases} \dot{x} = a_1(\delta)\left(z - y^2\right) + a_3(\delta)xy + zh_1(x, y, z, \delta) + y^2 h_2(x, y, z, \delta), \\ \dot{y} = b_1(\delta)x + zh_4(x, y, z, \delta), \\ \dot{z} = 0, \end{cases} \tag{3.4}$$

with $a_1(\delta) > 0$, $b_1(\delta) < 0$. The coordinate transformation which brings Eq. (2.6) into Eq. (3.4) transforms Eq. (1.8) into the system of the form

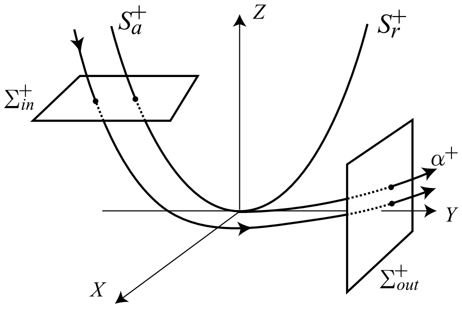

Fig. 7. Transition map Π_{loc}^+ and the heteroclinic orbit α^+.

$$\begin{cases} \dot{x} = a_1(\delta)\big(z - y^2\big) + a_3(\delta)xy + zh_1(x, y, z, \delta) + y^2 h_2(x, y, z, \delta) + \varepsilon h_3(x, y, z, \varepsilon, \delta), \\ \dot{y} = b_1(\delta)x + zh_4(x, y, z, \delta) + \varepsilon h_5(x, y, z, \varepsilon, \delta), \\ \dot{z} = \varepsilon\big(g_1(\delta) + h_6(x, y, z, \varepsilon, \delta)\big), \end{cases} \tag{3.5}$$

where h_3, h_5, h_6 are C^∞ functions such that $h_6 \sim O(x, y, z, \varepsilon)$, and where $g_1(\delta) := g(L^+, 0, \delta)$ is a negative constant on account of the assumption (C4). Finally, changing coordinates and time scales as

$$x = -X\frac{a_1(\delta)}{g_1(\delta)}\left(-\frac{g_1(\delta)^2}{a_1(\delta)b_1(\delta)}\right)^{4/5}, \qquad y = Y\left(-\frac{g_1(\delta)^2}{a_1(\delta)b_1(\delta)}\right)^{1/5}, \qquad z = Z\left(-\frac{g_1(\delta)^2}{a_1(\delta)b_1(\delta)}\right)^{2/5},$$
$$t \mapsto -\frac{t}{g_1(\delta)}\left(-\frac{g_1(\delta)^2}{a_1(\delta)b_1(\delta)}\right)^{2/5}, \tag{3.6}$$

and modifying the definitions of $h_i's$ $(i = 1, \dots, 6)$ appropriately, we obtain Eq. (3.1). Note that since $g_1(\delta), a_1(\delta), b_1(\delta) \neq 0$ for $\delta \in [0, \delta_0)$, this transformation is a local diffeomorphism for every $\delta \in [0, \delta_0)$. □

Let ρ_1 be a small positive number and let

$$\Sigma_{in}^+ = \big\{\big(X, Y, \rho_1^4\big) \bigm| (X, Y) \in \mathbf{R}^2\big\}, \qquad \Sigma_{out}^+ = \big\{\big(X, \rho_1^2, Z\big) \bigm| (X, Z) \in \mathbf{R}^2\big\} \tag{3.7}$$

be Poincaré sections in the (X, Y, Z) space defined near the origin (see Fig. 7). The purpose of this section is to construct a transition map from Σ_{in}^+ to Σ_{out}^+. Recall that there exists an orbit $\alpha^+(\delta)$ emerging from $L^+(\delta)$, where $L^+(\delta)$ corresponds to the origin in the (X, Y, Z) space.

Theorem 3.2. *Suppose* (C1), (C2) *and* (C4) *to* (C6). *If $\rho_1 > 0$ is sufficiently small, there exists $\varepsilon_0 > 0$ such that the followings hold for $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta_0$:*

(I) *There exists an open set $U_\varepsilon \subset \Sigma_{in}^+$ near the point $\Sigma_{in}^+ \cap S_a^+(\delta)$ such that the transition map $\Pi_{loc}^+ : U_\varepsilon \to \Sigma_{out}^+$ along the flow of Eq.* (3.1) *is well-defined, C^∞ with respect to X and Y, and expressed as*

$$\Pi_{loc}^+\begin{pmatrix} X \\ Y \\ \rho_1^4 \end{pmatrix} = \begin{pmatrix} G_1(\rho_1, \delta) \\ \rho_1^2 \\ 0 \end{pmatrix} + \begin{pmatrix} G_2(\mathcal{X}, \mathcal{Y}, \rho_1, \delta)\varepsilon^{4/5} + O(\varepsilon \log \varepsilon) \\ 0 \\ (\Omega + H(\mathcal{X}, \mathcal{Y}))\varepsilon^{4/5} + O(\varepsilon \log \varepsilon) \end{pmatrix}, \tag{3.8}$$

where $\Omega \sim -3.416$ is a negative constant, and G_1, G_2, H are C^∞ functions with respect to $\mathcal{X}, \mathcal{Y}, \delta$. The arguments $\mathcal{X}, \mathcal{Y}$ are defined by

$$\begin{cases} \mathcal{X} = \hat{D}_1(X, Y, \rho_1, \varepsilon, \delta)\varepsilon^{-3/5} \exp\left[-\hat{d}(\rho_1, \delta)\dfrac{\delta}{\varepsilon}\right], \\ \mathcal{Y} = \hat{D}_2(X, Y, \rho_1, \varepsilon, \delta)\varepsilon^{-2/5} \exp\left[-\hat{d}(\rho_1, \delta)\dfrac{\delta}{\varepsilon}\right], \end{cases} \tag{3.9}$$

where $\hat{D}_1$, $\hat{D}_2$ and $\hat{d}$ are C^∞ functions with respect to X, Y, δ such that $\hat{d} > 0$ for $\delta \geqslant 0$. Functions $\hat{D}_1$ and $\hat{D}_2$ are not smooth in ε, however, they are bounded and nonzero as $\varepsilon \to 0$ and $\delta \to 0$.

(II) *The point $(G_1(\rho_1, \delta), \rho_1^2, 0)$ is the intersection of $\alpha^+(\delta)$ and Σ^+_{out}.*

(III) *The function H satisfies*

$$H(0, 0) = 0, \qquad \frac{\partial H}{\partial \mathcal{X}}(\mathcal{X}, \mathcal{Y}) \neq 0. \tag{3.10}$$

(IV) *If U_ε is sufficiently small, for each $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, we can suppose that*

$$\frac{\partial \hat{D}_1}{\partial X}(X, Y, \rho_1, \varepsilon, \delta) \neq 0, \tag{3.11}$$

by changing the value of ρ_1 if necessary.

This theorem means that an orbit of Eq. (1.8) or Eq. (3.1) running around $S_a^+(\delta)$ jumps near $L^+(\delta)$, goes to the right of $L^+(\delta)$ and the distance of the orbit and the orbit $\alpha^+(\delta)$ is of $O(\varepsilon^{4/5})$ (see Fig. 7). In particular, it converges to $\alpha^+(\delta)$ as $\varepsilon \to 0$. We use the blow-up method to prove this theorem. In Section 3.2, we introduce the blow-up coordinates and outline the strategy of the proof of Theorem 3.2. Analysis of our system in the blow-up coordinates is done after Section 3.3 and the proof is completed in Section 3.6. The constant Ω is a pole of the first Painlevé equation, as is shown in Section 3.3. The function H, which is actually an analytic function, also arises from the first Painlevé equation. To prove Theorems 1 and 2, it is sufficient to show that $\mathcal{X}$ and $\mathcal{Y}$ are exponentially small as $\varepsilon \to 0$. However, we need more precise decay rate for proving Theorem 3. For this purpose, the factors $\varepsilon^{-3/5}$ and $\varepsilon^{-2/5}$ will be derived by means of the WKB theory. Eq. (3.10) and (3.11) are also used to prove Theorem 3. Thus our analysis involves a harder calculation than a usual treatment of fold points in fast–slow systems. The assumption (C6) is used to assure that the domain U_ε of the transition map is independent of $\delta \in (0, \delta_0)$. The assumption (C5) is used to show that the argument of $\exp[\cdots]$ in Eq. (3.9) is of order $O(\delta)$. For other parts of the theorem, we need only (C1), (C2) and (C4).

3.2. Blow-up coordinates

In this subsection, we introduce the blow-up coordinates to "desingularize" the fixed point $L^+(\delta)$ having a nilpotent linear part. Regarding ε as a dependent variable on t, we rewrite Eq. (3.1) as

$$\begin{cases} \dot{X} = Z - Y^2 + c_1(\delta)XY + Zh_1(X, Y, Z, \delta) + Y^2 h_2(X, Y, Z, \delta) + \varepsilon h_3(X, Y, Z, \varepsilon, \delta), \\ \dot{Y} = -X + Zh_4(X, Y, Z, \delta) + \varepsilon h_5(X, Y, Z, \varepsilon, \delta), \\ \dot{Z} = -\varepsilon + \varepsilon h_6(X, Y, Z, \varepsilon, \delta), \\ \dot{\varepsilon} = 0, \end{cases} \tag{3.12}$$

with the estimate (3.2). For this system, we define the blow-up transformations K_1, K_2 and K_3 to be

$$(X, Y, Z, \varepsilon) = \left(r_1^3 x_1, r_1^2 y_1, r_1^4, r_1^5 \varepsilon_1\right), \tag{3.13}$$

$$(X, Y, Z, \varepsilon) = \left(r_2^3 x_2, r_2^2 y_2, r_2^4 z_2, r_2^5\right), \tag{3.14}$$

$$(X, Y, Z, \varepsilon) = \left(r_3^3 x_3, r_3^2, r_3^4 z_3, r_3^5 \varepsilon_3\right), \tag{3.15}$$

respectively, where K_1, K_2 and K_3 are defined on half spaces $\{Z \geqslant 0\}$, $\{\varepsilon \geqslant 0\}$ and $\{Y \geqslant 0\}$, respectively. In what follows, we refer to the coordinates $(x_1, y_1, r_1, \varepsilon_1)$, (x_2, y_2, z_2, r_2), $(x_3, r_3, z_3, \varepsilon_3)$ as K_1, K_2, K_3 coordinates, respectively. Transformations κ_{ij} from the K_i coordinates to the K_j coordinates are given by

$$\begin{cases} \kappa_{12}\colon\ (x_2, y_2, z_2, r_2) = \big(x_1\varepsilon_1^{-3/5}, y_1\varepsilon_1^{-2/5}, \varepsilon_1^{-4/5}, r_1\varepsilon_1^{1/5}\big), \\ \kappa_{21}\colon\ (x_1, y_1, r_1, \varepsilon_1) = \big(x_2 z_2^{-3/4}, y_2 z_2^{-1/2}, r_2 z_2^{1/4}, z_2^{-5/4}\big), \\ \kappa_{32}\colon\ (x_2, y_2, z_2, r_2) = \big(x_3\varepsilon_3^{-3/5}, \varepsilon_3^{-2/5}, z_3\varepsilon_3^{-4/5}, r_3\varepsilon_3^{1/5}\big), \\ \kappa_{23}\colon\ (x_3, r_3, z_3, \varepsilon_3) = \big(x_2 y_2^{-3/2}, r_2 y_2^{1/2}, z_2 y_2^{-2}, y_2^{-5/2}\big), \end{cases} \tag{3.16}$$

respectively. Our next task is to write out Eq. (3.12) in the K_i coordinate. Eqs. (3.13) and (3.12) are put together to provide

$$\begin{cases} \dot{x}_1 = r_1\bigg(1 - y_1^2 + c_1(\delta)r_1x_1y_1 + h_8(x_1, y_1, r_1, \delta) + y_1^2 h_9(x_1, y_1, r_1, \delta) \\ \qquad + r_1\varepsilon_1 h_{10}(x_1, y_1, r_1, \varepsilon_1, \delta) + \dfrac{3}{4}x_1\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big)\bigg), \\ \dot{y}_1 = r_1\bigg(-x_1 + r_1h_{11}(x_1, y_1, r_1, \delta) + r_1^2\varepsilon_1 h_{12}(x_1, y_1, r_1, \varepsilon_1, \delta) \\ \qquad + \dfrac{1}{2}y_1\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big)\bigg), \\ \dot{r}_1 = -\dfrac{1}{4}r_1^2\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big), \\ \dot{\varepsilon}_1 = \dfrac{5}{4}r_1\varepsilon_1^2\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big), \end{cases} \tag{3.17}$$

where h_i $(i = 7, \ldots, 12)$ are C^∞ functions such that

$$h_7(x_1, y_1, r_1, \varepsilon_1, \delta) = h_6\big(r_1^3x_1, r_1^2y_1, r_1^4, r_1^5\varepsilon_1, \delta\big), \tag{3.18}$$

and $h_8, \ldots, h_{12}$ are defined in a similar manner through $h_1, \ldots, h_5$, respectively. Thus in these functions, x_1, y_1, ε_1 are always with the factors r_1^3, r_1^2, r_1^5, respectively. This fact will be used in later calculations. Note that $h_i \sim O(r_1^2)$ for $i = 7, 8, 9, 11$ because of (3.2). By changing the time scale appropriately, we can factor out r_1 in the right-hand side of the above equations:

$$(\mathrm{K}_1)\quad \begin{cases} \dot{x}_1 = 1 - y_1^2 + c_1(\delta)r_1x_1y_1 + h_8(x_1, y_1, r_1, \delta) + y_1^2 h_9(x_1, y_1, r_1, \delta) \\ \qquad + r_1\varepsilon_1 h_{10}(x_1, y_1, r_1, \varepsilon_1, \delta) + \dfrac{3}{4}x_1\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big), \\ \dot{y}_1 = -x_1 + r_1h_{11}(x_1, y_1, r_1, \delta) + r_1^2\varepsilon_1 h_{12}(x_1, y_1, r_1, \varepsilon_1, \delta) \\ \qquad + \dfrac{1}{2}y_1\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big), \\ \dot{r}_1 = -\dfrac{1}{4}r_1\varepsilon_1\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big), \\ \dot{\varepsilon}_1 = \dfrac{5}{4}\varepsilon_1^2\big(1 - h_7(x_1, y_1, r_1, \varepsilon_1, \delta)\big). \end{cases} \tag{3.19}$$

Since the time scale transformation does not change the phase portrait of Eq. (3.17), we can use Eq. (3.19) to calculate the transition map.

In a similar manner (*i.e.* changing the coordinates and dividing by the common factors), we obtain the systems of equations written in the K_2, K_3 coordinates as

$$(\mathrm{K}_2)\quad \begin{cases} \dot{x}_2 = z_2 - y_2^2 + r_2 h_{13}(x_2, y_2, z_2, r_2, \delta), \\ \dot{y}_2 = -x_2 + r_2^2 h_{14}(x_2, y_2, z_2, r_2, \delta), \\ \dot{z}_2 = -1 + r_2^2 h_{15}(x_2, y_2, z_2, r_2, \delta), \\ \dot{r}_2 = 0, \end{cases} \tag{3.20}$$

and

$$(\mathrm{K}_3)\quad \begin{cases} \dot{x}_3 = -1 + z_3 + c_1(\delta) r_3 x_3 + \dfrac{3}{2} x_3 h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta) + r_3^2 h_{17}(x_3, r_3, z_3, \varepsilon_3, \delta), \\ \dot{r}_3 = -\dfrac{1}{2} r_3 h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta), \\ \dot{z}_3 = -\varepsilon_3 + 2 z_3 h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta) + r_3^2 \varepsilon_3 h_{18}(x_3, r_3, z_3, \varepsilon_3, \delta), \\ \dot{\varepsilon}_3 = \dfrac{5}{2} \varepsilon_3 h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta), \end{cases} \tag{3.21}$$

respectively, where $h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta) := x_3 + r_3^2 h_{19}(x_3, r_3, z_3, \varepsilon_3, \delta)$ and h_i $(i = 13, \ldots, 19)$ are C^∞ functions satisfying

$$h_{17}, h_{18}, h_{19} \sim O(x_3, r_3, z_3, \varepsilon_3).$$

Our strategy for understanding the flow of Eq. (3.1) near the fold point $L^+(\delta)$ is as follows: In Section 3.3, we analyze Eq. (3.20) in the K_2 coordinates. We will find it to be a perturbed first Painlevé equation. Since asymptotic behavior of the first Painlevé equation is well studied, we can construct a transition map along the flow of it approximately. In Section 3.4, we analyze Eq. (3.19) in the K_1 coordinates. We will see that in the K_1 coordinates, $S_a^+(\delta)$ has a 2-dimensional attracting center manifold $W^c(\delta)$ for $\delta > 0$ (see Fig. 8). Since it is attracting, orbits passing nearby $S_a^+(\delta)$ approaches $W^c(\delta)$. Thus if we construct the invariant manifold $W^c(\delta)$ globally, we can well understand asymptotic behavior of orbits passing through nearby $S_a^+(\delta)$. Although usual center manifold theory provides the center manifold $W^c(\delta)$ only locally, we will show that there exists an orbit γ, called the Boutroux's tritronquée solution, of the first Painlevé equation in the K_2 coordinates such that if it is transformed into the K_1 coordinates, it is attached on the edge of $W^c(\delta)$ (see Fig. 8). This means that the orbit γ of the first Painlevé equation guides the manifold $W^c(\delta)$ and provides a global structure of it. In Section 3.5, we analyze Eq. (3.21) in the K_3 coordinates. We will see that there exists a fixed point whose unstable manifold is 1-dimensional. Since the orbit γ of the first Painlevé equation written in the K_3 coordinates approaches the fixed point, the manifold $W^c(\delta)$ put on the γ is also attached on the unstable manifold (see Fig. 8). The unstable manifold corresponds to the heteroclinic orbit $\alpha^+(\delta)$ in the (X, Y, Z) coordinates if it is blown down. This means that orbits of Eq. (3.1) coming from a region above $L^+(\delta)$ go to the right of $L^+(\delta)$ (see Fig. 7) and pass near the heteroclinic orbit $\alpha^+(\delta)$. Thus the transition map Π_{loc}^+ is well defined. The fixed point in the K_3 coordinates corresponds to a pole of the solution γ in the K_2 coordinates. In this way, the value Ω of the pole appears in the transition map (3.8).

Combining transition maps constructed on each K_i coordinates and blowing it down to the (X, Y, Z) coordinates, we can prove Theorem 3.2.

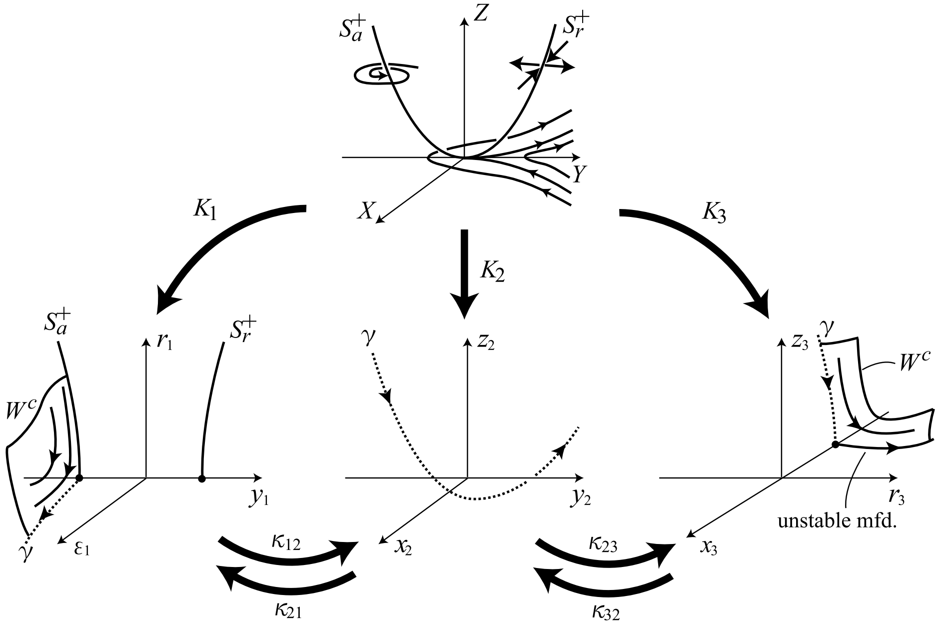

Fig. 8. The flow in the (X, Y, Z) coordinates and the blow-up coordinates. The dotted line denotes the orbit γ of the first Painlevé equation.

