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E_∞-MONOIDS WITH COHERENT HOMOTOPY INVERSES ARE ABELIAN GROUPS

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LOOSELY speaking, an A_∞ monoid is an H-space X whose multiplication is homotopy associative with a homotopy unit in a coherent way, i.e. the homotopies fit together up to higher homotopies which in turn fit together up to homotopies etc. If X in addition is coherently homotopy commutative it is called an E_∞ monoid. Interest in such homotopy monoid structures arose from the following well-known fact.

1.1. X is an A_∞ monoid admitting a homotopy inverse iff X is of the homotopy type of a loop space (and hence of the weak homotopy type of a topological group). It is an E_∞ monoid admitting a homotopy inverse iff it is of the homotopy type of an infinite loop space.

People working with E_∞ structures usually disregard the canonical and apparently coherent (?) homotopy inverses which (iterated) loop spaces possess, go through the constructions they want to do and regain the homotopy inverses by adding a group completion process. There have been attempts to study E_∞ structures with incorporated homotopy inverses (e.g. see [2]), which all turned out to be not quite satisfying.

One could live quite well with this situation until the appearance of Waldhausen's algebraic K-theory $A(X)$ of a topological space X (see [4]), which can be defined as the algebraic K-theory of the A_∞ ring $Q(\Omega X_+)$, where Q is the functor $\Omega^\infty\Sigma^\infty$ and $Y_+ = Y_\cup$ point. An A_∞ ring is a semiring up to homotopy with an additive E_∞ monoid structure and a multiplicative A_∞ monoid structure, which are related by types of distributive laws. The coherently commutative analogue is called an E_∞ ring.

Quillen's definition of the algebraic K-theory of a ring R as the plus construction on $BGl(R)$ can be extended to A_∞ rings to give $A(X)$. The close relation of $A(X)$ to the stable pseudo isotopy space of X, if X is a manifold, made it an interesting object to study. One central question is how much of classical algebraic K-theory can be transferred to the A_∞ and E_∞ ring case. Here one immediately runs into the problem of the existence of coherent additive inverses. The following example, due to Waldhausen, indicates trouble:

Example 1.2. Suppose QS^0 admits sufficiently coherent additive inverses one would be able to construct a determinant map over QS^0. As in classical K-theory this determinant implies a splitting of $A(*) = K(QS^0)$ into the "units" F of QS^0, where F is the space of stable selfhomotopy equivalences of the sphere, and a factor $SK(QS^0)$. Calculations of [4] show that this is impossible.

It is the purpose of this paper to show that (1.2) is not surprising at all: We will prove that one cannot expect to have coherent homotopy inverses in an E_∞ monoid and hence in the additive structure of an A_∞ or E_∞ ring. Moreover, the situation does not improve if one localizes at a set of primes unless one rationalizes (with little effort one can show that an E_∞ monoid with homotopy inverses is rationally equivalent to an abelian topological group; the analogous statements hold in both ring cases, too).

DEFINITIONS

It is understood that we work in a good category of topological spaces (e.g. see [3]). We start describing E_∞ monoid structures using the language of universal algebra. We refer to [1; chapter II] for background material.

Definition 2.1. A *theory* is a category Θ with one object $[n]$ for each integer $n \geq 0$ and topologized morphism sets such that $[n]$ is the n- fold product of $[1]$, the composition is continuous, and the natural map $\Theta([k],[n]) \cong (\Theta([k],[1]))^n$ is a homeomorphism. A Θ-*space* is a continuous, product preserving functor $X:\Theta \to Top$. We call $X[1]$ the *underlying space* of X. However, we frequently refer to X when we mean $X[1]$ and vice-versa.

Examples 2.2. (i) $\Theta_{\mathscr{S}}$-the theory of set operations. It is the dual of the category $\mathscr{S}$ of finite sets $\{1,2,\ldots,n\}, n \geq o$. We denote the morphism corresponding to the map $f:\{1,\ldots,n\} \to \{1,\ldots,m\}$ by $(x_{f(1)},\ldots,x_{f(n)})$. Composition is by substitution. The functor $\Theta_{\mathscr{S}} \to Top$ sending $(x_{f(1)},\ldots,x_{f(n)})$ to the map $(x_1,\ldots,x_m) \to (x_{f(1)},\ldots,x_{f(n)})$ makes any topological space X into a $\Theta_{\mathscr{S}}$-space. Moreover, $\Theta_{\mathscr{S}}$ is initial in the category of theories: The functor $\Theta_{\mathscr{S}} \to \Theta$ sends $(x_{f(1)},\ldots,x_{f(n)})$ to $(p_{f(1)},\ldots,p_{f(n)}):[m] \to [1]^n = [n]$, where $p_j:[m] \to [1]$, $j = 1,\ldots,m$ are the projections in Θ. The morphisms in its image are called the set operations of Θ.

