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Goussarov–Habiro theory for string links and the Milnor–Johnson correspondence

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Abstract

We study the Goussarov–Habiro finite type invariants theory for framed string links in homology balls. Their degree 1 invariants are computed: they are given by Milnor's triple linking numbers, the mod 2 reduction of the Sato–Levine invariant, Arf and Rochlin's μ invariant. These invariants are seen to be naturally related to invariants of homology cylinders through the Milnor–Johnson correspondence: in particular, an analogue of the Birman–Craggs homomorphism for string links is computed. The relation with Vassiliev theory is studied.

Keywords: Finite type invariant; String link; Homology cylinder; Clasper

1. Motivations

In the late 90's, M. Goussarov and K. Habiro independently developed a finite type invariant theory for compact oriented 3-manifolds. The theory makes use of an efficient surgical calculus machinery called *calculus of claspers* [6,3,8]. In particular the Y_k*-equivalence*, an equivalence relation for 3-manifolds arising from calculus of claspers, plays an important role in the understanding of the invariants.

Though it is also well-defined for manifolds with links, this aspect of the theory remains so far almost non-existing in the literature. In the present paper, we study the case of framed n-string links in homology balls. For $n = 1$, this is equivalent to studying homology spheres with framed knots. String links play an important role in the study of knots and links [10] and happen to have nice properties in the theory of claspers. Here, we compute explicitly the degree 1 invariants (in the Goussarov–Habiro sense) for framed string links in homology balls, using some versions of classical invariants, such as Milnor numbers, Sato–Levine, Arf and Rochlin invariants. This is the outcome of the characterization of the Y_2-equivalence relation for these objects.

String links are very closely related to *homology cylinders* [4,14]. Homology cylinders over a compact connected oriented surface Σ can be seen as a generalization of the Torelli group of Σ. G. Massuyeau and the author explicitely computed their degree 1 invariants [19]; they are given by the natural extensions of the *first Johnson homomorphism* and the *Birman–Craggs homomorphism*, initially defined for the Torelli group [2,12,13]. On the other hand,

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N. Habegger showed in [9] how homology cylinders are geometrically related to string links in homology balls, such that the extension of the first Johnson homomorphism agrees with Milnor's triple linking numbers. So the problem which naturally arises is to compute, likewise, the analogue of the Birman–Craggs homomorphism for this so-called Milnor–Johnson correspondence. Our computation of degree 1 invariants of string links in homology balls allows us to answer this question.

Like Goussarov–Habiro theory, the Vassiliev theory for (classical) string links can be defined using claspers. This viewpoint allows us to compare both theories. More precisely, we can relate the computation of degree 1 invariants of string links in homology balls to an analogous results obtained by the author on Vassiliev invariants [21]. We also consider the link case, where a similar statement exists [27].

The paper is organized as follows. We will begin with some necessary preliminary material on clasper theory. We compute in Section 3 the Goussarov–Habiro degree 1 invariants for framed string links in homology balls. Section 3.3 is devoted to the proof of this result, and Section 3.2 contains a precise definition of the invariants it involves. In Section 4, we introduce homology cylinders and study the Milnor–Johnson correspondence. The last section deals with Vassiliev invariants.

2. Preliminaries

Throughout this paper, all 3-manifolds will be supposed to be compact, connected and oriented.

2.1. A brief review of the Goussarov–Habiro theory

Let us briefly recall from [8,3,6] the basic notions of clasper theory for 3-manifolds with links.

Definition 1. Let γ be a n-component link in a 3-manifold M. A *clasper* G for (M,γ) is the embedding

$$G : F \to M$$

of a surface F which is the thickening of a (non-necessarily connected) unitrivalent graph having a copy of S^1 attached to each of its univalent vertices. G is disjoint from the link γ.

The (thickened) circles are called the *leaves* of G, the trivalent vertices are called the *nodes* of G and we still call the thickened edges of the graph the *edges* of G.

In particular, a *tree clasper* is a connected clasper obtained from the thickening of a simply connected unitrivalent graph (with circles attached).

The *degree* of a clasper G is the minimal number of nodes of its connected components.

A clasper G for (M,γ) is the instruction for a modification on this pair. There is indeed a precise procedure to construct, in a regular neighbourhood $N(G)$ of the clasper, an associated framed link L_G. The *surgery along the clasper* G is defined to be surgery along L_G. Though the procedure for the construction of L_G will not be explained here, it is well illustrated by the two examples of Fig. 1.[1]

We respectively call these two particular types of claspers Y-*graphs* and H-*graphs*.