3.3. Analysis in the K_2 coordinates

We consider Eq. (3.20). Since $r_2 = \varepsilon^{1/5}$ is a small constant, we are allowed to take the system

$$\begin{cases} \dot{x}_2 = z_2 - y_2^2, \\ \dot{y}_2 = -x_2, \\ \dot{z}_2 = -1, \end{cases} \tag{3.22}$$

as the unperturbed system of Eq. (3.20). This is equivalent to the first Painlevé equation:

$$\begin{cases} \dfrac{dx_2}{dz_2} = -z_2 + y_2^2, \\ \dfrac{dy_2}{dz_2} = x_2, \end{cases} \quad \text{or} \quad \frac{d^2 y_2}{dz_2^2} = -z_2 + y_2^2. \tag{3.23}$$

It is known that there exists a two parameter family of solutions of the first Painlevé equation whose asymptotic expansions are given by

$$\begin{pmatrix} x_2(z_2) \\ y_2(z_2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} z_2^{-1/2} - (\frac{C_1}{8} z_2^{-9/8} - \sqrt{2} C_2 z_2^{1/8}) \cos\phi - (\frac{C_2}{8} z_2^{-9/8} + \sqrt{2} C_1 z_2^{1/8}) \sin\phi + O(z_2^{-3}) \\ -z_2^{1/2} + C_1 z_2^{-1/8} \cos\phi + C_2 z_2^{-1/8} \sin\phi + O(z_2^{-2}) \end{pmatrix}, \tag{3.24}$$

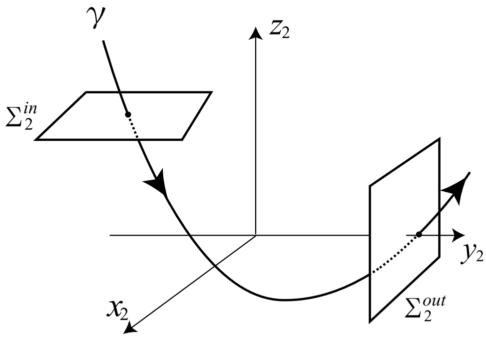

Fig. 9. The solution γ of the first Painlevé equation and the Poincaré sections.

as $z_2 \to \infty$ and

$$\begin{pmatrix} x_2(z_2) \\ y_2(z_2) \end{pmatrix} = \begin{pmatrix} \frac{-12}{(z_2-z_0)^3} + \frac{z_0}{5}(z_2-z_0) + \frac{1}{2}(z_2-z_0)^2 + 4C_3(z_2-z_0)^3 + O((z_2-z_0)^4) \\ \frac{6}{(z_2-z_0)^2} + \frac{z_0}{10}(z_2-z_0)^2 + \frac{1}{6}(z_2-z_0)^3 + C_3(z_2-z_0)^4 + O((z_2-z_0)^5) \end{pmatrix}, \tag{3.25}$$

as $z_2 \to z_0 + 0$, where $\phi \sim \frac{4\sqrt{2}}{5} z_2^{5/4}$ $(z_2 \to \infty)$, and where C_1, C_2, C_3 and z_0 are constants which depend on an initial value. The value z_0 is a movable pole of the first Painlevé equation (see Ince [17], Noonburg [26], Conte [5]). In particular, there exists a unique solution γ, which corresponds to the case $C_1 = C_2 = 0$, whose asymptotic expansions as $z_2 \to \infty$ and as $z_2 \to \Omega + 0$ are of the form

$$\gamma : \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2(z_2) \\ y_2(z_2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} z_2^{-1/2} + O(z_2^{-3}) \\ -z_2^{1/2} + O(z_2^{-2}) \end{pmatrix}, \tag{3.26}$$

and

$$\gamma : \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2(z_2) \\ y_2(z_2) \end{pmatrix} = \begin{pmatrix} \frac{-12}{(z_2-\Omega)^3} + \frac{\Omega}{5}(z_2-\Omega) + O((z_2-\Omega)^2) \\ \frac{6}{(z_2-\Omega)^2} + \frac{\Omega}{10}(z_2-\Omega)^2 + O((z_2-\Omega)^3) \end{pmatrix}, \tag{3.27}$$

respectively, where $\Omega \sim -3.416$. The γ is called the Boutroux's tritronquée solution [1,19].

Let ρ_2 and ρ_3 be small positive numbers and define Poincaré sections to be

$$\Sigma_2^{in} = \{z_2 = \rho_2^{-4/5}\}, \qquad \Sigma_2^{out} = \{y_2 = \rho_3^{-2/5}\}, \tag{3.28}$$

(see Fig. 9). By Eqs. (3.26), (3.27), the intersections $P_2 = \gamma \cap \Sigma_2^{out}$, $Q_2 = \gamma \cap \Sigma_2^{in}$ of γ and the sections are given by

$$P_2 = (p_x, p_y, p_z) = \left(-(2/3)^{1/2} \rho_3^{-3/5} + O(\rho_3^{1/5}), \rho_3^{-2/5}, \Omega + \sqrt{6}\rho_3^{1/5} + O(\rho_3)\right), \tag{3.29}$$

$$Q_2 = (q_x, q_y, q_z) = \left(-\rho_2^{2/5}/2 + O(\rho_2^{12/5}), -\rho_2^{-2/5} + O(\rho_2^{8/5}), \rho_2^{-4/5}\right), \tag{3.30}$$

respectively.

Proposition 3.3. *If ρ_2 and ρ_3 are sufficiently small positive numbers, there exists an open set $U_2 \subset \Sigma_2^{in}$ such that the transition map $\Pi_2^{loc} : U_2 \to \Sigma_2^{out}$ along the flow of Eq.* (3.20) *is well defined and expressed as*

$$\Pi_2^{loc}\begin{pmatrix} x_2 \\ y_2 \\ \rho_2^{-4/5} \\ r_2 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 0 \end{pmatrix} + \begin{pmatrix} H_1(x_2 - q_x, y_2 - q_y, \rho_2, r_2, \rho_3, \delta) \\ 0 \\ H_2(x_2 - q_x, y_2 - q_y, \rho_2, r_2, \rho_3, \delta) \\ r_2 \end{pmatrix}, \tag{3.31}$$

where $H_1(x, y, \rho_2, r, \rho_3, \delta)$ and $H_2(x, y, \rho_2, r, \rho_3, \delta)$ are C^∞ functions with respect to x, y, r and δ satisfying the equalities $H_1(0, 0, \rho_2, 0, \rho_3, \delta) = H_2(0, 0, \rho_2, 0, \rho_3, \delta) = 0$ for any small $\rho_2, \rho_3 > 0$ and $\delta \in [0, \delta_0)$.

Proof. This is an immediate consequence of the differentiability of solutions with respect to initial values x_2, y_2 and parameters r_2, δ. Note that at this time, we did not prove differentiability at $\rho_3 = 0$, which will be proved in the next lemma. □

Since H_1 and H_2 are C^∞ with respect to r and δ, we put them in the form

$$H_i(x, y, \rho_2, r, \rho_3, \delta) = \tilde{H}_i(x, y, \rho_2, \rho_3) + O(r), \quad i = 1, 2, \tag{3.32}$$

where we use the fact that when $r_2 = 0$, the system (3.20) is independent of δ. Then, the value $\lim_{\rho_3 \to 0}(p_z + \tilde{H}_2(x - q_x, y - q_y, \rho_2, \rho_3))$ gives a pole z_0 of a solution of Eq. (3.23) through an initial point $(x, y, \rho_2^{-4/5})$; that is, $x_2(z_2), y_2(z_2) \to \infty$ as $z_2 \to z_0$. Proposition 3.3 implies that $\tilde{H}_i$ are C^∞ in x and y when $\rho_3 > 0$. Now we show that $\tilde{H}_i$ can be expanded in $\rho_3^{1/5}$ and they are C^∞ even if $\rho_3 = 0$. This means that a position of a pole is also smooth with respect to initial values. In the proof, the Painlevé property will play a crucial role. Part (ii) of the next lemma is used to prove Theorem 3.2(III).

Lemma 3.4.

(i) *The functions $\tilde{H}_1$ and $\tilde{H}_2$ are analytic with respect to $(x, y) \in U_2$, $\rho_2^{1/5} > 0$ and $\rho_3^{1/5} \geqslant 0$, though they are singular at $\rho_2^{1/5} = 0$.*
(ii) $\tilde{H}_2(0, 0, \rho_2, 0) = 0$, $\frac{\partial}{\partial x}\tilde{H}_2(x, y, \rho_2, 0) \neq 0$.

Proof. Let $x_2 = x_2(z_2; \rho_2, x_0, y_0)$ and $y_2 = y_2(z_2; \rho_2, x_0, y_0)$ be a solution of the system (3.23) with the initial condition

$$x_2\big(\rho_2^{-4/5}; \rho_2, x_0, y_0\big) = x_0, \qquad y_2\big(\rho_2^{-4/5}; \rho_2, x_0, y_0\big) = y_0.$$

Suppose that $y_2(z) = \rho_3^{-2/5}$ for some $z = z(x_0, y_0, \rho_2, \rho_3)$. When $\rho_3 > 0$, the statement (i) immediately follows from the fundamental theorem of ODEs: Since the right-hand side of the system (3.23) is analytic, any solution is analytic in time z_2, initial time $\rho_2^{-4/5}$ and initial values (x_0, y_0). Applying the implicit function theorem to the equality

$$y_2\big(z(x_0, y_0, \rho_2, \rho_3); \rho_2, x_0, y_0\big) = \rho_3^{-2/5}, \tag{3.33}$$

one can verify that

$$z(x_0, y_0, \rho_2, \rho_3) = p_z + \tilde{H}_2(x_0 - q_x, y_0 - q_y, \rho_2, \rho_3) \tag{3.34}$$

is analytic in $x_0, y_0, \rho_2^{1/5} > 0$ and $\rho_3^{1/5} > 0$. Thus

$$x_2\big(z(x_0, y_0, \rho_2, \rho_3); \rho_2, x_0, y_0\big) = p_x + \tilde{H}_1(x_0 - q_x, y_0 - q_y, \rho_2, \rho_3) \tag{3.35}$$

is also analytic in the same region. Since $z \to \infty$ as $\rho_2 \to 0$, $\tilde{H}_1$ and $\tilde{H}_2$ are singular at $\rho_2^{1/5} = 0$.

When $\rho_3 = 0$, $z(x_0, y_0, \rho_2, 0)$ gives a pole and $x_2 = y_2 = \infty$ at $z_2 = z(x_0, y_0, \rho_2, 0)$. Thus we should change the coordinates so that a pole becomes a regular point. For (3.23), change the dependent variables (x_2, y_2) and the independent variable z_2 to (ξ, η) and τ by the relation

$$\begin{cases} x_2 = \dfrac{2\kappa^2}{\eta^3} + \dfrac{\kappa^2\tau}{2}\eta + \dfrac{\kappa^2}{2}\eta^2 - \kappa^2\eta^3\xi, \\ y_2 = -\dfrac{\kappa^3}{\eta^2}, \end{cases} \tag{3.36}$$

and $z_2 = \kappa\tau$, respectively, where $\kappa := (-6)^{1/5} < 0$. Then, (3.23) is brought into the analytic system

$$\begin{cases} \dfrac{d\eta}{d\tau} = 1 + \dfrac{\tau}{4}\eta^4 + \dfrac{1}{4}\eta^5 - \dfrac{1}{2}\eta^6\xi, \\ \dfrac{d\xi}{d\tau} = \dfrac{1}{8}\tau^2\eta + \dfrac{3}{8}\tau\eta^2 - \left(\tau\xi - \dfrac{1}{4}\right)\eta^3 - \dfrac{5}{4}\eta^4\xi + \dfrac{3}{2}\eta^5\xi^2. \end{cases} \tag{3.37}$$

Since any pole of $y_2(z_2)$ is second order [17], a pole of y_2 is transformed into a zero of $\eta(\tau)$ of first order. Let $\eta = \eta(\tau; s, \eta_0, \xi_0)$ and $\xi = \xi(\tau; s, \eta_0, \xi_0)$ be a solution of the system satisfying the initial condition

$$\eta(s; s, \eta_0, \xi_0) = \eta_0, \qquad \xi(s; s, \eta_0, \xi_0) = \xi_0,$$

where (η_0, ξ_0) and the initial time s correspond to (x_0, y_0) and $\rho_2^{-4/5}$, respectively, by the transformation (3.36). Suppose that

$$\eta\big(\hat{\tau}(s, \eta_0, \xi_0, \rho_3); s, \eta_0, \xi_0\big) = \big(-\kappa^3\big)^{1/2}\rho_3^{1/5}$$

for some $\tau = \hat{\tau}(s, \eta_0, \xi_0, \rho_3)$, which corresponds to a value of $z(x_0, y_0, \rho_2, \rho_3)$ by the relation $z = \kappa\tau$ so that $y_2(z) = \rho_3^{-2/5}$ (note that when $y_2 = \rho_3^{-2/5}$, then $\eta = (-\kappa^3)^{1/2}\rho_3^{1/5}$). Since

$$\left.\frac{\partial\eta}{\partial\tau}\right|_{\eta=(-\kappa^3)^{1/2}\rho_3^{1/5}} = 1 + O\big(\rho_3^{4/5}\big),$$

the implicit function theorem proves that $\hat{\tau}$ is analytic in s, η_0, ξ_0 and small $\rho_3^{1/5} \geqslant 0$. Since the transformation $(\eta_0, \xi_0) \mapsto (x_0, y_0)$ defined through (3.36) is analytic when $y_0 \neq 0$, it turns out that $z(x_0, y_0, \rho_2, \rho_3)$ is analytic in $(x_0, y_0) \in U_2$, $\rho_2^{1/5} > 0$ and $\rho_3^{1/5} \geqslant 0$. Now Eqs. (3.34), (3.35) prove the part (i) of lemma.

To prove (ii), let us calculate the asymptotic expansion of $\hat{\tau}(s, \eta_0, \xi_0, 0)$, at which $\eta = 0$. We rewrite (3.37) as

$$\begin{cases} \dfrac{d\tau}{d\eta} = \dfrac{1}{1 + \frac{\tau}{4}\eta^4 + \frac{1}{4}\eta^5 - \frac{1}{2}\eta^6\xi}, \\ \dfrac{d\xi}{d\eta} = \dfrac{\frac{1}{8}\tau^2\eta + \frac{3}{8}\tau\eta^2 - (\tau\xi - \frac{1}{4})\eta^3 - \frac{5}{4}\eta^4\xi + \frac{3}{2}\eta^5\xi^2}{1 + \frac{\tau}{4}\eta^4 + \frac{1}{4}\eta^5 - \frac{1}{2}\eta^6\xi}. \end{cases} \tag{3.38}$$

A general solution of this system is obtained in a power series of η as

$$\begin{cases} \tau = \tau_1 + \eta - \dfrac{\tau_1}{20}\eta^5 - \dfrac{1}{12}\eta^6 + \dfrac{\xi_1}{14}\eta^7 + O\big(\eta^8\big), \\ \xi = \xi_1 + \dfrac{\tau_1^2}{16}\eta^2 + \dfrac{5\tau_1}{24}\eta^3 + O(\eta^4). \end{cases} \tag{3.39}$$

where τ_1 and ξ_1 are constants to be determined from an initial condition. By using the initial condition $(\tau, \eta, \xi) = (s, \eta_0, \xi_0)$, τ_1 is determined as

$$\tau_1 = s - \eta_0 + \frac{s}{20}\eta_0^5 + \frac{1}{30}\eta_0^6 - \frac{\xi_0}{14}\eta_0^7 + O\big(\eta_0^8\big). \tag{3.40}$$

When $\eta = 0$, $\tau = \tau_1$. This means that the above τ_1 gives the expansion of $\hat{\tau}(s, \eta_0, \xi_0, 0)$. Then we obtain

$$\begin{aligned} \frac{\partial \tilde{H}_2}{\partial x_0}(x_0 - q_x, y_0 - q_y, \rho_2, 0) &= \frac{\partial z}{\partial x_0}(x_0, y_0, \rho_2, 0) \\ &= \kappa \frac{\partial \hat{\tau}}{\partial x_0}(s, \eta_0, \xi_0, 0) \\ &= \kappa \frac{\partial \hat{\tau}}{\partial \eta_0}\frac{\partial \eta_0}{\partial x_0} + \kappa \frac{\partial \hat{\tau}}{\partial \xi_0}\frac{\partial \xi_0}{\partial x_0} \\ &= \kappa\left(-\frac{1}{14}\eta_0^7 + O\big(\eta_0^8\big)\right) \cdot -\frac{1}{\kappa^2\eta_0^3}, \end{aligned}$$

which is not zero for small η_0 (thus for large y_0). The equality $\tilde{H}_2(0, 0, \rho_2, 0) = 0$ is obvious from the definition. □

Remark. Since $\tilde{H}_i$ is analytic in $\rho_3^{1/5} \geqslant 0$, it is expanded as

$$\tilde{H}_i(x, y, \rho_2, \rho_3) = \hat{H}_i(x, y, \rho_2) + O\big(\rho_3^{1/5}\big), \tag{3.41}$$

for $i = 1, 2$. Indeed, one can verify that

$$\tilde{H}_i(x, y, \rho_2, \rho_3) = \tilde{H}_i(x, y, \rho_2, 0) + \sqrt{6}\rho_3^{1/5} + \frac{3\sqrt{6}}{10}\big(\tilde{H}_i(x, y, \rho_2, 0) + p_z\big)\rho_3 + 3\rho_3^{6/5} + O\big(\rho_3^{7/5}\big)$$

by using the expansion (3.25). Further, $\tilde{H}_i$ are expanded in a Laurent series of $\rho_2^{1/5}$. In particular, Eq. (3.40) show that the expansions are of the form

$$\hat{H}_i(x, y, \rho_2) = \hat{\hat{H}}_i(x, y) + \rho_2^{-4/5}F_i\big(x, y, \rho_2^{-4/5}\big), \tag{3.42}$$

because $s = \rho_2^{-4/5}/\kappa$, where F_1, F_2 are analytic functions. The proof of the above lemma is based on the fact that a pole of (3.23) can be transformed into a zero of the analytic system by the analytic transformation. This property is common to Painlevé equations, and the transformation (3.36) is used to prove that (3.23) has the Painlevé property [5,17].

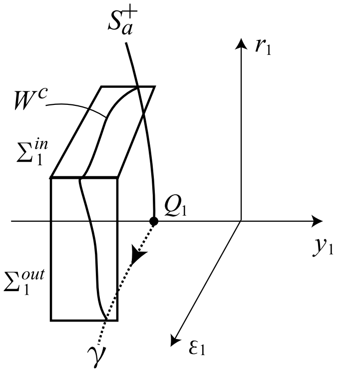

Fig. 10. Poincaré sections to define the transition map Π_1^{loc}.