(ii) $\mathscr{CM}$–the theory of commutative monoids. $\mathscr{CM}([n],[1])$ is the free commutative monoid on n generators $x_1,\ldots,x_n$, $\mathscr{CM}([n],[k]) = (\mathscr{CM}([n],[1])^k$, composition is by substitution. If X is a commutative monoid, the functor $\mathscr{CM} \to Top$ sending $k_1x_1 + \ldots + k_nx_n$ to

$$X^n \to X, (x_1,\ldots,x_n) \to k_1x_1 + \ldots + k_nx_n$$

makes X a $\mathscr{CM}$-space. Conversely, any $\mathscr{CM}$-space is a commutative monoid.

(iii) $\mathscr{CG}$–the theory of commutative groups is defined analogously.

Note that $\mathscr{CM}$ as a theory is generated by the morphisms $\lambda_0 = 0:[0] \to [1]$ and $\lambda_n = x_1 + \ldots + x_n:[n] \to [1]$, $n > 0$, and $\mathscr{CG}$ by these morphisms and the inversion $i = -x:[1] \to [1]$.

Definition 2.3. An E_∞ *theory* is a theory Θ with an augmentation $F:\Theta \to \mathscr{CM}$ which is bijective on path components (of morphism spaces) and for which $F^{-1}(\lambda_n)$ is contractible for each n. An E_∞ theory with *weakly coherent homotopy inverses* is a theory Ψ with augmentation $G:\Psi \to \mathscr{CG}$ which is bijective on path components and for which $G^{-1}(\lambda_n)$ is contractible for each n. If in addition $G^{-1}(i)$ is contractible, we call Ψ an E_∞ theory with *coherent homotopy inverses.* A Θ-space (Ψ-space) is called an E_∞ *monoid* (E_∞ *monoid with* (*weakly*) *coherent homotopy inverses*).

Example 2.4. The little cubes theory Θ_∞ of [1; (2.49)]. Let $Q_k([n],[1])$ be the space of embeddings of $\mathring{I}^k \sqcup \ldots \sqcup \mathring{I}^k$ (n copies) into $\mathring{I}^k$, where $\mathring{I}^k$ is the open unit k-cube. Composition of embeddings defines a composition of $a \in Q_k([n],[1])$ with an n-tuple of elements in $Q_k([r],[1])$. Let $Q_k([n],[1]) \to Q_{k+1}([n],[1])$ be the product with $\mathring{I}^1$ and let $Q([n],[1])$ be the limit over k. Then $Q([n],[1])$ is a free contractible Σ_n-space. Define

$$\Theta_\infty([n],[1]) = \coprod_r Q([r],[1]) \times_{\Sigma_r} \Theta_S([n],[r])$$

The map sending $Q([r],[1])$ to $\{\lambda_r\}$ induces an augmentation $F:\Theta_\infty \to \mathscr{CM}$. It is easy to check that Θ_∞ is an E_∞ theory. It is universal in the sense that each infinite loop space is homotopy equivalent to a Θ_∞-space and that each Θ_∞-space with a homotopy inverse is an infinite loop space. Observe that $F^{-1}(kx) = B\Sigma_k$ because $kx = \lambda_k \circ \Delta$ where Δ is the k-fold diagonal. If $\pi \in \Sigma_k$, then $\pi \circ \Delta = \Delta$, so that $Q([k],[1]) \times_{\Sigma_r} \{\Delta\} = B\Sigma_k$.

STATEMENTS AND PROOFS

THEOREM 3.1. *An E_∞ monoid with weakly coherent homotopy inverses is weakly equivalent to an abelian topological group and hence to a weak product of Eilenberg MacLane spaces.*

Remark. In particular, if one extends the E_∞ monoid structure of an infinite loop space by incorporating the existing canonical homotopy inverses into the theory, one loses control over the morphism spaces $F^{-1}(\lambda_n)$: They cease to be contractible. Constructions from classical ring theory can be transferred to the A_∞ and E_∞ case as long as they can be expressed universally in terms of theories and as long as the operations $c \in \operatorname{mor}\mathscr{CM}$ involved in the constructions satisfy that $F^{-1}(c)$ is contractible. The theorem follows from (3.2) and (3.3) below.

PROPOSITION 3.2. *If $G:\Theta \to \mathscr{CG}$ is a theory functor which is a homotopy equivalence, then each Θ-space X is weakly equivalent to a $\mathscr{CG}$-space and hence to an abelian topological group.*

Proof. Take CW-approximations of Θ and of X and then apply [1; Thm. 4.58].