3.4. Analysis in the K_1 coordinates

We turn to Eq. (3.19). It is easy to verify that Eq. (3.19) has fixed points $(x_1, y_1, r_1, \varepsilon_1) = (0, \pm 1, 0, 0)$. By virtue of the implicit function theorem, we can show that there exist two sets of fixed points which form two curves emerging from $(0, \pm 1, 0, 0)$, and they correspond to $S_a^+(\delta)$ and $S_r^+(\delta)$, respectively (see Fig. 8). On the fixed points, the Jacobian matrix of the right-hand side of Eq. (3.19) has eigenvalues given by

$$0,\ 0,\ \frac{1}{2}\Big(c_1(\delta)r_1y_1 + O\big(r_1^3\big) \pm \sqrt{8y_1 - 4c_1(\delta)r_1x_1 + O\big(r_1^2\big)}\Big). \tag{3.43}$$

In particular, the eigenvalues become $0, 0, \pm\sqrt{2}i$ at the fixed point $Q_1 = (0, -1, 0, 0)$, but at fixed points in $S_a^+(\delta)\backslash Q_1$, they have two eigenvalues whose real parts are negative if r_1 is small and $\delta > 0$. Eigenvectors associated with the two zero eigenvalues at points on $S_a^+(\delta)\backslash Q_1$ converge to those at Q_1, which are given by $(0, 0, 1, 0)$ and $(-1, 0, 0, 2)$, as $r_1 \to 0$. The vector $(0, 0, 1, 0)$ is tangent to $S_a^+(\delta)$. Thus $(-1, 0, 0, 2)$ is a nontrivial center direction.

Lemma 3.5. *If $\delta > 0$, there exists an attracting 2-dimensional center manifold $W^c(\delta)$ which includes $S_a^+(\delta)$ and the orbit γ of the first Painlevé equation written in the K_1 coordinates (see Fig. 10).*

Proof. Let $B(a)$ be the open ball of radius a centered at Q_1. Since at points in $S_a^+(\delta)\backslash B(a)$ the Jacobian matrix has two zero eigenvalues and the other two eigenvalues with negative real parts, there exists an attracting 2-dimensional center manifold $W^c(\delta, a)$ emerging from $S_a^+(\delta)\backslash B(a)$ for any small $a > 0$. Let γ be the solution of the first Painlevé equation described in the previous subsection. Its asymptotic expansion (3.26) is written in the K_1 coordinates as

$$\gamma : \begin{pmatrix} x_1 \\ y_1 \\ r_1 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z_2^{-5/4} + O(z_2^{-15/4}) \\ -1 + O(z_2^{-5/2}) \\ 0 \\ z_2^{-5/4} \end{pmatrix} \quad (\text{as } z_2 \to \infty), \tag{3.44}$$

by the coordinate change κ_{21} (3.16). The curve (3.44) approaches the point Q_1 as $z_2 \to \infty$ and its tangent vector converges to the eigenvector $(-1, 0, 0, 2)$ at Q_1 as $z_2 \to \infty$. Thus $W^c(\delta) := \lim_{a\to 0} W^c(\delta, a) \cup \gamma$ forms an invariant manifold. □

Note that γ is included in the subspace $\{r_1 = 0\}$. This lemma means that the orbit γ guides global behavior of the center manifold $W^c(\delta)$.

Let $\rho_1, \rho_2 > 0$ be the small constants referred to in Theorem 3.2 and Proposition 3.3, respectively. Take two Poincaré sections Σ_1^{in} and Σ_1^{out} defined to be

$$\begin{aligned}\Sigma_1^{in} &= \big\{(x_1, y_1, r_1, \varepsilon_1) \,\big|\, r_1 = \rho_1,\ |x_1| \leqslant \rho_1,\ |y_1 + 1| \leqslant \rho_1,\ 0 < \varepsilon_1 \leqslant \rho_2\big\}, \\ \Sigma_1^{out} &= \big\{(x_1, y_1, r_1, \varepsilon_1) \,\big|\, 0 \leqslant r_1 \leqslant \rho_1,\ |x_1| \leqslant \rho_1,\ |y_1 + 1| \leqslant \rho_1,\ \varepsilon_1 = \rho_2\big\},\end{aligned} \tag{3.45}$$

respectively. Note that Σ_1^{in} is included in the section Σ_{in}^{+} (see Eq. (3.7)) if written in the (X, Y, Z) coordinates and Σ_1^{out} in the section Σ_2^{in} (see Eq. (3.28)) if written in the K_2 coordinates.

Proposition 3.6. *Suppose* (C1), (C2) *and* (C4) *to* (C6).

(I) *If ρ_1 and ρ_2 are sufficiently small, the transition map $\Pi_1^{loc} : \Sigma_1^{in} \to \Sigma_1^{out}$ along the flow of Eq.* (3.19) *is well defined for every $\delta \in (0, \delta_0)$ and expressed as*

$$\Pi_1^{loc}\begin{pmatrix} x_1 \\ y_1 \\ \rho_1 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} \varphi_1(\rho_1 \varepsilon_1^{1/5} \rho_2^{-1/5}, \rho_2, \delta) \\ \varphi_2(\rho_1 \varepsilon_1^{1/5} \rho_2^{-1/5}, \rho_2, \delta) \\ \rho_1 \varepsilon_1^{1/5} \rho_2^{-1/5} \\ \rho_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ Y_1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.46}$$

where φ_1 and φ_2 are C^∞ functions such that the graph of $x_1 = \varphi_1(r_1, \varepsilon_1, \delta)$ and $y_1 = \varphi_2(r_1, \varepsilon_1, \delta)$ gives the center manifold $W^c(\delta)$. The second term denotes the deviation from $W^c(\delta)$, and X_1 and Y_1 are defined to be

$$\begin{cases} X_1 = D_1(x_1, y_1, \rho_1, \varepsilon_1, \rho_2, \delta)\left(\dfrac{\rho_2}{\varepsilon_1}\right)^{3/5} \exp\left[-d(\rho_1, \varepsilon_1, \rho_2, \delta)\dfrac{\delta}{\varepsilon_1}\right], \\ Y_1 = D_2(x_1, y_1, \rho_1, \varepsilon_1, \rho_2, \delta)\left(\dfrac{\rho_2}{\varepsilon_1}\right)^{2/5} \exp\left[-d(\rho_1, \varepsilon_1, \rho_2, \delta)\dfrac{\delta}{\varepsilon_1}\right], \end{cases} \tag{3.47}$$

where D_1, D_2 and d are C^∞ functions with respect to x_1, y_1, ρ_1 and δ. Although D_1, D_2 and d are not C^∞ in ε_1 and ρ_2, they are bounded and nonzero as $\varepsilon_1 \to 0$ and $\delta \to 0$. Further, they admit the expansions of the form

$$D_i(x_1, y_1, \rho_1, \varepsilon_1, \rho_2, \delta) = \hat{D}_i(x_1, y_1, \rho_1, \varepsilon_1, \delta) + O\big((\varepsilon_1/\rho_2)^{1/5}\big), \tag{3.48}$$

$$d(\rho_1, \varepsilon_1, \rho_2, \delta) = \hat{d}(\rho_1, \delta) + O\big((\varepsilon_1/\rho_2)^{1/5}\big), \tag{3.49}$$

for $i = 1, 2$.

(II) *The first term in the right-hand side of Eq.* (3.46) *is on the intersection of Σ_1^{out} and the center manifold $W^c(\delta)$. In particular, as $\varepsilon_1 \to 0$, $\Pi_1^{loc}(x_1, y_1, \rho_1, \varepsilon_1)$ converges to the intersection point of Σ_1^{out} and γ.*

(III) *If the initial point $(x_1, y_1, \rho_1, \varepsilon_1)$ is sufficiently close to $W^c(\delta)$,*

$$\frac{\partial \hat{D}_1}{\partial x_1}(x_1, y_1, \rho_1, \varepsilon_1, \delta) \neq 0 \tag{3.50}$$

except for a countable set of values of ε_1.

Remark. To prove the existence of a periodic orbit, it is sufficient to show that X_1 and Y_1 are exponentially small as $\varepsilon_1 \to 0$. However, to prove the existence of chaos, we need more precise estimate as the factors $(\rho_2/\varepsilon_1)^{3/5}$ and $(\rho_2/\varepsilon_1)^{2/5}$. Eq. (3.50) is used to prove Eq. (3.11).

Proof. At first, we divide the right-hand side of Eq. (3.19) by $1-h_7$ and change the time scale accordingly. Note that this does not change the phase portrait. Then we obtain

$$\begin{cases} \dot{x}_1 = 1 - y_1^2 + c_1(\delta) r_1 x_1 y_1 + \dfrac{3}{4} x_1 \varepsilon_1 + h_8 + y_1^2 h_9 + r_1 \varepsilon_1 h_{10} \\ \qquad + \big(1 - y_1^2 + c_1(\delta) r_1 x_1 y_1 + h_8 + y_1^2 h_9 + r_1 \varepsilon_1 h_{10}\big) h_{21}, \\ \dot{y}_1 = -x_1 + \dfrac{1}{2} y_1 \varepsilon_1 + r_1 h_{11} + r_1^2 \varepsilon_1 h_{12} + \big(-x_1 + r_1 h_{11} + r_1^2 \varepsilon_1 h_{12}\big) h_{21}, \\ \dot{r}_1 = -\dfrac{1}{4} r_1 \varepsilon_1, \\ \dot{\varepsilon}_1 = \dfrac{5}{4} \varepsilon_1^2, \end{cases} \tag{3.51}$$

where $h_{21} = \sum_{k=1}^{\infty} h_7^k$, and arguments of functions are omitted. Equations for r_1 and ε_1 are solved as

$$r_1(t) = r_1(0) \left(\frac{4 - 5\varepsilon_1(0) t}{4} \right)^{1/5}, \qquad \varepsilon_1(t) = \frac{4\varepsilon_1(0)}{4 - 5\varepsilon_1(0) t}, \tag{3.52}$$

respectively. Let T be a transition time from Σ_1^{in} to Σ_1^{out}. Since $\varepsilon_1(T) = \rho_2$, T is given by

$$T = \frac{4}{5\varepsilon_1(0)} \left(1 - \frac{\varepsilon_1(0)}{\rho_2} \right). \tag{3.53}$$

To estimate $x_1(T)$ and $y_1(T)$, let us introduce the new time variable τ by

$$\tau = \left(\frac{4 - 5\varepsilon_1(0) t}{4} \right)^{1/5}. \tag{3.54}$$

Then, $r_1(t) = r_1(0)\tau$, $\varepsilon_1(t) = \varepsilon_1(0)\tau^{-5}$. Note that when $t=0$, $\tau = 1$ and when $t = T$, one has $\tau = (\varepsilon_1(0)/\rho_2)^{1/5}$.

Claim 1. Any solutions (x_1, y_1) of (3.51) are of the form $x_1 = \tau^{-3} u_1(\tau)$, $y_1 = \tau^{-2} u_2(\tau)$, where u_1 and u_2 are C^∞ with respect to τ.

Proof. Changing the time t to τ, the system (3.51) is rewritten as

$$\begin{cases} -\dfrac{1}{4}\varepsilon_1(0)\tau^{-4} \dfrac{dx_1}{d\tau} = 1 - y_1^2 + c_1(\delta) r_1(0)\tau x_1 y_1 + \dfrac{3}{4} x_1 \varepsilon_1(0)\tau^{-5} + h_8 + y_1^2 h_9 + r_1(0)\varepsilon_1(0)\tau^{-4} h_{10} \\ \qquad + \big(1 - y_1^2 + c_1(\delta) r_1(0)\tau x_1 y_1 + h_8 + y_1^2 h_9 + r_1(0)\varepsilon_1(0)\tau^{-4} h_{10}\big) h_{21}, \\ -\dfrac{1}{4}\varepsilon_1(0)\tau^{-4} \dfrac{dy_1}{d\tau} = -x_1 + \dfrac{1}{2}\varepsilon_1(0)\tau^{-5} y_1 + r_1(0)\tau h_{11} + r_1(0)^2 \varepsilon_1(0)\tau^{-3} h_{12} \\ \qquad + \big(-x_1 + r_1(0)\tau h_{11} + r_1(0)^2 \varepsilon_1(0)\tau^{-3} h_{12}\big) h_{21}. \end{cases}$$

Putting $x_1 = \tau^{-3} u_1$, $y_1 = \tau^{-2} u_2$ yields

$$\begin{cases} -\dfrac{1}{4}\varepsilon_1(0) \dfrac{du_1}{d\tau} = \big(\tau^7 - \tau^3 u_2^2 + c_1(\delta) r_1(0)\tau^3 u_1 u_2 + \tau^7 h_8 + \tau^3 u_2^2 h_9 \\ \qquad + r_1(0)\varepsilon_1(0)\tau^3 h_{10}\big)(1 + h_{21}), \\ -\dfrac{1}{4}\varepsilon_1(0) \dfrac{du_2}{d\tau} = \big(-\tau^3 u_1 + r_1(0)\tau^7 h_{11} + r_1(0)^2 \varepsilon_1(0)\tau^3 h_{12}\big)(1 + h_{21}). \end{cases} \tag{3.55}$$

Recall that h_7 is defined through (3.18), and thus

$$h_7(x_1, y_1, r_1, \varepsilon_1, \delta) = h_6\big(r_1(0)^3 u_1, r_1(0)^2 u_2, r_1(0)^4 \tau^4, r_1(0)^5 \varepsilon_1(0), \delta\big), \tag{3.56}$$

which implies that h_7 is C^∞ with respect to u_1, u_2, $r_1(0)$, $\varepsilon_1(0)$, δ and τ. Functions $h_8, \dots, h_{12}$ and h_{21} have the same property. Hence the right-hand side of Eq. (3.55) is C^∞ with respect to u_1, u_2, $r_1(0)$, δ and τ, which proves that solutions $u_1(\tau)$ and $u_2(\tau)$ are C^∞ with respect to $r_1(0)$, δ and τ. □

Next thing to do is to derive the center manifold and how $x_1(t)$ and $y_1(t)$ approach to it. The local center manifold $W^c(\delta)$ is given as a graph of C^∞ functions $x_1 = \varphi_1(r_1, \varepsilon_1, \delta)$, $y_1 = \varphi_2(r_1, \varepsilon_1, \delta)$. By using the standard center manifold theory, we can calculate φ_1 and φ_2 as

$$\varphi_1(r_1, \varepsilon_1, \delta) = -\frac{1}{2}\varepsilon_1 + O\big(r_1^2, r_1\varepsilon_1, \varepsilon_1^2\big), \qquad \varphi_2(r_1, \varepsilon_1, \delta) = -1 + O\big(r_1^2, r_1\varepsilon_1, \varepsilon_1^2\big). \tag{3.57}$$

To see the behavior of solutions x_1 and y_1 near the center manifold $W^c(\delta)$, we put x_1 and y_1 in the form

$$x_1(\tau) = \varphi_1\big(r_1(\tau), \varepsilon_1(\tau), \delta\big) + \tau^{-3} v_1(\tau), \qquad y_1(\tau) = \varphi_2\big(r_1(\tau), \varepsilon_1(\tau), \delta\big) + \tau^{-2} v_2(\tau). \tag{3.58}$$

Since $\tau^3 x_1(\tau)$ and $\tau^2 y_1(\tau)$ are C^∞ in τ for every solutions x_1 and y_1, so are solutions $\tau^3\varphi_1(r_1(\tau), \varepsilon_1(\tau), \delta)$ and $\tau^2\varphi_2(r_1(\tau), \varepsilon_1(\tau), \delta)$ on the center manifold multiplied by τ^3 and τ^2, respectively. This implies that $v_1(\tau)$ and $v_2(\tau)$ are also C^∞ in τ. Substituting Eq. (3.58) into (3.51) and expanding it in v_1, v_2 and $\varepsilon_1(0)$, we obtain the system of the form

$$\begin{cases} \varepsilon_1 \dfrac{dv_1}{d\tau} = -8\tau^5 v_2 + 4c_1 r_1 \tau^5 v_1 + r_1^3 \tau^7 h_{22}(r_1, \tau, \delta) v_1 + r_1^2 \tau^7 h_{23}(r_1, \tau, \delta) v_2 \\ \qquad\qquad + g_1(v_1, v_2, r_1, \varepsilon_1, \delta, \tau), \\ \varepsilon_1 \dfrac{dv_2}{d\tau} = 4\tau^3 v_1 + r_1^4 \tau^7 h_{24}(r_1, \tau, \delta) v_1 + r_1^3 \tau^7 h_{25}(r_1, \tau, \delta) v_2 + g_2(v_1, v_2, r_1, \varepsilon_1, \delta, \tau), \end{cases} \tag{3.59}$$

where $g_1, g_2 \sim O(v_1^2, v_1 v_2, v_2^2, \varepsilon_1)$ denote higher order terms, $h_{22}, \dots, h_{25}$ are C^∞ functions, and where $r_1(0), \varepsilon_1(0)$ and $c_1(\delta)$ are denoted by r_1, ε_1 and c_1, respectively. This is a singular perturbed problem with respect to ε_1.

Claim 2. Any nonzero solutions of this system are expressed as

$$\begin{aligned} v_1 &= D_1^*(\tau, r_1, \varepsilon_1, \delta; v_{10}, v_{20}) \exp\left[-\frac{d^*(\tau, r_1, \delta)}{\varepsilon_1}\right], \\ v_2 &= D_2^*(\tau, r_1, \varepsilon_1, \delta; v_{10}, v_{20}) \exp\left[-\frac{d^*(\tau, r_1, \delta)}{\varepsilon_1}\right], \end{aligned} \tag{3.60}$$

where $v_{10} = v_1(1)$ and $v_{20} = v_2(1)$ are initial values, and where D_1^*, D_2^* and d^* are C^∞ in τ, r_1, v_{10}, v_{20} and δ. Although D_1^* and D_2^* are not C^∞ in ε_1, they are bounded and nonzero as $\varepsilon_1 \to 0$, $\delta \to 0$. If v_{10}, v_{20}, r_1 and τ are sufficiently small,

$$\frac{\partial D_1^*}{\partial v_{10}}(\tau, r_1, \varepsilon_1, \delta; v_{10}, v_{20}) \neq 0, \tag{3.61}$$

except for a countable set of values of ε_1.

Proof. At first, we consider the linearized system of (3.59) as

$$\begin{cases} \varepsilon_1 \dfrac{dv_1}{d\tau} = -8\tau^5 v_2 + 4c_1 r_1 \tau^5 v_1 + r_1^3 \tau^7 h_{22}(r_1, \tau, \delta) v_1 + r_1^2 \tau^7 h_{23}(r_1, \tau, \delta) v_2 + O(\varepsilon_1), \\ \varepsilon_1 \dfrac{dv_2}{d\tau} = 4\tau^3 v_1 + r_1^4 \tau^7 h_{24}(r_1, \tau, \delta) v_1 + r_1^3 \tau^7 h_{25}(r_1, \tau, \delta) v_2 + O(\varepsilon_1), \end{cases} \tag{3.62}$$

which yields the equation of v_1 as

$$\varepsilon_1^2 \frac{d^2 v_1}{d\tau^2} - \varepsilon_1 \big(4c_1 r_1 \tau^5 + 4r_1^3 \tau^7 h_{26} + O(\varepsilon_1)\big) \frac{dv_1}{d\tau} + \big(32\tau^8 - 4r_1^2 \tau^7 h_{27} + O(\varepsilon_1)\big) v_1 = 0, \tag{3.63}$$

where $h_{26}(r_1, \tau, \delta)$ and $h_{27}(r_1, \tau, \delta)$ are C^∞ functions. According to the WKB theory, we construct a solution of this equation in the form

$$v_1(\tau) = \exp\left[\frac{1}{\varepsilon_1} \sum_{n=0}^{\infty} \varepsilon_1^n S_n(\tau)\right].$$

Substituting this into Eq. (3.63), we obtain the equation of $S_0(\tau)$,

$$\left(\frac{dS_0}{d\tau}\right)^2 - \big(4c_1 r_1 \tau^5 + 4r_1^3 \tau^7 h_{26}\big) \frac{dS_0}{d\tau} + 32\tau^8 - 4r_1^2 \tau^7 h_{27} = 0,$$

which is solved as $S_0 = S_0^{\pm}(\tau) = V(\tau) \pm iW(\tau)$, where

$$V(\tau) = \int_1^\tau \big(2c_1 r_1 s^5 + 2r_1^3 s^7 h_{26}\big)\, ds,$$

$$W(\tau) = \int_1^\tau \big(2c_1 r_1 s^5 + 2r_1^3 s^7 h_{26}\big) \sqrt{\frac{8s^8 - r_1^2 s^7 h_{27}}{(c_1 r_1 s^5 + r_1^3 s^7 h_{26})^2} - 1}\, ds,$$

are real-valued functions for small r_1. If $r_1 > 0$ is sufficiently small and if $c_1(\delta) > 0$, $0 < \tau < 1$, then $V(\tau) < 0$. For these $S_0^+(\tau)$ and $S_0^-(\tau)$, $S_1^{\pm}(\tau)$, $S_2^{\pm}(\tau)$, $\ldots$ are uniquely determined by induction, respectively. Thus a general solution $v_1(\tau)$ is of the form

$$\begin{aligned} v_1(\tau) = {}& k^+ \exp\big[V(\tau)/\varepsilon_1\big] \exp\big[iW(\tau)/\varepsilon_1\big] \exp\big[S_1^+ + \varepsilon_1 S_2^+ + \cdots\big] \\ &+ k^- \exp\big[V(\tau)/\varepsilon_1\big] \exp\big[-iW(\tau)/\varepsilon_1\big] \exp\big[S_1^- + \varepsilon_1 S_2^- + \cdots\big], \end{aligned}$$

where $k^+, k^- \in \mathbf{C}$ are arbitrary constants. Put

$$D_{1+}^* = \exp\big[iW(\tau)/\varepsilon_1\big] \exp\big[S_1^+ + \varepsilon_1 S_2^+ + \cdots\big], \qquad D_{1-}^* = \exp\big[-iW(\tau)/\varepsilon_1\big] \exp\big[S_1^- + \varepsilon_1 S_2^- + \cdots\big].$$

Then, v_1 is rewritten as

$$v_1(\tau) = k^+ \exp\big[V(\tau)/\varepsilon_1\big] D_{1+}^* + k^- \exp\big[V(\tau)/\varepsilon_1\big] D_{1-}^*,$$

where D^*_{1+} and D^*_{1-} are C^∞ in v_{10}, v_{20}, τ, r_1 and δ. They are not C^∞ in ε_1 because of the factor $1/\varepsilon_1$, however, they are bounded and nonzero as $\varepsilon_1 \to 0$. In a similar manner, it turns out that v_2 is expressed as

$$v_2(\tau) = k^+ \exp\bigl[V(\tau)/\varepsilon_1\bigr]D^*_{2+} + k^- \exp\bigl[V(\tau)/\varepsilon_1\bigr]D^*_{2-},$$

where D^*_{2+} and D^*_{2-} are C^∞ in $v_{10}, v_{20}, \tau, r_1, \delta$, and are bounded and nonzero as $\varepsilon_1 \to 0$. Therefore, the fundamental matrix of the linear system (3.62) is given as

$$F(\tau) = \begin{pmatrix} D^*_{1+} & D^*_{1-} \\ D^*_{2+} & D^*_{2-} \end{pmatrix} \exp\bigl[V(\tau)/\varepsilon_1\bigr]. \tag{3.64}$$

Now we come back to the nonlinear system (3.59). We rewrite it in the abstract form as

$$\varepsilon_1 \frac{d\boldsymbol{v}}{d\tau} = A(\tau)\boldsymbol{v} + g(\boldsymbol{v}, \tau),$$

where $\boldsymbol{v} = (v_1, v_2)$, $g = (g_1, g_2)$, and $A(\tau)$ is a matrix defining the linear part of the system. To estimate the nonlinear terms, the variation-of-constants formula is applied. Put $\boldsymbol{v} = F(\tau)\boldsymbol{c}(\tau)$ with $\boldsymbol{c}(\tau) = (c_1(\tau), c_2(\tau)) \in \mathbf{C}^2$. Then, $\boldsymbol{c}(\tau)$ satisfies the equation

$$\frac{d\boldsymbol{c}}{d\tau} = \frac{1}{\varepsilon_1} F(\tau)^{-1} g\bigl(F(\tau)\boldsymbol{c}, \tau\bigr). \tag{3.65}$$

Let $\boldsymbol{c} = \boldsymbol{c}(\tau, \varepsilon_1)$ be a solution of this equation. Since $F(\tau) \sim O(e^{V(\tau)/\varepsilon_1})$ tends to zero exponentially as $\varepsilon_1 \to 0$ and since g is nonlinear, the time-dependent vector field defined by the right-hand side of (3.65) tends to zero as $\varepsilon_1 \to 0$. Since solutions $\boldsymbol{c}(\tau, \varepsilon_1)$ are continuous with respect to the parameter ε_1, it turns out that $\boldsymbol{c}(\tau, \varepsilon_1)$ tends to a constant as $\varepsilon_1 \to 0$, which is not zero except for the trivial solution $\boldsymbol{c}(\tau, \varepsilon_1) \equiv 0$. This proves Eq. (3.60) with the desired properties by putting $d^* = -V(\tau)$ and $D^*_i = D^*_{i+}c_1 + D^*_{i-}c_2$ $(i = 1, 2)$. Note that since the right-hand side of (3.59) is not zero at $\delta = 0$, $D^*_1 \not\equiv 0$, $D^*_2 \not\equiv 0$ as $\delta \to 0$.

When $r_1 = v_{10} = v_{20} = 0$, the derivatives $\partial v_i/\partial v_{10}$, $(i = 1, 2)$ with respect to the initial value v_{10} satisfy the initial value problem

$$\begin{cases} \varepsilon_1 \dfrac{d}{d\tau}\dfrac{\partial v_1}{\partial v_{10}}(\tau, 0, \varepsilon_1, \delta; 0; 0) = -8\tau^5 \dfrac{\partial v_2}{\partial v_{10}}(\tau, 0, \varepsilon_1, \delta; 0; 0), & \dfrac{\partial v_1}{\partial v_{10}}(1, 0, \varepsilon_1, \delta; 0; 0) = 1, \\ \varepsilon_1 \dfrac{d}{d\tau}\dfrac{\partial v_2}{\partial v_{10}}(\tau, 0, \varepsilon_1, \delta; 0; 0) = 4\tau^3 \dfrac{\partial v_1}{\partial v_{10}}(\tau, 0, \varepsilon_1, \delta; 0; 0), & \dfrac{\partial v_2}{\partial v_{10}}(1, 0, \varepsilon_1, \delta; 0; 0) = 0. \end{cases} \tag{3.66}$$

This is exactly solved as

$$\frac{\partial v_1}{\partial v_{10}}(\tau, 0, \varepsilon_1, \delta; 0; 0) = \cos\left(\frac{4\sqrt{2}}{5\varepsilon_1}\bigl(\tau^5 - 1\bigr)\right). \tag{3.67}$$

In particular,

$$\frac{\partial v_1}{\partial v_{10}}(0, 0, \varepsilon_1, \delta; 0; 0) = \cos\left(\frac{4\sqrt{2}}{5\varepsilon_1}\right) \tag{3.68}$$

is not zero except for a countable set of values of ε_1. This and the continuity of solutions of ODE prove Eq. (3.61). □

Let us proceed the proof of Proposition 3.6. For $v_1(\tau)$ and $v_2(\tau)$ in (3.60), $x_1(\tau)$ and $y_1(\tau)$ are given as (3.58). Since $\tau = (\varepsilon_1(0)/\rho_2)^{1/5}$ when $t = T$, we obtain

$$\begin{aligned} x_1(T) &= \varphi_1\big(r_1(0)\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, \rho_2, \delta\big) \\ &\quad + \left(\frac{\rho_2}{\varepsilon_1(0)}\right)^{3/5} D_1^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \varepsilon_1(0), \delta; v_{10}, v_{20}\big) \\ &\quad \times \exp\left[-\frac{d^*((\varepsilon_1(0)/\rho_2)^{1/5}, r_1(0), \delta)}{\varepsilon_1(0)}\right], \\ y_1(T) &= \varphi_2\big(r_1(0)\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, \rho_2, \delta\big) \\ &\quad + \left(\frac{\rho_2}{\varepsilon_1(0)}\right)^{2/5} D_2^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \varepsilon_1(0), \delta; v_{10}, v_{20}\big) \\ &\quad \times \exp\left[-\frac{d^*((\varepsilon_1(0)/\rho_2)^{1/5}, r_1(0), \delta)}{\varepsilon_1(0)}\right]. \end{aligned}$$

Put

$$D_i^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \varepsilon_1(0), \delta; v_{10}, v_{20}\big) = D_i\big(x_1(0), y_1(0), r_1(0), \varepsilon_1(0), \rho_2, \delta\big)$$

for $i = 1, 2$. Since D_i^* is C^∞ in $v_{10}, v_{20}, r_1(0)$ and δ, D_i is also C^∞ in $x_1(0), y_1(0), r_1(0)$ and δ. Since D_i^* is C^∞ in τ, D_i is bounded and nonzero as $\varepsilon_1(0) \to 0$. Finally, let us calculate

$$d^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \delta\big) = \int_{(\varepsilon_1(0)/\rho_2)^{1/5}}^{1} \big(2c_1(\delta)r_1(0)\tau^5 + 2r_1(0)^3\tau^7 h_{26}\big(r_1(0), \tau, \delta\big)\big)\, d\tau.$$

Due to the mean value theorem, there exists a number $\tau^* > 0$ such that

$$\begin{aligned} d^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \delta\big) &= \frac{1}{3}c_1(\delta)r_1(0)\left(1 - \left(\frac{\varepsilon_1(0)}{\rho_2}\right)^{6/5}\right) \\ &\quad + h_{26}\big(r_1(0), \tau^*, \delta\big)\frac{r_1(0)^3}{4}\left(1 - \left(\frac{\varepsilon_1(0)}{\rho_2}\right)^{8/5}\right). \end{aligned}$$

By the assumption (C5), an orbit of (3.51) near the center manifold $W^c(\delta)$ approaches to $W^c(\delta)$ with the rate $O(e^{-\delta\mu^+ t})$. By the assumption (C6), such an attraction region (basin) of $W^c(\delta)$ exists uniformly in $\delta > 0$ at least near the branch $S_a^+(\delta)$. Thus $h_{26}(r_1(0), \tau^*, \delta)$ is of order $O(\delta)$ as well as $c_1(\delta)$ if $\rho_2 > 0$ is sufficiently small. Therefore, there exists a function d, which is C^∞ with respect to $r_1(0)$ and δ, such that

$$d^*\big(\big(\varepsilon_1(0)/\rho_2\big)^{1/5}, r_1(0), \delta\big) = d\big(r_1(0), \varepsilon_1(0), \rho_2, \delta\big) \cdot \delta.$$

Since $\mu^+(z, 0) \neq 0$, $d(r_1(0), \varepsilon_1(0), \rho_2, 0) \neq 0$. Since D_i^* and d^* are C^∞ in $\tau = (\varepsilon_1/\rho_2)^{1/5}$, they admit the expansions (3.48, 3.49). This proves (I) of Proposition 3.6. Proposition 3.6 (II) is clear from the definition of φ_1, φ_2, and (III) follows from Eq. (3.61). □

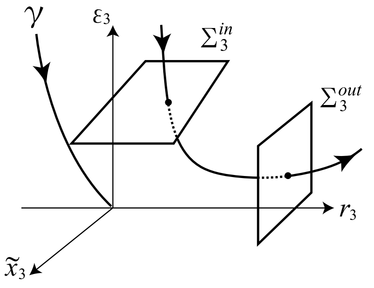

Fig. 11. Poincaré sections to define the transition map Π_3^{loc}.

3.5. Analysis in the K_3 coordinates

We come to the system (3.21). This system has the fixed point $(x_3, r_3, z_3, \varepsilon_3) = (-\sqrt{2/3}, 0, 0, 0)$ (see Fig. 8). To analyze the system, we divide the right-hand side of Eq. (3.21) by $-h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta)$ and change the time scale accordingly. Note that this does not change the phase portrait. At first, note that the equality

$$\frac{1}{h_{16}(x_3, r_3, z_3, \varepsilon_3, \delta)} = -\sqrt{\frac{3}{2}}\left(1 + \sqrt{\frac{3}{2}}\left(x_3 + \sqrt{\frac{2}{3}}\right) + h_{31}(x_3 + \sqrt{2/3}, r_3, z_3, \varepsilon_3, \delta)\right) \quad (3.69)$$

holds, where $h_{31} \sim O_p(2)$ is a C^∞ function. Using Eq. (3.69) and introducing the new coordinate by $x_3 + \sqrt{2/3} = \tilde{x}_3$, we eventually obtain

$$\begin{cases} \dot{\tilde{x}}_3 = -3\tilde{x}_3 + \sqrt{\frac{3}{2}} z_3 - c_1(\delta) r_3 + h_{32}(\tilde{x}_3, r_3, z_3, \varepsilon_3, \delta), \\ \dot{r}_3 = \frac{1}{2} r_3, \\ \dot{z}_3 = -2z_3 - \sqrt{\frac{3}{2}}\varepsilon_3 + \varepsilon_3 h_{33}(\tilde{x}_3, z_3, \varepsilon_3, \delta) + \varepsilon_3 r_3 h_{34}(\tilde{x}_3, r_3, z_3, \varepsilon_3, \delta), \\ \dot{\varepsilon}_3 = -\frac{5}{2}\varepsilon_3, \end{cases} \quad (3.70)$$

where $h_{32} \sim O_p(2)$ and $h_{33}, h_{34} \sim O_p(1)$ are C^∞ functions. Note that h_{33} is independent of r_3. This system has a fixed point at the origin, and eigenvalues of the Jacobian matrix at the origin of the right-hand side of Eq. (3.70) are given by $-3, 1/2, -2, -5/2$. In particular, the eigenvector associated with the positive eigenvalue $1/2$ is given by $(-2c_1(\delta)/7, 1, 0, 0)$ and the origin has a 1-dimensional unstable manifold which is tangent to the eigenvector. The asymptotic expansion (3.27) of the solution γ of the first Painlevé equation is rewritten in the present coordinates as

$$(\tilde{x}_3, r_3, z_3, \varepsilon_3) = \left(O\left((z_2 - \Omega)^4\right), 0, O\left((z_2 - \Omega)^4\right), O\left((z_2 - \Omega)^5\right)\right), \quad (3.71)$$

which converges to the origin as $z_2 \to \Omega$ (see Fig. 11).

Let ρ_1 and ρ_3 be the small constants introduced in Section 3.1 and Section 3.3, respectively. Define Poincaré sections Σ_3^{in} and Σ_3^{out} to be

$$\Sigma_3^{in} = \left\{(\tilde{x}_3, r_3, z_3, \varepsilon_3) \mid |\tilde{x}_3| < \rho_1,\ 0 < r_3 \leqslant \rho_1,\ |z_3| \leqslant \rho_1,\ \varepsilon_3 = \rho_3\right\}, \quad (3.72)$$

$$\Sigma_3^{out} = \left\{(\tilde{x}_3, r_3, z_3, \varepsilon_3) \mid |\tilde{x}_3| < \rho_1,\ r_3 = \rho_1,\ |z_3| \leqslant \rho_1,\ 0 < \varepsilon_3 \leqslant \rho_3\right\}, \quad (3.73)$$

respectively (see Fig. 11). Note that Σ_3^{in} is included in the section Σ_2^{out} (see Eq. (3.28)) if written in the K_2 coordinates and Σ_3^{out} in the section Σ_{out}^{+} (see Eq. (3.7)) if written in the (X, Y, Z) coordinates.

Proposition 3.7. (I) *If ρ_1 and ρ_3 are sufficiently small, the transition map $\Pi_3^{loc} : \Sigma_3^{in} \to \Sigma_3^{out}$ along the flow of Eq.* (3.70) *is well defined and expressed as*

$$\Pi_3^{loc}\begin{pmatrix} \tilde{x}_3 \\ r_3 \\ z_3 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} \beta_1(\rho_1, \delta) + r_3^4\beta_2(\tilde{x}_3, r_3, z_3, \rho_3, \rho_1, \delta) \\ \rho_1 \\ (z_3 - \sqrt{6}\rho_3 + \rho_3\beta_3(\tilde{x}_3, z_3, \rho_3, \delta))(\frac{r_3}{\rho_1})^4 + r_3^5 \cdot \log r_3 \cdot \beta_4(\tilde{x}_3, r_3, z_3, \rho_3, \rho_1, \delta) \\ \rho_3(\frac{r_3}{\rho_1})^5 \end{pmatrix}, \tag{3.74}$$

where β_1 and β_3 are C^∞ in their arguments, β_2 and β_4 are C^∞ with respect to $\tilde{x}_3$, z_3, ρ_3 and δ with the property that β_2 and β_4 are bounded as $r_3 \to 0$.

(II) *As $r_3 \to 0$, $\Pi_3^{loc}(\tilde{x}_3, r_3, z_3, \rho_3)$ converges to the intersection point $(\beta(\rho_1, \delta), \rho_1, 0, 0)$ of Σ_3^{out} and the unstable manifold of the origin.*

Before proving Proposition 3.7, we need to derive the normal form of Eq. (3.70).

Lemma 3.8. *In the vicinity of the origin, there exists a C^∞ coordinate transformation*

$$\begin{pmatrix} \tilde{x}_3 \\ r_3 \\ z_3 \\ \varepsilon_3 \end{pmatrix} = \Phi(X_3, r_3, Z_3, \varepsilon_3, \delta) := \begin{pmatrix} X_3 + \psi_1(X_3, Z_3, \varepsilon_3, \delta) \\ r_3 \\ Z_3 + \varepsilon_3\psi_2(X_3, Z_3, \varepsilon_3, \delta) \\ \varepsilon_3 \end{pmatrix} \tag{3.75}$$

such that Eq. (3.70) *is transformed into*

$$\begin{cases} \dot{X}_3 = -3X_3 + \sqrt{\dfrac{3}{2}}Z_3 - c_1(\delta)r_3 + r_3h_{35}(X_3, r_3, Z_3, \varepsilon_3, \delta), \\ \dot{r}_3 = \dfrac{1}{2}r_3, \\ \dot{Z}_3 = -2Z_3 - \sqrt{\dfrac{3}{2}}\varepsilon_3 + \varepsilon_3 r_3 h_{36}(X_3, r_3, Z_3, \varepsilon_3, \delta), \\ \dot{\varepsilon}_3 = -\dfrac{5}{2}\varepsilon_3, \end{cases} \tag{3.76}$$

where $\psi_2, h_{35}, h_{36} \sim O_p(1)$ and $\psi_1 \sim O_p(2)$ are C^∞ functions.