PROPOSITION 3.3. *If $G:\Theta \to \mathscr{CG}$ is an E_∞ theory with weakly coherent homotopy inverses, then G is a homotopy equivalence.*

COROLLARY 3.4. *In order to extend the E_∞ structure of an infinite loop space to an E_∞ structure with weakly coherent homotopy inverses one, in general, has to rationalize.*

Proof of 3.4. In general, the E_∞-structure is codified by Θ_∞ of Example 2.4. By (3.3) the augmentation $F:\Theta_\infty \to \mathscr{CM}$ has to become a homotopy equivalence. In particular, all the $B\Sigma_k = F^{-1}(k \cdot x)$ have to be changed to contractible spaces.

Proof of 3.3. Since $G^{-1}(\lambda_n)$ is contractible by assumption, (3.3) is a consequence of the following lemma, whose proof is based on the fact that adding $+x_i$ can up to homotopy be reversed by any kind of subtraction of x_i.

LEMMA 3.5. $G^{-1}(k_1x_1 + \dots + k_nx_n) \simeq G^{-1}(k_1x_1 + \dots + (k_i+1)x_i + \dots + k_nx_n)$ *for all* $k_1, \dots, k_n \in \mathbb{Z}$.

Proof. Since composition with a permutation set operation is a homeomorphism it suffices to prove

$$G^{-1}(kx_1 + k_2x_2 + \dots + k_nx_n) \simeq G^{-1}((k+1)x_1 + k_2x_2 + \dots + k_nx_n)$$

Choose $\lambda_n^* \in G^{-1}(\lambda_n)$ and $i^* \in G^{-1}(i)$, and define

$$\alpha_k\colon G^{-1}(kx_1 + k_2x_2 + \ldots + k_nx_n) \to G^{-1}((k+1)x_1 + k_2x_2 + \ldots + k_nx_n)$$
$$\to \lambda_2^* \circ (id_1 \times f) \circ (\Delta \times id_{n-1})$$
$$\beta_k\colon G^{-1}((k+1)x_1 + k_2x_2 + \ldots + k_nx_n) \to G^{-1}(kx_1 + k_2x_2 + \ldots + k_nx_n)$$
$$\to \lambda_2^* \circ (i^* \times g) \circ (\Delta \times id_{n-1})$$

where Δ is the diagonal $1 \to 2$. Then

$$\beta_k \circ \alpha_k\colon f \to \lambda_2^* \circ (i^* \times [\lambda_2^* \circ (id_1 \times f) \circ (\Delta \times id_{n-1})]) \circ (\Delta \times id_{n-1})$$
$$= \lambda_2^* \circ (id_1 \times \lambda_2^*) \circ (i^* \times id_2) \circ (id_2 \times f) \circ (id_1 \times \Delta \times id_{n-1}) \circ (\Delta \circ id_{n-1})$$

Since $(id_1 \times \Delta) \circ \Delta = (\Delta \times id_1) \circ \Delta$, this is equal to

$$f \to \lambda_2^* \circ (id_1 \times \lambda_2^*) \circ (i^* \times id_2) \circ (id_2 \times f) \circ (\Delta \times id_1 \times id_{n-1}) \circ (\Delta \times id_{n-1})$$
$$= \lambda_2^* \circ (id_1 \times \lambda_2^*) \circ (i^* \times id_2) \circ (\Delta \times id_1) \circ (id_1 \times f) \circ (\Delta \times id_{n-1})$$

Since $G^{-1}(\lambda_3)$ is contractible, there is a path from $\lambda_2^* \circ (id_1 \times \lambda_2^*)$ to $\lambda_2^* \circ (\lambda_2^* \times id_1)$, so that $\beta_k \circ \alpha_k$ is homotopic to

$$f \to \lambda_2^* \circ (\lambda_2^* \times id_1) \circ (i^* \times id_2) \circ (\Delta \times id_1) \circ (id_1 \times f) \circ (\Delta \times id_{n-1})$$

Since $G(\lambda_2^* \circ (i^* \times id_1) \circ \Delta) = \lambda_2 \circ (i \times id_1) \circ \Delta = \lambda_0 \circ 0_1$, where 0_1 is the set operation $1 \to 0$ induced by $\varnothing \to \{1\}$, and since G is bijective on path components, there is a path from $\lambda_2^* \circ (i^* \times id_1) \circ \Delta$ to $\lambda_0^* \circ 0_1$ in $mor\,\Psi$. Hence $\beta_k \circ \alpha_k$ is homotopic to

$$f \to \lambda_2^* \circ (\lambda_0^* \circ 0_1 \times id_1) \circ (id_1 \times f) \circ (\Delta \times id_{n-1})$$

By a similar argument, there is a path from $\lambda_2^* \circ (\lambda_0^* \circ 0_1 \times id)$ to the projection $pr_2\colon 2 \to 1$ onto the second factor. Hence $\beta_k \circ \alpha_k$ is homotopic to

$$f \to pr_2 \circ (id_1 \times f) \circ (\Delta \times id_{n-1}) = f$$

so that $\beta_k \circ \alpha_k \simeq id$. Similarly, one shows that $\alpha_k \circ \beta_k \simeq id$.

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