Proof of Lemma 3.8. When $r_3 = 0$, Eq. (3.70) is written as

$$\begin{cases} \dot{\tilde{x}}_3 = -3\tilde{x}_3 + \sqrt{\dfrac{3}{2}}z_3 + h_{32}(\tilde{x}_3, 0, z_3, \varepsilon_3, \delta), \\ \dot{z}_3 = -2z_3 - \sqrt{\dfrac{3}{2}}\varepsilon_3 + \varepsilon_3 h_{33}(\tilde{x}_3, z_3, \varepsilon_3, \delta), \\ \dot{\varepsilon}_3 = -\dfrac{5}{2}\varepsilon_3. \end{cases} \tag{3.77}$$

Since eigenvalues of the Jacobian matrix at the origin of the right-hand side of the above are $-3, -2, -5/2$ and satisfy the non-resonance condition, there exists a C^∞ transformation of the form

$(\tilde{x}_3, z_3, \varepsilon_3) \mapsto (X_3 + \psi_1(X_3, Z_3, \varepsilon_3, \delta), Z_3 + \tilde{\psi}_2(X_3, Z_3, \varepsilon_3, \delta), \varepsilon_3)$ such that Eq. (3.77) is linearized (see Chow, Li and Wang [4]). The $\tilde{\psi}_2$ is of the form $\tilde{\psi}_2 = \varepsilon_3\psi_2$, where ψ_2 is a C^∞ function, because if $\varepsilon_3 = 0$, Eq. (3.77) gives $\dot{z}_3 = -2z_3$ and it follows that $Z_3 = z_3$ when $\varepsilon_3 = 0$. This transformation brings Eq. (3.70) into Eq. (3.76). □

Proof of Proposition 3.7. Note that even in the new coordinates $(X_3, r_3, Z_3, \varepsilon_3)$, the sections Σ_3^{in} and Σ_3^{out} are included in the hyperplanes $\{\varepsilon_3 = \rho_3\}$ and $\{r_3 = \rho_1\}$, respectively.

Let us calculate the transition time T from Σ_3^{in} to Σ_3^{out}. Since $r_3(t) = r_3(0)e^{t/2}$ and $\varepsilon_3(t) = \varepsilon_3(0)e^{-5t/2}$ from Eq. (3.76), T is given by

$$T = \log\left(\frac{\rho_1}{r_3(0)}\right)^2. \tag{3.78}$$

By integrating the third equation of Eq. (3.76), $Z_3(t)$ is calculated as

$$Z_3(t) = Z_3(0)e^{-2t} + \sqrt{6}\rho_3\left(e^{-5t/2} - e^{-2t}\right) + e^{-2t}\int_0^t \rho_3 r_3(0)h_{36}\left(\boldsymbol{X}_3(s), \delta\right) ds, \tag{3.79}$$

where $\boldsymbol{X}_3(s) = (X_3(s), r_3(s), Z_3(s), \varepsilon_3(s))$. Owing to the mean value theorem, there exists $0 \leqslant \tau = \tau(t) \leqslant t$ such that Eq. (3.79) is rewritten as

$$Z_3(t) = \left(Z_3(0) - \sqrt{6}\rho_3\right)e^{-2t} + \sqrt{6}\rho_3 e^{-5t/2} + \rho_3 r_3(0)e^{-2t}h_{36}\left(\boldsymbol{X}_3(\tau), \delta\right)t. \tag{3.80}$$

This and Eq. (3.78) are put together to yield

$$Z_3(T) = \left(Z_3(0) - \sqrt{6}\rho_3\right)\left(\frac{r_3(0)}{\rho_1}\right)^4 + \sqrt{6}\rho_3\left(\frac{r_3(0)}{\rho_1}\right)^5 + \rho_3\frac{r_3(0)^5}{\rho_1^4}h_{36}\left(\boldsymbol{X}_3\left(\tau(T)\right), \delta\right)T. \tag{3.81}$$

Next, let us estimate $X_3(T)$. Since (X_3, r_3)-plane is invariant, the unstable manifold of the origin is included in this plane and given as a graph of the C^∞ function

$$X_3 = \phi(r_3, \delta) = -\frac{2}{7}c_1(\delta)r_3 + O\left(r_3^2\right). \tag{3.82}$$

To measure the distance between $X_3(t)$ and the unstable manifold, put $X_3 = \phi(r_3, \delta) + u$. Then, the first equation of (3.76) is rewritten as

$$\dot{u} = \left(-3 + h_{37}(u, r_3, Z_3, \varepsilon_3, \delta)\right)u + Z_3 h_{38}(u, r_3, Z_3, \varepsilon_3, \delta) + \varepsilon_3 h_{39}(u, r_3, Z_3, \varepsilon_3, \delta),$$

where $h_{37} \sim O_p(1)$ and h_{38}, h_{39} are C^∞ functions. This is integrated as

$$u(t) = e^{-3t}E(t)\left(u(0) + \int_0^t e^{3s}E(s)^{-1}\left(Z_3(s)h_{38}\left(\boldsymbol{u}(s), \delta\right) + \varepsilon_3(s)h_{39}\left(\boldsymbol{u}(s), \delta\right)\right) ds\right), \tag{3.83}$$

where $\boldsymbol{u}(s) = (u(s), r_3(s), Z_3(s), \varepsilon_3(s))$ and $E(t) = \exp[\int_0^t h_{37}(\boldsymbol{u}(s), \delta)\, ds]$. Substituting Eq. (3.80) and $\varepsilon_3(t) = \rho_3 e^{-5t/2}$ and estimating with the aid of the mean value theorem, one can verify that $u(T)$ is of the form

$$u(T) = r_3(0)^4 h_{40}\big(X_3(0), r_3(0), Z_3(0), \rho_3, \rho_1, \delta\big), \tag{3.84}$$

where h_{40} is bounded as $r_3(0) \to 0$ (the factor $Z_3(s)$ in Eq. (3.83) yields the factor $r_3(0)^4$, and other terms are of $O(r_3(0)^5 \log r_3(0))$). Since the transition time T is not C^∞ in ρ_1 and $r_3(0)$, h_{40} is C^∞ only in $X_3(0)$, $Z_3(0)$, ρ_3 and δ. Thus the transition map $\tilde{\Pi}_3^{loc}$ from Σ_3^{in} to Σ_3^{out} along the flow of Eq. (3.76) is given by

$$\tilde{\Pi}_3^{loc}\begin{pmatrix} X_3 \\ r_3 \\ Z_3 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} \phi(\rho_1,\delta) + r_3^4 h_{40}(X_3, r_3, Z_3, \rho_3, \rho_1, \delta) \\ \rho_1 \\ (Z_3 - \sqrt{6}\rho_3)(\frac{r_3}{\rho_1})^4 + \sqrt{6}\rho_3(\frac{r_3}{\rho_1})^5 - 2\rho_3 \frac{r_3^5}{\rho_1^4}\log(\frac{r_3}{\rho_1}) h_{41}(X_3, r_3, Z_3, \rho_3, \rho_1, \delta) \\ \rho_3(\frac{r_3}{\rho_1})^5 \end{pmatrix}, \tag{3.85}$$

where $h_{41}(X_3, r_3, Z_3, \rho_3, \rho_1, \delta) = h_{36}(\boldsymbol{X}_3(\tau(T)), \delta)$ is bounded as $r_3 \to 0$ because $\boldsymbol{X}_3(\tau(T))$ is bounded. Since the transition time T is not C^∞ in ρ_1 and $r_3(0)$, h_{41} is C^∞ in $X_3(0)$, $Z_3(0)$, ρ_3 and δ. Now Eq. (3.74) is verified by calculating $\Phi \circ \tilde{\Pi}_3^{loc} \circ \Phi^{-1}$. Note that β_3 in Eq. (3.74) is independent of r_3 and ρ_1 because it comes from the inverse of the transformation (3.75), which is of the form

$$\Phi^{-1}(\tilde{x}_3, r_3, z_3, \rho_3) = \begin{pmatrix} \tilde{x}_3 + \beta_5(\tilde{x}_3, z_3, \rho_3, \delta) \\ r_3 \\ z_3 + \rho_3\beta_3(\tilde{x}_3, z_3, \rho_3, \delta) \\ \rho_3 \end{pmatrix}$$

with C^∞ functions β_3 and β_5. The unstable manifold $\beta_1(\rho_1, \delta)$ in $(\tilde{x}_3, r_3, z_3, \varepsilon_3)$ coordinate is obtained from that in $(X_3, r_3, Z_3, \varepsilon_3)$ coordinate as $\beta_1(\rho_1, \delta) = \phi(\rho_1, \delta) + \psi_1(\phi(\rho_1, \delta), 0, 0, \delta)$. This proves Proposition 3.7(I). To prove (II) of Proposition 3.7, note that the hyperplane $\{r_3 = 0\}$ is invariant and included in the stable manifold of the origin. Since a point $(\tilde{x}_3, r_3, z_3, \rho_3)$ converges to the stable manifold as $r_3 \to 0$, $\Pi_3^{loc}(\tilde{x}_3, r_3, z_3, \rho_3)$ converges to the unstable manifold as $r_3 \to 0$ on account of the λ-lemma. This proves Proposition 3.7(II). □

3.6. Proof of Theorem 3.2

We are now in a position to prove Theorem 3.2. Let $\tau_x : (x, r, z, \varepsilon) \mapsto (x - \sqrt{2/3}, r, z, \varepsilon)$ be the translation in the x direction introduced in Section 3.5. Eq. (3.8) is obtained by writing out the map $\tilde{\Pi}_{loc}^+ := \tau_x \circ \Pi_3^{loc} \circ \tau_x^{-1} \circ \kappa_{23} \circ \Pi_2^{loc} \circ \kappa_{12} \circ \Pi_1^{loc}$ and blowing it down to the (X, Y, Z) coordinates. At first, $\Pi_2^{loc} \circ \kappa_{12} \circ \Pi_1^{loc}$ is calculated as

$$\begin{pmatrix} x_1 \\ y_1 \\ \rho_1 \\ \varepsilon_1 \end{pmatrix} \overset{\Pi_1^{loc}}{\longmapsto} \begin{pmatrix} \varphi_1 + X_1 \\ \varphi_2 + Y_1 \\ \rho_1\varepsilon_1^{1/5}\rho_2^{-1/5} \\ \rho_2 \end{pmatrix} \overset{\kappa_{12}}{\longmapsto} \begin{pmatrix} \rho_2^{-3/5}\varphi_1 + \rho_2^{-3/5}X_1 \\ \rho_2^{-2/5}\varphi_2 + \rho_2^{-2/5}Y_1 \\ \rho_2^{-4/5} \\ \rho_1\varepsilon_1^{1/5} \end{pmatrix}$$

$$\overset{\Pi_2^{loc}}{\longmapsto} \begin{pmatrix} p_x + H_1(\rho_2^{-3/5}\varphi_1 + \rho_2^{-3/5}X_1 - q_x, \rho_2^{-2/5}\varphi_2 + \rho_2^{-2/5}Y_1 - q_y, \rho_2, \rho_1\varepsilon_1^{1/5}, \rho_3, \delta) \\ \rho_3^{-2/5} \\ p_z + H_2(\rho_2^{-3/5}\varphi_1 + \rho_2^{-3/5}X_1 - q_x, \rho_2^{-2/5}\varphi_2 + \rho_2^{-2/5}Y_1 - q_y, \rho_2, \rho_1\varepsilon_1^{1/5}, \rho_3, \delta) \\ \rho_1\varepsilon_1^{1/5} \end{pmatrix}, \tag{3.86}$$

where $\varphi_1 = \varphi_1(\rho_1\varepsilon_1^{1/5}\rho_2^{-1/5}, \rho_2, \delta)$, $\varphi_2 = \varphi_2(\rho_1\varepsilon_1^{1/5}\rho_2^{-1/5}, \rho_2, \delta)$, and X_1, Y_1 are defined by Eq. (3.47). In what follows, we omit the arguments of H_1 and H_2. The last term in the above is further mapped to

$$\overset{\kappa_{23}}{\longmapsto} \begin{pmatrix} \rho_3^{3/5}p_x + \rho_3^{3/5}H_1 \\ \rho_1\rho_3^{-1/5}\varepsilon_1^{1/5} \\ \rho_3^{4/5}p_z + \rho_3^{4/5}H_2 \\ \rho_3 \end{pmatrix} \overset{\tau_x^{-1}}{\longmapsto} \begin{pmatrix} \sqrt{2/3} + \rho_3^{3/5}p_x + \rho_3^{3/5}H_1 \\ \rho_1\rho_3^{-1/5}\varepsilon_1^{1/5} \\ \rho_3^{4/5}p_z + \rho_3^{4/5}H_2 \\ \rho_3 \end{pmatrix} := \begin{pmatrix} \tilde{x}_3 \\ r_3 \\ z_3 \\ \rho_3 \end{pmatrix}. \tag{3.87}$$

Let us denote the resultant as $(\tilde{x}_3, r_3, z_3, \rho_3)$ as above. Then, $\tilde{\Pi}_{loc}^+$ proves to be given by

$$\begin{aligned}\tilde{\Pi}_{loc}^+\begin{pmatrix} x_1 \\ y_1 \\ \rho_1 \\ \varepsilon_1 \end{pmatrix} &= \begin{pmatrix} -\sqrt{2/3} + \beta_1(\rho_1, \delta) + r_3^4\beta_2(\tilde{x}_3, r_3, z_3, \rho_3, \rho_1, \delta) \\ \rho_1 \\ (z_3 - \sqrt{6}\rho_3 + \rho_3\beta_3(\tilde{x}_3, z_3, \rho_3, \delta))(\frac{r_3}{\rho_1})^4 + r_3^5 \cdot \log r_3 \cdot \beta_4(\tilde{x}_3, r_3, z_3, \rho_3, \rho_1, \delta) \\ \rho_3(\frac{r_3}{\rho_1})^5 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2/3} + \beta_1(\rho_1, \delta) + \rho_1^4\rho_3^{-4/5}\varepsilon_1^{4/5}\beta_2(\tilde{x}_3, \rho_1\rho_3^{-1/5}\varepsilon_1^{1/5}, z_3, \rho_3, \rho_1, \delta) \\ \rho_1 \\ (z_3 - \sqrt{6}\rho_3 + \rho_3\beta_3(\tilde{x}_3, z_3, \rho_3, \delta))(\frac{\varepsilon_1}{\rho_3})^{4/5} + O(\varepsilon_1 \log \varepsilon_1) \\ \varepsilon_1 \end{pmatrix}. \end{aligned} \tag{3.88}$$

By using the definition of p_z in (3.29), the third component of the above is calculated as

$$\begin{aligned} &\big(z_3 - \sqrt{6}\rho_3 + \rho_3\beta_3(\tilde{x}_3, z_3, \rho_3, \delta)\big)\left(\frac{\varepsilon_1}{\rho_3}\right)^{4/5} + O(\varepsilon_1 \log \varepsilon_1) \\ &\quad = \big(\Omega + O(\rho_3) + H_2\big(\hat{X}, \hat{Y}, \rho_2, \rho_1\varepsilon_1^{1/5}, \rho_3, \delta\big) + \rho_3^{1/5}\beta_3(\tilde{x}_3, z_3, \rho_3, \delta)\big)\varepsilon_1^{4/5} \\ &\qquad + O(\varepsilon_1 \log \varepsilon_1), \end{aligned} \tag{3.89}$$

where

$$\hat{X} = \rho_2^{-3/5}\varphi_1 + \rho_2^{-3/5}X_1 - q_x, \qquad \hat{Y} = \rho_2^{-2/5}\varphi_2 + \rho_2^{-2/5}Y_1 - q_y.$$

From Eqs. (3.32) and (3.41), Eq. (3.89) is rewritten as

$$\big(\Omega + O(\rho_3) + \hat{H}_2(\hat{X}, \hat{Y}, \rho_2) + O\big(\rho_3^{1/5}\big) + \rho_3^{1/5}\beta_3(\tilde{x}_3, z_3, \rho_3, \delta)\big)\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1). \tag{3.90}$$

Since $\tilde{\Pi}_{loc}^+(x_1, y_1, \rho_1, \varepsilon_1)$ is independent of ρ_3, which is introduced to define the intermediate sections Σ_2^{out} and Σ_3^{in}, all terms including ρ_3 are canceled out and Eq. (3.90) has to be of the form

$$\big(\Omega + \hat{H}_2(\hat{X}, \hat{Y}, \rho_2)\big)\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1). \tag{3.91}$$

Now we look into $\hat{X}$ and $\hat{Y}$. Since $\varphi_1(r_1, \varepsilon_1, \delta)$ and $\varphi_2(r_1, \varepsilon_1, \delta)$ give the graph of the center manifold $W^c(\delta)$ and since the orbit γ of the first Painlevé equation is attached on the edge of $W^c(\delta)$,

$x_1 = \varphi_1(0, \varepsilon_1, \delta)$ and $y_1 = \varphi_2(0, \varepsilon_1, \delta)$ coincide with γ written in the K_1 coordinates. Thus we obtain

$$\begin{aligned}
\hat{X} &= \rho_2^{-3/5}\varphi_1\big(\rho_1\varepsilon_1^{1/5}\rho_2^{-1/5}, \rho_2, \delta\big) + \rho_2^{-3/5}X_1 - q_x \\
&= \big(\rho_2^{-3/5}\varphi_1(0, \rho_2, \delta) - q_x\big) + \rho_2^{-3/5}X_1 + \rho_2^{-3/5}O\big((\varepsilon_1/\rho_2)^{1/5}\big) \\
&= \rho_2^{-3/5}X_1 + \rho_2^{-3/5}O\big((\varepsilon_1/\rho_2)^{1/5}\big) \\
&= \rho_2^{-3/5}D_1\left(\frac{\rho_2}{\varepsilon_1}\right)^{3/5}\exp\left[-\frac{d\delta}{\varepsilon_1}\right] + \rho_2^{-3/5}O\big((\varepsilon_1/\rho_2)^{1/5}\big) \\
&= D_1\varepsilon_1^{-3/5}\exp\left[-\frac{d\delta}{\varepsilon_1}\right] + \rho_2^{-3/5}O\big((\varepsilon_1/\rho_2)^{1/5}\big). \qquad (3.92)
\end{aligned}$$

The $\hat{Y}$ is calculated in the same manner. Functions D_1, D_2 and d are expanded as Eqs. (3.48), (3.49), and $\hat{H}_2$ is expanded as (3.42). Since Eq. (3.91) should be independent of ρ_2, which is introduced to define the intermediate sections Σ_1^{out} and Σ_2^{in}, Eq. (3.91) is rewritten as

$$\big(\Omega + \hat{\hat{H}}_2(\mathcal{X}, \mathcal{Y})\big)\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1). \qquad (3.93)$$

where

$$\begin{aligned}
\mathcal{X} &= \hat{D}_1(x_1, y_1, \rho_1, \varepsilon_1, \delta)\varepsilon_1^{-3/5}\exp\left[-\frac{\hat{d}(\rho_1, \delta)\delta}{\varepsilon_1}\right], \\
\mathcal{Y} &= \hat{D}_2(x_1, y_1, \rho_1, \varepsilon_1, \delta)\varepsilon_1^{-2/5}\exp\left[-\frac{\hat{d}(\rho_1, \delta)\delta}{\varepsilon_1}\right]. \qquad (3.94)
\end{aligned}$$

Similarly, since the first component of Eq. (3.88) is independent of ρ_2 and ρ_3, we find that it is expressed as

$$-\sqrt{2/3} + \beta_1(\rho_1, \delta) + \hat{G}(\mathcal{X}, \mathcal{Y}, \rho_1, \delta)\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1) \qquad (3.95)$$

with some C^∞ function $\hat{G}$.

Our final task is to blow down Eq. (3.88) with Eqs. (3.93), (3.95) to the (X, Y, Z) coordinates to obtain Eq. (3.8). By the transformation (3.13), a point $(X, Y, \rho_1^4, \varepsilon)$ in (X, Y, Z, ε)-space is mapped to the point $(X\rho_1^{-3}, Y\rho_1^{-2}, \rho_1, \varepsilon\rho_1^{-5})$ in K_1-space. Further, it is mapped by the transition map $\tilde{\Pi}_{loc}^{+}$ to

$$\begin{pmatrix}
-\sqrt{2/3} + \beta_1(\rho_1, \delta) + \hat{G}(\mathcal{X}, \mathcal{Y}, \rho_1, \delta)\varepsilon^{4/5}\rho_1^{-4} + O(\varepsilon \log \varepsilon) \\
\rho_1 \\
(\Omega + \hat{\hat{H}}_2(\mathcal{X}, \mathcal{Y}))\varepsilon^{4/5}\rho_1^{-4} + O(\varepsilon \log \varepsilon) \\
\varepsilon\rho_1^{-5}
\end{pmatrix},$$

in K_3-space, in which

$$\begin{aligned}
\mathcal{X} &= \hat{D}_1\big(X\rho_1^{-3}, Y\rho_1^{-2}, \rho_1, \varepsilon\rho_1^{-5}, \delta\big)\varepsilon^{-3/5}\rho_1^3\exp\left[-\frac{\hat{d}(\rho_1, \delta)\delta}{\varepsilon\rho_1^{-5}}\right], \\
\mathcal{Y} &= \hat{D}_2\big(X\rho_1^{-3}, Y\rho_1^{-2}, \rho_1, \varepsilon\rho_1^{-5}, \delta\big)\varepsilon^{-2/5}\rho_1^2\exp\left[-\frac{\hat{d}(\rho_1, \delta)\delta}{\varepsilon\rho_1^{-5}}\right].
\end{aligned}$$

Finally, it is blown down by (3.15) as

$$\begin{pmatrix} -\sqrt{2/3}\rho_1^3 + \beta_1(\rho_1,\delta)\rho_1^3 + \rho_1^{-1}\hat{G}(\mathcal{X},\mathcal{Y},\rho_1,\delta)\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1) \\ \rho_1^2 \\ (\Omega + \hat{\hat{H}}_2(\mathcal{X},\mathcal{Y}))\varepsilon_1^{4/5} + O(\varepsilon_1 \log \varepsilon_1) \\ \varepsilon \end{pmatrix}.$$

By changing the definitions of $\hat{D}_1, \hat{D}_2$ and $\hat{d}$ appropriately, we obtain Theorem 3.2 (I) with

$$G_1 = -\sqrt{2/3}\rho_1^3 + \beta_1(\rho_1,\delta)\rho_1^3, \qquad G_2 = \rho_1^{-1}\hat{G}, \qquad H = \hat{\hat{H}}_2.$$

Theorem 3.2(II) follows from the fact that the unstable manifold described in Proposition 3.7(II) coincides with the heteroclinic orbit $\alpha^+(\delta)$ if written in the (X, Y, Z) coordinates. Theorem 3.2(III) follows from Lemma 3.4, and (IV) follows from Eq. (3.50) because ε_1 in Eq. (3.50) is now replaced by $\varepsilon\rho_1^{-5}$. This complete the proof of Theorem 3.2

4. Global analysis and the proof of main theorems

In this section, we construct a global Poincaré map by combining a succession of transition maps (see Fig. 5) and prove Theorems 1, 2 and 3.

4.1. Global coordinate

Let us introduce a global coordinate to calculate the global Poincaré map. In what follows, we suppose without loss of generality that the branch S^+ and S^- of the critical manifold are convex downward and upward, respectively, as is shown in Fig. 1. Recall that (X, Y, Z) coordinate is defined near the fold point L^+ and that the sections Σ_{in}^+ and Σ_{out}^+ are defined in Eq. (3.7). We define a global coordinate transformation $(x, y, z) \mapsto (X, Y, Z)$ satisfying following: We suppose that in the (X, Y, Z) coordinate, $L^+(\delta) = (0, 0, 0)$, $L^-(\delta) = (0, y_0, z_0)$ with $y_0 > 0$, $z_0 > 0$, and that Y coordinates of S_a^- are larger than those of S_a^+ just as shown in Fig. 12. Let $z_1 > z_0$ be a number and put $z_2 = \rho_1^4 + e^{-1/\varepsilon^2}$. Define the new section

$$\Sigma_I^+ = \{Z = \rho_1^4 + e^{-1/\varepsilon^2}\}, \tag{4.1}$$

which lies slightly above Σ_{in}^+. Change the coordinates so that the segment of S_a^+ in the region $z_2 \leqslant Z \leqslant z_1$ is expressed as

$$\{X = 0,\ Y = -\eta,\ z_2 \leqslant Z \leqslant z_1\}, \tag{4.2}$$

where η is a sufficiently small positive constant (if ρ_1 is sufficiently small). We can define such a coordinate without changing the local coordinate near L^+ and the expression of Π_{loc}^+ given in Eq. (3.8) by using a partition of unity. We can change the coordinates near $S_a^- \cup \{L^-\}$ in a similar manner without changing the coordinate expression near $S_a^+ \cup \{L^+\}$. Let

$$\begin{cases} \dot{X} = f_1(X, Y, Z, \varepsilon, \delta), \\ \dot{Y} = f_2(X, Y, Z, \varepsilon, \delta), \\ \dot{Z} = \varepsilon g(X, Y, Z, \varepsilon, \delta), \end{cases} \tag{4.3}$$

be the system (1.8) written in the resultant coordinate, where the definitions of f_1, f_2 and g are accordingly changed.

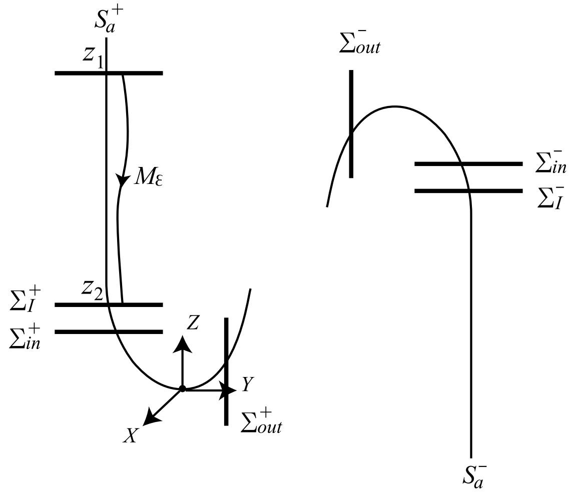

Fig. 12. Coordinate for calculating the global Poincaré map, and a slow manifold M_ε corresponding to the segment $S_a^+(z_2, z_1) = \{(X, Y, Z) \in S_a^+ \mid z_2 \leqslant Z \leqslant z_1\}$ of the critical manifold.

4.2. Flow near the slow manifold

Put $S_a^+(z_2, z_1) = \{(X, Y, Z) \in S_a^+ \mid z_2 \leqslant Z \leqslant z_1\}$. Then, $S_a^+(z_2, z_1)$ is a compact attracting normally hyperbolic invariant manifold of the unperturbed system of (4.3), see Fig. 12. In this subsection, we construct an approximate flow around the slow manifold M_ε corresponding to $S_a^+(z_2, z_1)$. If the parameter δ is a constant, the existence of the slow manifold immediately follows from Fenichel's theorem:

Theorem. *(See Fenichel [8].) Let N be a C^r manifold ($r \geqslant 1$), and $\mathcal{X}^r(N)$ the set of C^r vector fields on N with the C^1 topology. Let F be a C^r vector field on N and suppose that $M \subset N$ is a compact normally hyperbolic F-invariant manifold. Then, there exists a neighborhood $\mathcal{U} \subset \mathcal{X}^r(N)$ of the origin such that if ε is a small positive number so that $\varepsilon G \in \mathcal{U}$ for a given vector field $G \in \mathcal{X}^r(N)$, then the vector field $F + \varepsilon G$ has a locally invariant manifold M_ε within an ε-neighborhood of M. It is diffeomorphic to M and has the same stability as that of M.*

Further, Fenichel [9,10] proved that M_ε admits a fibration: there exists a family of smooth manifolds $\{\mathcal{F}_\varepsilon(p)\}_{p \in M_\varepsilon}$ such that

(i) if $p \neq p'$, then $\mathcal{F}_\varepsilon(p) \cap \mathcal{F}_\varepsilon(p') = \emptyset$;
(ii) $\mathcal{F}_\varepsilon(p) \cap M_\varepsilon = \{p\}$;
(iii) the family $\{\mathcal{F}_\varepsilon(p)\}$ is invariant in the sense that $\phi_t(\mathcal{F}_\varepsilon(p)) \subset \mathcal{F}_\varepsilon(\phi_t(p))$, where ϕ_t is a flow generated by $F + \varepsilon G \in \mathcal{X}^r(N)$;
(iv) there exist $C > 0$, $\lambda > 0$ such that for $q \in \mathcal{F}_\varepsilon(p)$, $\|\phi_t(p) - \phi_t(q)\| < Ce^{-\lambda t}$, where we suppose for simplicity that M (and thus M_ε) is attracting.

See also Wiggins [35] for Fenichel theory. These theorems are applied to fast–slow systems by Fenichel [11] to obtain a slow manifold M_ε and a flow around M_ε. Roughly speaking, these theorems state that for a fast–slow system, there is a locally invariant manifold M_ε, called the slow manifold, within an ε-neighborhood of the critical manifold M if $\varepsilon > 0$ is sufficiently small. A flow near M_ε is given as the sum of the slow motion (dynamics on M_ε) and the fast motion. If M_ε is attracting, the fast motion decays exponentially to zero and eventually a flow is well approximated by the dynamics on M_ε.

Applying these results to our fast–slow system (4.3), when δ is independent of ε, we obtain an attracting slow manifold M_ε and we can construct an approximate flow around M_ε. However, if δ depends on ε, Fenichel theory is no longer applicable in general even if $\varepsilon \ll \delta$. To see this, let us recall how the existence of M_ε is proved.

For simplicity of exposition, suppose that vector fields are defined on $\mathbf{R}^m \times \mathbf{R}^n$. We denote a point on this space as $(x, z) \in \mathbf{R}^m \times \mathbf{R}^n$. Suppose that a given unperturbed vector field F has an attracting compact normally hyperbolic invariant manifold M on the subspace $\{x = 0\}$. We denote a flow ϕ_t generated by the perturbed vector field $F + \varepsilon G$ by

$$\phi_t(x, z, \varepsilon) = \bigl(\phi_t^1(x, z, \varepsilon), \phi_t^2(x, z, \varepsilon)\bigr).$$

From the assumption of normal hyperbolicity, we can show that there exists a positive constant T such that

$$\left\| \frac{\partial \phi_T^1}{\partial x}(0, z, 0) \right\| \cdot \left\| \frac{\partial \phi_T^2}{\partial z}(0, z, 0)^{-1} \right\| < \frac{1}{4}, \quad \text{for } (0, z) \in M, \tag{4.4}$$

because $\partial \phi_t^1 / \partial x$ decays faster than $\partial \phi_t^2 / \partial z$. Since $M \subset \{x = 0\}$ is F-invariant, we have

$$\phi_T^1(0, z, 0) = 0, \qquad \frac{\partial \phi_T^1}{\partial z}(0, z, 0) = 0, \quad \text{for } (0, z) \in M. \tag{4.5}$$

Since the flow is continuous with respect to x, z and ε, for given small positive numbers η_1 and η_2, there exist $\varepsilon_0 > 0$ and an open set $V \supset M$ such that the inequalities

$$\left\| \frac{\partial \phi_T^1}{\partial x}(x, z, \varepsilon) \right\| \cdot \left\| \frac{\partial \phi_T^2}{\partial z}(x, z, \varepsilon)^{-1} \right\| < \frac{1}{2}, \tag{4.6}$$

$$\bigl\| \phi_T^1(x, z, \varepsilon) \bigr\| < \eta_1, \tag{4.7}$$

$$\left\| \frac{\partial \phi_T^1}{\partial z}(x, z, \varepsilon) \right\| < \eta_2, \tag{4.8}$$

hold for $0 < \varepsilon < \varepsilon_0$ and $(x, z) \in V$. Let S be the set of Lipschitz functions from M into the x-space with a suitable norm. Let S_C be the subset of S consisting of functions h such that $(h(z), z) \in V$ and their Lipschitz constants are smaller than some constant $C > 0$. We now define the map $G : S_c \to S$ through

$$(Gh)\bigl(\phi_T^2\bigl(h(z), z, \varepsilon\bigr)\bigr) = \phi_T^1\bigl(h(z), z, \varepsilon\bigr).$$

By using inequalities (4.6), (4.7), (4.8) (and several inequalities which trivially follow from compactness of M), we can show that G is a contraction map from S_C into S_C. See Lemma 3.2.9 of Wiggins [35], in which all inequalities for proving Fenichel's theorem are collected. Thus G has a fixed point h_ε satisfying $h_\varepsilon(\phi_T^2(h_\varepsilon(z), z, \varepsilon)) = \phi_T^1(h_\varepsilon(z), z, \varepsilon)$. This proves that the graph of $x = h_\varepsilon(z)$, which defines M_ε, is invariant under the flow $\phi_t(\cdot, \cdot, \varepsilon)$. The existence of a fibration $\{\mathcal{F}_\varepsilon(p)\}_{p \in M_\varepsilon}$ can be proved in a similar manner.

If the unperturbed vector field $F = F_\delta$ smoothly depends on δ and if δ depends on ε, the above discussion is not valid even if $\varepsilon \ll \delta$. The inequality (4.4) for F_δ does not imply the inequality (4.6) for $F_\delta + \varepsilon G$ in general. For example, consider the linear system $\dot{\boldsymbol{x}} = A_0 \boldsymbol{x} + \delta A_1 \boldsymbol{x}$ with matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose that $\delta = \sqrt{\varepsilon}$. Eigenvalues of $A_0 + \delta A_1$ are given by $-\delta$ (double root), so that the derivative of the flow at the origin is exponentially small for $t > 0$. Next, add the perturbation $\varepsilon A_2 \boldsymbol{x}$ to this system, where

$$A_2 = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}.$$

Although εA_2 is quite smaller than $A_0 + \delta A_1$ if ε is sufficiently small, the eigenvalues of $A_0 + \delta A_1 + \varepsilon A_2$ are δ and -3δ, so that the derivative of the flow of the perturbed system diverges as $t \to \infty$. This shows that Eq. (4.4) does not imply Eq. (4.6) in general if δ depends on ε. Further, the open set V above also depends on ε through δ and it may shrink as $\varepsilon \to 0$. For this linear system, it is easy to see that such a stability change does not occur if A_0 has no Jordan block. For our fast–slow system, the assumption (C5) allows us to prove that such a stability change does not occur.

Lemma 4.1. *Let $A(\delta, z)$ and $B(\varepsilon, \delta, z)$ be 2×2 matrices which are C^∞ in their arguments. Suppose that eigenvalues of $A(\delta, z)$ are given by $-\delta\mu(z, \delta) \pm \sqrt{-1}\omega(z, \delta)$ with the conditions $\mu(z, \delta) > 0$ and $\omega(z, \delta) \neq 0$ for $\delta \geqslant 0$. Further suppose that δ depends on ε as $\varepsilon \sim o(\delta)$ (that is, $\varepsilon \ll \delta$ as $\varepsilon \to 0$). Then, eigenvalues of $A(\delta, z) + \varepsilon B(\varepsilon, \delta, z)$ are given by*

$$-\delta\mu(z, \delta) \pm \sqrt{-1}\omega(z, \delta) + O(\varepsilon) \tag{4.9}$$

as $\varepsilon \to 0$.

Proof. Straightforward calculation. □

Now we return to our fast–slow system (4.3). Put $\boldsymbol{X} = (X, Y)$, $\boldsymbol{f} = (f_1, f_2)$ and rewrite Eq. (4.3) as

$$\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}, Z, \varepsilon, \delta), \qquad \dot{Z} = \varepsilon g(\boldsymbol{X}, Z, \varepsilon, \delta). \tag{4.10}$$

The flow generated by this system is denoted as

$$\phi_t(\boldsymbol{X}, Z, \varepsilon, \delta) = \big(\phi_t^1(\boldsymbol{X}, Z, \varepsilon, \delta), \phi_t^2(\boldsymbol{X}, Z, \varepsilon, \delta)\big). \tag{4.11}$$

Recall that $S_a^+(z_2, z_1)$ is expressed as $X = 0$, $Y = -\eta$; that is, $\boldsymbol{f}(0, -\eta, Z, 0, \delta) = 0$ for $z_2 \leqslant Z \leqslant z_1$. When $\varepsilon = 0$, $\phi_t^2(\boldsymbol{X}, Z, 0, \delta) = Z$, which proves that $\|(\partial\phi_t^2(\boldsymbol{X}, Z, 0, \delta)/\partial Z)^{-1}\| = 1$. Next, the derivative of ϕ_t^1 satisfies the variational equation

$$\frac{d}{dt}\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(\boldsymbol{X}, Z, 0, \delta) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}\big(\phi_t^1(\boldsymbol{X}, Z, 0, \delta), Z, 0, \delta\big)\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(\boldsymbol{X}, Z, 0, \delta).$$

On $S_a^+(z_2, z_1)$, this is reduced to the autonomous system

$$\frac{d}{dt}\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta)\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta).$$

The assumption (C5) implies that the eigenvalues of the matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta)$ are given by $-\delta\mu^+(z, \delta) \pm \sqrt{-1}\omega^+(z, \delta)$. Thus

$$\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta) \sim O\big(e^{-\delta t}\big)$$

on $S_a^+(z_2, z_1)$. This proves the inequality

$$\left\| \frac{\partial \phi_T^1}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta) \right\| \cdot \left\| \frac{\partial \phi_T^2}{\partial Z}(0, -\eta, Z, 0, \delta)^{-1} \right\| < \frac{1}{4}, \tag{4.12}$$

for some large $T > 0$. In general, this does not imply Eq. (4.6) as was explained. However, in our situation, by applying Lemma 4.1 to

$$A(\delta, z) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}(0, -\eta, Z, 0, \delta),$$

it turns out that eigenvalues of the matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}(0, -\eta, Z, \varepsilon, \delta)$ are of the form (4.9) for small $\varepsilon > 0$. Therefore, $\frac{\partial \phi_t^1}{\partial \boldsymbol{X}}(0, -\eta, Z, \varepsilon, \delta)$ also decays with the rate $O(e^{-\delta t})$ on $S_a^+(z_2, z_1)$. Further, the assumption (C6) proves that there exists a neighborhood V^+ of $S_a^+(z_2, z_1)$, which is independent of δ, such that real parts of eigenvalues of $\partial \boldsymbol{f}/\partial \boldsymbol{X}$ are also of order $O(-\delta)$ on V^+. This yields the inequality (4.6) on V^+. Inequalities (4.7) and (4.8) are easily obtained. In this manner, all inequalities for proving Fenichel's theorem are obtained, and the existence of the slow manifold M_ε and a fibration on M_ε for our system are proved in the standard way as long as $\varepsilon \ll \delta$ (to prove Theorem 3, we will suppose that $\delta \sim O(\varepsilon(-\log \varepsilon)^{1/2}) \gg \varepsilon$). Note that the existence of a neighborhood V^+ of the critical manifold, on which eigenvalues of $\partial \boldsymbol{f}/\partial \boldsymbol{X}$ have negative real parts, are also assumed in the classical approach for singular perturbed problems to estimate the dynamics of fast motion, see O'Malley [27] and Smith [31].

Remark. Another way to construct an approximate flow near $S_a^\pm$ is to use the blow-up method near cylinders by adding the equation $\dot{\delta} = 0$, which may allow one to obtain approximate solutions even for $\delta \sim O(\varepsilon)$. In this paper, we adopt Fenichel's argument by noting the assumption $\varepsilon \ll \delta$ because the extension of Fenichel's theorem itself is important.

We have seen that a solution of (4.10) on V^+ is written as the sum of the slow motion on the slow manifold and the fast motion which decays exponentially. To calculate them, it is convenient to introduce the slow time scale by $\tau = \varepsilon t$, which provides

$$\varepsilon \frac{d\boldsymbol{X}}{d\tau} = \boldsymbol{f}(\boldsymbol{X}, Z, \varepsilon, \delta), \qquad \frac{dZ}{d\tau} = g(\boldsymbol{X}, Z, \varepsilon, \delta). \tag{4.13}$$

A solution of this system is given by

$$\begin{cases} \boldsymbol{X}(\tau, \varepsilon, \delta) = \boldsymbol{x}_s(\tau, \varepsilon, \delta) + \boldsymbol{x}_f(\tau, \varepsilon, \delta), \\ Z(\tau, \varepsilon, \delta) = z_s(\tau, \varepsilon, \delta) + z_f(\tau, \varepsilon, \delta), \end{cases} \tag{4.14}$$

where $\boldsymbol{x}_s, z_s$ describe the slow motion and $\boldsymbol{x}_f, z_f$ describe the fast motion. They are C^∞ in ε (see Fenichel [11]) and their expansions with respect to ε are obtained step by step according to O'Malley [27] as follows: We expand them as

$$\boldsymbol{x}_s(\tau, \varepsilon, \delta) = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{x}_s^{(k)}(\tau, \delta), \qquad \boldsymbol{x}_f(\tau, \varepsilon, \delta) = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{x}_f^{(k)}(\tau, \delta),$$

$$z_s(\tau, \varepsilon, \delta) = \sum_{k=0}^{\infty} \varepsilon^k z_s^{(k)}(\tau, \delta), \qquad z_f(\tau, \varepsilon, \delta) = \sum_{k=0}^{\infty} \varepsilon^k z_f^{(k)}(\tau, \delta),$$

with the initial condition

$$\boldsymbol{X}(0,\varepsilon,\delta)=\boldsymbol{x}_0(\delta)+O(\varepsilon),\qquad Z(0,\varepsilon,\delta)=z_0(\delta)+O(\varepsilon),$$

in V^+. At first, $\boldsymbol{x}_s^{(0)}$ and $z_s^{(0)}$ are determined to satisfy the system (4.13) for $\varepsilon=0$. Thus $\boldsymbol{x}_s^{(0)}$ is given by $\boldsymbol{x}_s^{(0)}=(0,-\eta)$ and $z_s^{(0)}$ is given as the solution of the equation

$$\frac{dz_s^{(0)}}{d\tau}=g\big(0,-\eta,z_s^{(0)},0,\delta\big) \tag{4.15}$$

with the initial condition $z_s^{(0)}(0,\delta)=z_0(\delta)$. This system is called the slow system. Next, from the system (4.10) for $\varepsilon=0$, we obtain $z_f^{(0)}\equiv 0$, and $\boldsymbol{x}_f^{(0)}$ is governed by the system

$$\frac{d\boldsymbol{x}_f^{(0)}}{dt}=\frac{d\boldsymbol{X}}{dt}(t,0,\delta)-\frac{d\boldsymbol{x}_s^{(0)}}{dt}(\tau,\delta)=\boldsymbol{f}\big((0,-\eta)+\boldsymbol{x}_f^{(0)},z_s^{(0)}(\tau),0,\delta\big) \tag{4.16}$$

with the initial condition

$$\boldsymbol{x}_f^{(0)}(0,\delta)=\boldsymbol{x}_0(\delta)-\boldsymbol{x}_s(0,0,\delta)=\boldsymbol{x}_0(\delta)-(0,-\eta). \tag{4.17}$$

Fenichel's theorem (Part (iv) above) shows that if $\boldsymbol{x}_f^{(0)}(0,\delta)\in V^+$, then $\boldsymbol{x}_f^{(0)}$ decays exponentially as $t\to\infty$. In the classical approach [27], the existence of V^+ is used to estimate Eq. (4.16) directly to prove that $\boldsymbol{x}_f^{(0)}$ decays exponentially, see also Smith [31]. To investigate behavior of a solution as $\varepsilon\to 0$, we rewrite Eq. (4.16) as

$$\frac{d\boldsymbol{x}_f^{(0)}}{d\tau}=\frac{1}{\varepsilon}\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{X}}\big(0,-\eta,z_s^{(0)}(\tau),0,\delta\big)\boldsymbol{x}_f^{(0)}+\frac{1}{\varepsilon}Q_1\big(\boldsymbol{x}_f^{(0)},\delta\big), \tag{4.18}$$

where $Q_1\sim O((\boldsymbol{x}_f^{(0)})^2)$ is a C^∞ function.

Lemma 4.2. *A solution of the system* (4.18) *is given by*

$$\begin{pmatrix} K_1(\tau,\varepsilon)\cos[\frac{1}{\varepsilon}W(\tau)]+K_2(\tau,\varepsilon)\sin[\frac{1}{\varepsilon}W(\tau)] & K_3(\tau,\varepsilon)\cos[\frac{1}{\varepsilon}W(\tau)]+K_4(\tau,\varepsilon)\sin[\frac{1}{\varepsilon}W(\tau)] \\ K_5(\tau,\varepsilon)\cos[\frac{1}{\varepsilon}W(\tau)]+K_6(\tau,\varepsilon)\sin[\frac{1}{\varepsilon}W(\tau)] & K_7(\tau,\varepsilon)\cos[\frac{1}{\varepsilon}W(\tau)]+K_8(\tau,\varepsilon)\sin[\frac{1}{\varepsilon}W(\tau)] \end{pmatrix}$$
$$\times\exp\left[-\frac{\delta}{\varepsilon}\int_0^\tau \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\big(\boldsymbol{x}_f^{(0)}(0,\delta)+\boldsymbol{u}\big(\tau,\varepsilon,\delta;\boldsymbol{x}_f^{(0)}(0,\delta)\big)\big), \tag{4.19}$$

where $W(\tau)=\int_0^\tau\omega^+(z_s^{(0)}(s),\delta)\,ds$, K_i $(i=1,\dots,8)$ *are* C^∞ *functions, and* $\boldsymbol{u}\sim O(\boldsymbol{x}_f^{(0)}(0,\delta)^2)$ *denotes higher order terms with respect to the initial value.*

Proof. We use the WKB analysis. Put $\boldsymbol{x}_f^{(0)}=(v_1,v_2)$ and

$$\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{X}}\big(0,-\eta,z_s^{(0)}(\tau),0,\delta\big)=\begin{pmatrix} a(\tau) & b(\tau)\\ c(\tau) & d(\tau)\end{pmatrix}. \tag{4.20}$$

Let us consider the linearized system

$$\frac{d}{d\tau}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}\big(0, -\eta, z_s^{(0)}(\tau), 0, \delta\big)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{4.21}$$

Then, $v_1(\tau)$ proves to satisfy the equation

$$\varepsilon^2 v_1'' - \left(\varepsilon(a+d) + \varepsilon^2 \frac{b'}{b}\right)v_1' + \left(ad - bc + \varepsilon\left(\frac{ab'}{b} - a'\right)\right)v_1 = 0. \tag{4.22}$$

We construct a formal solution of the form

$$v_1(\tau) = \exp\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n(\tau)\right].$$

Substituting it into Eq. (4.22), we obtain an equation of S_0,

$$\big(S_0'\big)^2 - (a+d)S_0' + (ad - bc) = 0.$$

This is solved as

$$S_0(\tau) = \int_0^\tau \lambda_+(s)\,ds, \qquad \int_0^\tau \lambda_-(s)\,ds,$$

where

$$\lambda_\pm(\tau) = -\delta\mu\big(z_s^{(0)}(\tau), \delta\big) \pm \sqrt{-1}\omega^+\big(z_s^{(0)}(\tau), \delta\big)$$

are eigenvalues of the matrix (4.20). For each $\int_0^\tau \lambda_+(s)\,ds$ and $\int_0^\tau \lambda_-(s)\,ds$, $S_1, S_2, \ldots$ are uniquely determined. Thus a general solution $v_1(\tau)$ is given by

$$v_1(\tau) = C_1 \exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds\right]K_{11}(\tau, \varepsilon) + C_2 \exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds\right]K_{12}(\tau, \varepsilon),$$

where $C_1, C_2 \in \mathbf{C}$ and K_{11}, K_{12} are C^∞ functions. In a similar manner, it turns out that v_2 is expressed as

$$v_2(\tau) = C_1 \exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds\right]K_{21}(\tau, \varepsilon) + C_2 \exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds\right]K_{22}(\tau, \varepsilon).$$

Therefore, a general solution of the system (4.21) is written as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds]K_{11}(\tau, \varepsilon) & \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds]K_{12}(\tau, \varepsilon) \\ \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds]K_{21}(\tau, \varepsilon) & \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds]K_{22}(\tau, \varepsilon) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The fundamental matrix of (4.21) is given by

$$\begin{pmatrix} \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds]K_{11}(\tau,\varepsilon) & \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds]K_{12}(\tau,\varepsilon) \\ \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_+(s)\,ds]K_{21}(\tau,\varepsilon) & \exp[\frac{1}{\varepsilon}\int_0^\tau \lambda_-(s)\,ds]K_{22}(\tau,\varepsilon) \end{pmatrix} \begin{pmatrix} K_{11}(0,\varepsilon) & K_{12}(0,\varepsilon) \\ K_{21}(0,\varepsilon) & K_{22}(0,\varepsilon) \end{pmatrix}^{-1}.$$

This shows that each component of the fundamental matrix is a linear combination of

$$\exp\left[-\frac{\delta}{\varepsilon}\int_0^\tau \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\cos\left[\frac{1}{\varepsilon}W(\tau)\right] \quad \text{and} \quad \exp\left[-\frac{\delta}{\varepsilon}\int_0^\tau \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\sin\left[\frac{1}{\varepsilon}W(\tau)\right].$$

Finally, the variation-of-constants formula is applied to the nonlinear system (4.18) to prove Lemma 4.2. □

With this $\boldsymbol{x}_f^{(0)}$, the zeroth order approximate solution is constructed as

$$\begin{pmatrix} \boldsymbol{X}(\tau,\varepsilon,\delta) \\ Z(\tau,\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} O(\varepsilon) \\ -\eta + O(\varepsilon) \\ z_s^{(0)}(\tau,\delta) + O(\varepsilon), \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}_f^{(0)}(\tau,\delta) + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix}, \tag{4.23}$$

as long as the orbit is in V^+. The first term in the right-hand side denotes the position on M_ε and the second term denotes the deviation from M_ε. It is known that all terms $\boldsymbol{x}_f^{(k)}, z_f^{(k)}$ in the expansions of the fast motion decay exponentially as well as $\boldsymbol{x}_f^{(0)}$ [11,27,31].

Combining this approximate solution near the slow manifold with the transition map near the fold point, Theorem 1 is easily proved.

Proof of Theorem 1. To prove Theorem 1, δ is assumed to be fixed. For the system (2.1), take an initial value in V^+. Then, a solution is given by (4.23) with (4.19). These expressions show that when $t > 0$, the solution lies sufficiently close to the critical manifold S_a^+ if ε is sufficiently small. Because of the assumption (A3), z_s decreases (where we suppose that S^+ is convex downward) with the velocity of order ε (with respect to the original time scale t). Thus the solution reaches the section Σ_{in}^+ after some time, which is of order $O(1/\varepsilon)$. The intersection point is mapped into Σ_{out}^+ by the transition map Π_{loc}^+ given in Theorem 3.2, and it proves that after passing through Σ_{out}^+ the distance between the solution and the orbit α^+ is of order $O(\varepsilon^{4/5})$. □

4.3. Global Poincaré map

In Section 3, the transition map Π_{loc}^+ around the fold point $L^+(\delta)$ had been constructed. The transition map around the fold point $L^-(\delta)$ is obtained in the same way. The sections Σ_{in}^- and Σ_{out}^- are defined in a similar way to Σ_{in}^+ and Σ_{out}^+ (see Fig. 5), respectively, and the transition map Π_{loc}^- from an open set in Σ_{in}^- into Σ_{out}^- along the flow of (4.3) proves to take the same form as Π_{loc}^+, although functions G_1, G_2 and higher order terms denoted as $O(\varepsilon\log\varepsilon)$ may be different from one another (note that Ω and H are common for Π_{loc}^+ and Π_{loc}^- because they arise from the first Painlevé equation).

Since the unperturbed system has a heteroclinic orbit α^- connecting $L^-(\delta)$ with a point on $S_a^+(\delta)$ and since $S_a^+(\delta)$ has an attraction basin V^+ which is independent of δ, there is an open set $U_{out}^- \subset \Sigma_{out}^-$, which is independent of δ and ε, such that orbits of (4.3) starting from U_{out}^- go into V^+ and are eventually approximated by Eq. (4.23). Let z_0 be the Z coordinate of $L^-(\delta)$. Define the section Σ_{II}^+ to be

$$\Sigma_{II}^+ = V^+ \cap \big\{(X,Y,Z)\ \big|\ Y = -\eta,\ |Z - z_0| \leqslant \rho_4\big\}, \tag{4.24}$$

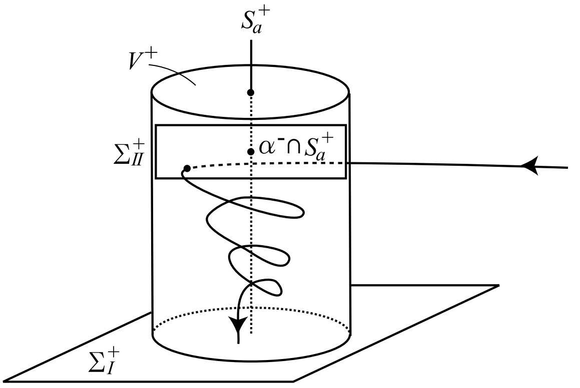

Fig. 13. The sections Σ_I^+, Σ_{II}^+ and an orbit of Eq. (4.3).

where ρ_4 is a small positive number so that a solution of (4.3) starting from U_{out}^- intersects Σ_{II}^+ only once (see Fig. 13).

The global Poincaré map is constructed as follows: Let $\Pi_{II,out}^+$, $\Pi_{I,II}^+$, $\Pi_{in,I}^+$ be transition maps from $U_{out}^- \subset \Sigma_{out}^-$ into Σ_{II}^+, Σ_{II}^+ into Σ_I^+, Σ_I^+ into Σ_{in}^+, respectively. Then, the transition map Π^+ from U_{out}^- into Σ_{out}^+ is given by

$$\Pi^+ = \Pi_{loc}^+ \circ \Pi_{in,I}^+ \circ \Pi_{I,II}^+ \circ \Pi_{II,out}^+.$$

The transition map Π^- from an open set in Σ_{out}^+ into Σ_{in}^- is calculated in a similar manner and it has the same form as Π^+. The global Poincaré map is given by $\Pi^+ \circ \Pi^-$. However, it is sufficient to investigate one of them by identifying Σ_{out}^+ and Σ_{out}^-. If $\Pi^+ : U_{out}^- \to \Sigma_{out}^+$ is a contraction map, so is $\Pi^+ \circ \Pi^-$, and if Π^+ has a horseshoe, so is $\Pi^+ \circ \Pi^-$ because Π^+ and Π^- have the same properties. To identify two sections Σ_{out}^- and Σ_{out}^+, recall that $L^- = (0, y_0, z_0)$ in the (X, Y, Z)-coordinate, and define Σ_{out}^- to be $\{Y = y_0 - \rho_1^2\}$. Let U_{out}^- be an open set in Σ_{out}^- such that the transition map $\Pi_{II,out}^+ : U_{out}^- \to \Sigma_{II}^+$ is well defined. The set U_{out}^- includes the point $\Sigma_{out}^- \cap \alpha^-$. We identify U_{out}^- with an open set U_{out}^+ in Σ_{out}^+ by the translation

$$\mathcal{T} : \begin{pmatrix} X \\ \rho_1^2 \\ Z \end{pmatrix} \mapsto \begin{pmatrix} X \\ y_0 - \rho_1^2 \\ Z + z_0 \end{pmatrix}. \tag{4.25}$$

Then, the transition map $\tilde{\Pi}_{II,out}^+$ from $U_{out}^+ \subset \Sigma_{out}^+$ into Σ_{II}^+ is obtained by combining the translation and $\Pi_{II,out}^+$. Since the velocity in the Z direction is of order ε, it is expressed as

$$\tilde{\Pi}_{II,out}^+ \begin{pmatrix} X \\ \rho_1^2 \\ Z \end{pmatrix} = \Pi_{II,out}^+ \circ \mathcal{T} \begin{pmatrix} X \\ \rho_1^2 \\ Z \end{pmatrix} = \begin{pmatrix} P^+(X, Z, \varepsilon, \delta) \\ -\eta \\ Z + z_0 + O(\varepsilon) \end{pmatrix}, \tag{4.26}$$

where P^+ is a C^∞ function. Since $\tilde{\Pi}_{II,out}^+$ is C^∞, we expand it as

$$\tilde{\Pi}_{II,out}^+ \begin{pmatrix} X \\ \rho_1^2 \\ Z \end{pmatrix} = \begin{pmatrix} p(\delta) + O(X, Z, \varepsilon) \\ -\eta \\ Z + z_0 + O(\varepsilon) \end{pmatrix}. \tag{4.27}$$

To prove Theorem 3, we will use the fact that there exists a positive constant $p_0 > 0$ such that $|p(\delta)| > p_0$ for $0 < \delta < \delta_0$, which is proved as follows: Since δ controls the strength of the stability of S_a^+, if δ is sufficiently small, orbits which converge to $(0, -\eta, z_0)$ (the intersection of the heteroclinic orbit α^- and S_a^+) rotate around this point so many times. In particular, they intersect with Σ_{II}^+ before reaching $(0, -\eta, z_0)$. If $p(\delta)$ were zero, the right-hand side above tends to $(0, -\eta, z_0)$ as $X, Z, \varepsilon \to 0$, which yields a contradiction.

Next thing to do is to combine the above $\tilde{\Pi}_{II,out}^+$ with $\Pi_{I,II}^+$. By Eq. (4.23), the transition map $\Pi_{I,II}^+$ from Σ_{II}^+ into Σ_I^+ is given by

$$\Pi_{I,II}^+ \begin{pmatrix} X \\ -\eta \\ Z \end{pmatrix} = \begin{pmatrix} O(\varepsilon) \\ -\eta + O(\varepsilon) \\ z_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}_f^{(0)}(\tau(X, Z, \varepsilon, \delta), \delta) + O(\varepsilon) \\ 0 \end{pmatrix}, \tag{4.28}$$

where $\boldsymbol{x}_f^{(0)} = \boldsymbol{x}_f^{(0)}(\tau, \delta)$ is given by (4.19) with the initial condition $\boldsymbol{x}_f^{(0)}(0, \delta) = (X, 0)$, $z_2 = \rho_1^4 + e^{-1/\varepsilon^2}$ is the Z coordinate of the section Σ_I^+ as defined before, and $\tau = \tau(X, Z, \varepsilon, \delta)$ is a transition time (with respect to the slow time scale) from a point $(X, -\eta, Z)$ to Σ_I^+. This transition time τ is determined as follows: Let $z_s^{(0)}(\tau, \delta)$ be a solution of Eq. (4.15) with the initial condition $z_s^{(0)}(0, \delta) = Z$. Then, Eq. (4.23) implies that $\tau = \tau(X, Z, \varepsilon, \delta)$ is given as a root of the equation

$$z_2 = z_s^{(0)}(\tau, \delta) + O(\varepsilon).$$

Let $\hat{\tau}$ be a root of the equation $z_2 = z_s^{(0)}(\tau, \delta)$. By virtue of the implicit function theorem, τ is written as $\tau = \hat{\tau} + O(\varepsilon)$. Since Eq. (4.15) is independent of X and ε, so is $\hat{\tau}$. Thus we obtain

$$\tau(X, Z, \varepsilon, \delta) = \hat{\tau}(Z, \delta) + O(\varepsilon). \tag{4.29}$$

Further, $\hat{\tau}$ is bounded as $\delta \to 0$ because $g \neq 0$ on S_a^+ uniformly in $0 \leqslant \delta < \delta_0$. Therefore, $\Pi_{I,II}^+$ proves to be of the form

$$\begin{aligned} \Pi_{I,II}^+ \begin{pmatrix} X \\ -\eta \\ Z \end{pmatrix} &= \begin{pmatrix} O(\varepsilon) \\ -\eta + O(\varepsilon) \\ z_2 \end{pmatrix} \\ &+ \begin{pmatrix} X(K_1(\hat{\tau}, \varepsilon)\cos[\frac{1}{\varepsilon}W(\hat{\tau})] + K_2(\hat{\tau}, \varepsilon)\sin[\frac{1}{\varepsilon}W(\hat{\tau})])\exp[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+(z_s^{(0)}(s), \delta)\,ds](1 + O(\varepsilon, X)) \\ X(K_5(\hat{\tau}, \varepsilon)\cos[\frac{1}{\varepsilon}W(\hat{\tau})] + K_6(\hat{\tau}, \varepsilon)\sin[\frac{1}{\varepsilon}W(\hat{\tau})])\exp[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+(z_s^{(0)}(s), \delta)\,ds](1 + O(\varepsilon, X)) \\ 0 \end{pmatrix}. \end{aligned} \tag{4.30}$$

The first line denotes the intersection point $M_\varepsilon \cap \Sigma_I^+$ and thus it is independent of X and Z. The second line denotes the deviation from the intersection. Note that the transition map $\Pi_{in,I}^+$ from Σ_I^+ into Σ_{in}^+ is $O(e^{-1/\varepsilon^2})$-close to the identity map. Thus $\Pi_{in,I}^+ \circ \Pi_{I,II}^+ \circ \Pi_{II,out}^+ \circ \mathcal{T}$ is calculated as

$$\begin{aligned} \Pi_{in,I}^+ \circ \Pi_{I,II}^+ \circ \Pi_{II,out}^+ \circ \mathcal{T} \begin{pmatrix} X \\ \rho_1^2 \\ Z \end{pmatrix} &= \begin{pmatrix} O(\varepsilon) \\ -\eta + O(\varepsilon) \\ \rho_1^4 \end{pmatrix} \\ &+ \begin{pmatrix} p(\delta)(K_1(\hat{\tau}, \varepsilon)\cos[\frac{1}{\varepsilon}W(\hat{\tau})] + K_2(\hat{\tau}, \varepsilon)\sin[\frac{1}{\varepsilon}W(\hat{\tau})])\exp[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+(z_s^{(0)}(s), \delta)\,ds](1 + O(\varepsilon, X, Z)) \\ p(\delta)(K_5(\hat{\tau}, \varepsilon)\cos[\frac{1}{\varepsilon}W(\hat{\tau})] + K_6(\hat{\tau}, \varepsilon)\sin[\frac{1}{\varepsilon}W(\hat{\tau})])\exp[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+(z_s^{(0)}(s), \delta)\,ds](1 + O(\varepsilon, X, Z)) \\ 0 \end{pmatrix}, \end{aligned} \tag{4.31}$$

where $\hat{\tau} = \hat{\tau}(Z + z_0, \delta)$ and $z_s^{(0)}(\tau)$ is a solution of (4.15) satisfying the initial condition $z_s^{(0)}(0) = Z + z_0$. Finally, the transition map

$$\Pi^+ = \Pi^+_{loc} \circ \Pi^+_{in,I} \circ \Pi^+_{I,II} \circ \Pi^+_{II,out} \circ \mathcal{T}$$

from U^-_{out} into Σ^+_{out} is obtained by combining the above map with Π^+_{loc}.

At this stage, we can prove Theorem 2.

Proof of Theorem 2. To prove Theorem 2, it is sufficient to show that the map Π^+ has a hyperbolically stable fixed point. Then, the global Poincaré map (without identifying Σ^+_{out} and Σ^-_{out}) has the same property because Π^- takes the same form as Π^+. Indeed, if ε is sufficiently small for fixed δ, Theorem 3.2 and Eq. (4.31) show that the image of the map Π^+ is exponentially small, and thus Π^+ is a contraction map. Further, eigenvalues of the derivative of Π^+ is of order $O(e^{-1/\varepsilon})$, which proves that Π^+ has a hyperbolically stable fixed point. □

4.4. Derivative of the transition map

If δ is fixed, it is obvious that the transition map Π^+ is of order $O(e^{-1/\varepsilon})$ as $\varepsilon \to 0$. However, when δ is small as well as ε, the action of Π^+ becomes more complex. In what follows, we suppose that δ depends on ε and $\varepsilon \sim o(\delta)(\varepsilon \ll \delta)$ as $\varepsilon \to 0$. A straightforward calculation shows that the derivative of Π^+ is of the form

$$\frac{\partial \Pi^+}{\partial(X,Z)} = \begin{pmatrix} L_1(X,Z,\varepsilon,\delta)\varepsilon^{1/5} & L_2(X,Z,\varepsilon,\delta)\varepsilon^{-4/5} \\ L_3(X,Z,\varepsilon,\delta)\varepsilon^{1/5} & L_4(X,Z,\varepsilon,\delta)\varepsilon^{-4/5} \end{pmatrix}$$
$$\times \exp\left[-\hat{d}(\rho,\delta)\frac{\delta}{\varepsilon}\right] \cdot \exp\left[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\big(1 + L_5(X,Z,\varepsilon,\delta)\big), \tag{4.32}$$

where L_i $(i = 1,\dots,4)$ are bounded as $\varepsilon \to 0$, and L_5 denotes higher order terms such that $L_5 \sim o(1)$ as $X, Z, \varepsilon \to 0$.

Eigenvalues of the derivative are given by

$$\lambda_1 = L_4\varepsilon^{-4/5}\exp\left[-\hat{d}(\rho,\delta)\frac{\delta}{\varepsilon}\right] \cdot \exp\left[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\big(1 + o(1)\big), \tag{4.33}$$

and

$$\lambda_2 = \frac{L_1L_4 - L_2L_3}{L_4}\varepsilon^{1/5}\exp\left[-\hat{d}(\rho,\delta)\frac{\delta}{\varepsilon}\right] \cdot \exp\left[-\frac{\delta}{\varepsilon}\int_0^{\hat{\tau}} \mu^+\big(z_s^{(0)}(s),\delta\big)\,ds\right]\big(1 + o(1)\big). \tag{4.34}$$

If δ is fixed, they are exponentially small as $\varepsilon \to 0$, although if δ is small as well as ε, $|\lambda_1|$ may become large. For example, if $\delta = C\varepsilon(-\log\varepsilon)^{1/2}$ with a positive constant C, and if $L_4(X,Z,\varepsilon,\delta) \neq 0$, $|\lambda_1|$ is of order $\varepsilon^{-4/5}e^{-C(-\log\varepsilon)^{1/2}}$, which is larger than 1 if ε is sufficiently small. On the other hand, $|\lambda_2|$ is always smaller than 1. The function L_4 is given by

$$L_4(X,Z,\varepsilon,\delta) = \frac{\partial H}{\partial X}\big(\hat{D}_1\varepsilon^{-3/5}e^{-\hat{d}\delta/\varepsilon}, \hat{D}_2\varepsilon^{-2/5}e^{-\hat{d}\delta/\varepsilon}\big)\cdot\frac{\partial\hat{D}_1}{\partial X}\cdot p(\delta)\cdot\frac{\partial}{\partial Z}W(\hat{\tau}) \times\left(-K_1(\hat{\tau},\varepsilon)\sin\left[\frac{1}{\varepsilon}W(\hat{\tau})\right] + K_2(\hat{\tau},\varepsilon)\cos\left[\frac{1}{\varepsilon}W(\hat{\tau})\right]\right), \tag{4.35}$$

in which arguments of $\hat{D}_i = \hat{D}_i(\cdot,\cdot,\rho_1,\varepsilon,\delta)$ are given by the first and second components of Eq. (4.31). From Theorem 3.2 (III) and (IV), we obtain $\partial H/\partial X \neq 0$, $\partial\hat{D}_1/\partial X \neq 0$. The value $p(\delta)$ is also not zero as was explained above. Recall that $\hat{\tau}(Z+z_0,\delta)$ is defined as a transition time along the flow of Eq. (4.15). Since $g<0$ uniformly on S_a^+ and $0 \leqslant \delta < \delta_0$, $\hat{\tau}$ is monotonically increasing with respect to Z. Further, $W(\hat{\tau})$ is monotonically decreasing or monotonically increasing because $\omega^+ \neq 0$ uniformly. This proves $\partial W(\hat{\tau})/\partial Z \neq 0$. Therefore, $L_4 = 0$ if and only if

$$-K_1\big(\hat{\tau}(Z+z_0,\delta),\varepsilon\big)\sin\left[\frac{1}{\varepsilon}W\big(\hat{\tau}(Z+z_0,\delta)\big)\right] + K_2\big(\hat{\tau}(Z+z_0,\delta),\varepsilon\big)\cos\left[\frac{1}{\varepsilon}W\big(\hat{\tau}(Z+z_0,\delta)\big)\right]$$
$$= -K_1\big(\hat{\tau}(z_0,\delta),\varepsilon\big)\sin\left[\frac{1}{\varepsilon}W\big(\hat{\tau}(Z+z_0,\delta)\big)\right] + K_2\big(\hat{\tau}(z_0,\delta),\varepsilon\big)\cos\left[\frac{1}{\varepsilon}W\big(\hat{\tau}(Z+z_0,\delta)\big)\right] + O(Z)$$

is zero. If there exists Z such that the above value is zero, then it is zero for a countable set of values of Z because of the periodicity. For these "bad" Z, λ_1 degenerates and $|\lambda_1|$ may become smaller than 1. Now we have the same situation as the proof of the existence of chaos in Silnikov's systems. In the proof of Silnikov's chaos, an eigenvalue of a transition map degenerates if and only if an expression $k_1\sin(\log(z/\varepsilon)) + k_2\cos(\log(z/\varepsilon))$ is zero, where k_1 and k_2 are some constants, see Wiggins [34].

4.5. *Proof of Theorem 3*

Now we are in a position to prove Theorem 3. The proof is done in the same way as the proof of Silnikov's chaos. At first, we show that the transition map Π^+ has a topological horseshoe: We show that an image of a rectangle under Π^+ becomes a ring-shaped area and it appropriately intersects with the rectangle. Next, to prove that the horseshoe is hyperbolic, we investigate the derivative of Π^+. We can avoid "bad" Z, at which the derivative degenerates, because they are at most countable.

Proof of Theorem 3. Suppose that $\delta = C_1\varepsilon(-\log\varepsilon)^{1/2}$ with some positive constant C_1. Recall that there exists a slow manifold within an ε neighborhood of S_a^+. Since it is one dimension, the slow manifold is a solution orbit of the system (4.10). By virtue of Theorem 3.2, this orbit intersects with Σ_{out}^+ near α^+. Let $Q \in \Sigma_{out}^+$ be the intersection point of this orbit and Σ_{out}^+. Take a rectangle R on Σ_{out}^+ including the point Q, whose boundaries are parallel to the X axis and the Z axis (see Fig. 5). Let $h_R = C_2\varepsilon$ be the height of R, where C_2 is a positive constant to be determined. The image of R under the map $\tilde{\Pi}_{out,II}^+ = \Pi_{out,II}^+ \circ \mathcal{T}$ is a deformed rectangle whose "height" is also of order $O(\varepsilon)$ since $dZ/dt \sim O(\varepsilon)$.

Next thing to consider is the shape of $\Pi_{II,I}^+ \circ \tilde{\Pi}_{out,II}^+(R)$. It is easy to show by using Eq. (4.30) that the image of $\tilde{\Pi}_{out,II}^+(R)$ under the map $\Pi_{II,I}^+$ becomes a ring-shaped area whose radius is of order $e^{-\delta/\varepsilon}$. Since the "height" of $\tilde{\Pi}_{out,II}^+(R)$ is of order ε, the rotation angle of the ring-shaped area is estimated as

$$\frac{1}{\varepsilon}W\big(\hat{\tau}\big(Z+z_0+O(\varepsilon)\big)\big) - \frac{1}{\varepsilon}W\big(\hat{\tau}(Z+z_0)\big) = \frac{1}{\varepsilon}\int_{\hat{\tau}(Z+z_0)}^{\hat{\tau}(Z+z_0+O(\varepsilon))}\omega^+\big(z_s^{(0)}(s),\delta\big)\,ds \sim O(1). \tag{4.36}$$

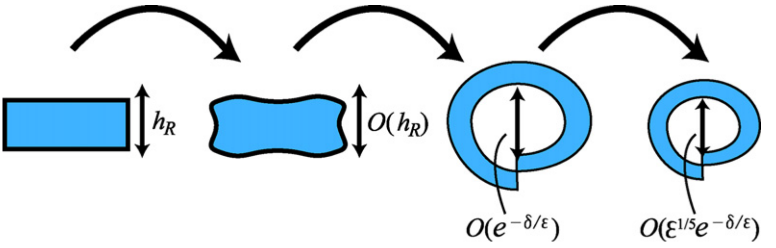

Fig. 14. Images of the rectangle R under a succession of transition maps.

Thus we can choose C_2 so that the rotation angle of the ring-shaped area is sufficiently close to 2π as is shown in Fig. 14.

Finally, we consider the shape of $\Pi^+(R)$ by using Theorem 3.2. Since $\partial H/\partial\mathcal{X}(0,0) \neq 0$, the expansion of H is estimated as

$$H(\mathcal{X},\mathcal{Y}) \sim \mathcal{X}\varepsilon^{-3/5}\exp[-\hat{d}\delta/\varepsilon]\bigl(1+O\bigl(\varepsilon^{1/5}\bigr)\bigr). \tag{4.37}$$

This and Eq. (3.8) show that the radius of $\Pi^+(R)$ is of order $O(\varepsilon^{1/5}e^{-\delta/\varepsilon})$. Since we put $\delta = C_1\varepsilon(-\log\varepsilon)^{1/2}$, the inequality

$$h_R = C_2\varepsilon \ll O\bigl(\varepsilon^{1/5}e^{-\delta/\varepsilon}\bigr) \tag{4.38}$$

holds if ε is sufficiently small. Further, the ring $\Pi^+(R)$ surrounds the point Q because the image of the rectangle R under the flow rotates around the slow manifold when passing between the section Σ^+_{II} and Σ^+_{in}. This means that two horizontal boundaries of R intersect with the ring $\Pi^+(R)$ as is shown in Fig. 6 (b). It is obvious that the vertical boundaries of R are mapped to the inner and outer boundaries of the ring, and the horizontal boundaries are mapped to the other boundaries in radial direction. This proves that the map Π^+ creates a horseshoe and thus has an invariant Cantor set.

To prove that this invariant set is hyperbolic, it is sufficient to show that there exist two disjoint rectangles H_1 and H_2 in R, whose horizontal boundaries are parallel to the X axis and vertical boundaries are included in those of R, such that the inequalities

$$\bigl\|D_x\Pi_1^+\bigr\| < 1, \tag{4.39}$$

$$\bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\| < 1, \tag{4.40}$$

$$1 - \bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\| \cdot \bigl\|D_x\Pi_1^+\bigr\| > 2\sqrt{\bigl\|D_z\Pi_1^+\bigr\| \cdot \bigl\|D_x\Pi_2^+\bigr\| \cdot \bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\|^2}, \tag{4.41}$$

$$1 - \bigl(\bigl\|D_x\Pi_1^+\bigr\| + \bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\|\bigr) + \bigl\|D_x\Pi_1^+\bigr\| \cdot \bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\|$$
$$> \bigl\|D_x\Pi_2^+\bigr\| \cdot \bigl\|D_z\Pi_1^+\bigr\| \cdot \bigl\|\bigl(D_z\Pi_2^+\bigr)^{-1}\bigr\|, \tag{4.42}$$

hold on $H_1 \cup H_2$, where Π_1^+ and Π_2^+ denote the X and Z components of Π^+, respectively, and D_x and D_z denote the derivatives with respect to X and Z, respectively. See Wiggins [34] for the proof. We can take such H_1 and H_2 so that "bad" Z, at which $L_4 = 0$, are not included. Then, inequalities

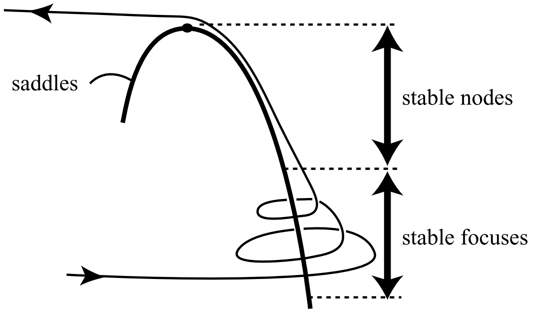

Fig. 15. Critical manifold consisting of a saddle-node type fold point, stable nodes, and stable focuses and an orbit near it.

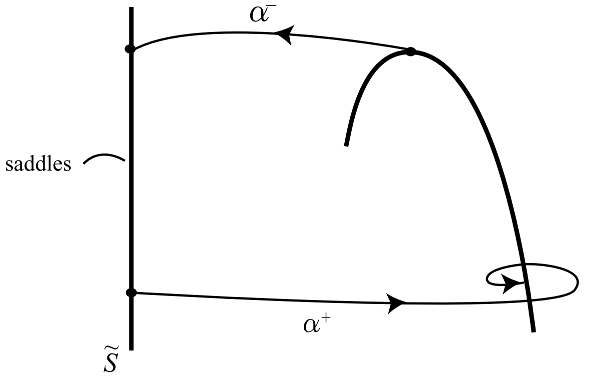

Fig. 16. Two connected components of critical manifolds. One is similar to that of our system (1.8) and the other consists of only saddles.

above immediately follows from Eq. (4.32): $\|D_x\Pi_1^+\|$ and $\|D_x\Pi_2^+\|$ are sufficiently small, and $\|D_z\Pi_1^+\|$ and $\|D_z\Pi_2^+\|$ are sufficiently large as $\varepsilon \to 0$. This proves Theorem 3. □

5. Concluding remarks

Our assumption of Bogdanov–Takens type fold points is not generic in the sense that the Jacobian matrix has two zero eigenvalues. However, this assumption is not essential for existence of periodic orbits or chaotic invariant sets.

At first, we remark that Theorems 2 and 3 hold even if we add a small perturbation to Eq. (1.8), since hyperbolic invariant sets remain to exist under small perturbations.

Second, we can consider the case that one of the connected components of critical manifolds consists of stable nodes, stable focuses and a saddle-node type fold point (*i.e.* a saddle-node bifurcation point of a unperturbed system), as in Fig. 15. In this case, Theorem 2 is proved in a similar way and Theorem 3 still holds if the length of the subset of the critical manifold consisting of stable focuses is of order $O(1)$. However, analysis of saddle-node type fold points is well performed in [20,25,12] and thus we do not deal with such a situation in this paper.

We can also consider the case that one of the connected components $\tilde{S}$ of critical manifolds has no fold points but consists of saddles with heteroclinic orbits $\alpha^{\pm}$, see Fig. 16. In this case, analysis around the $\tilde{S}$ is done by using the exchange lemma (see Jones [18]) and we can prove theorems similar to Theorems 2 and 3. Such a situation arises in an extended prey–predator system. In [23], a periodic orbit and chaos in an extended prey–predator system are numerically investigated with the aid of the theory of the present paper.

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