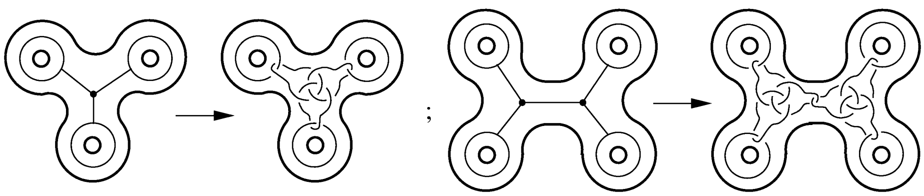

Fig. 1. A degree 1 and a degree 2 clasper and the associated framed links in their regular neighbourhoods.

[1] Here and throughout this paper, blackboard framing convention is used.

We denote by $(M,\gamma)_G = (M_G, \gamma_G)$ the result of a surgery move on (M,γ) along a clasper G:

$$\begin{cases} M_G = (M \setminus \mathrm{int}(N(G_0))) \cup_\partial N(G_0)_{L_G}, \\ \gamma_G \text{ is the link in } M_G \text{ defined by } \gamma \subset M \setminus \mathrm{int}(N(G_0)) \subset M_G. \end{cases}$$

Definition 2. Let $k \geqslant 1$ be an integer, and γ be a link in a 3-manifold M. A surgery move on (M,γ) along a connected clasper G of degree k is called a *Y_k-move*.

The *Y_k-equivalence*, denoted by $\sim_{Y_k}$, is the equivalence relation on 3-manifolds with links generated by the Y_k-moves and orientation-preserving diffeomorphisms (with respect to the boundary).

Note that Y_1-moves originally appear in [20] under the name of *Borromean surgery* (as Fig. 1 suggests). The next proposition outlines a couple of key facts about this equivalence relation.

Proposition 3.

(1) *Tree claspers do suffice to define the Y_k-equivalence.*
(2) *If $1 \leqslant k \leqslant n$, the Y_n-equivalence relation implies the Y_k-equivalence.*

We conclude this section with the definition of the Goussarov–Habiro theory, based on the notion of clasper. Consider a link γ_0 in a 3-manifold M_0, and the Y_1-equivalence class $\mathcal{M}_0$ of (M_0, γ_0).

Definition 4. Let A be an Abelian group, and $k \geqslant 0$ be an integer. A *finite type invariant of degree k* (in the Goussarov–Habiro sense) on $\mathcal{M}_0$ is a map $f : \mathcal{M}_0 \to A$ such that, for all $(M,\gamma) \in \mathcal{M}_0$ and all family $F = \{G_1, \ldots, G_{k+1}\}$ of $(k+1)$ disjoint Y-graphs for (M,γ), the following equality holds:

$$\sum_{F' \subseteq F} (-1)^{|F'|} f\big((M,\gamma)_{F'}\big) = 0.$$

2.2. Vassiliev theory using claspers

Another aspect of the theory of claspers is that it allows to redefine and study Vassiliev invariants of knots and links in a *fixed* manifold [8,7]. Here, for simplicity, we recall the definitions for the case of knots in S^3. For more about Vassiliev invariants, see [1].

Definition 5. Let K be a knot in S^3. A *clasper G* for K is the embedding

$$G : F \to S^3$$

of a surface F which is the planar thickening of a unitrivalent *tree* (a graph without loops). The (thickened) 1-vertices are called the *disk-leaves* of G, and the thickened trivalent vertices and edges of the graph are still called *nodes* and *edges* respectively. K is disjoint from G, except from the disk-leaves which it may intersect transversely once.

The *C-degree* of a connected clasper G is the number of nodes *plus* 1.

Again, a clasper G for K is the instruction for a surgical modification: it maps K to a new knot K_G in S^3. Examples are given for low C-degrees in Fig. 2.

Definition 6. Let $k \geqslant 1$ be an integer, and K be a knot in S^3. A surgery move on K along a connected C-degree k clasper G is called a *C_k-move*.

The *C_k-equivalence*, denoted by $\sim_{C_k}$, is the equivalence relation on knots generated by the C_k-moves and isotopies.

As in Proposition 3(2), the C_n-equivalence relation implies the C_k-equivalence if $1 \leqslant k \leqslant n$.

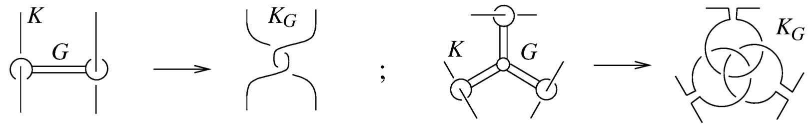

Fig. 2. A C_1-move and a C_2-move.

Remark 7.

(1) Note that a C_1-move is just a crossing change. As [22, Fig. 2.2] shows, a C_2-move is equivalent to a Δ-move. Moreover, a C_3-move is equivalent to a clasp-pass move (see Section 5 for a definition) [8].
(2) The C_{k+1}-equivalence implies the Y_k-equivalence, for all $K \geqslant 1$. More precisely, a C_{k+1}-move can be regarded as a special case of Y_k-move, where the leaves of the degree k clasper are (0-framed) copies of the meridian of the knot.

A C_1-move being equivalent to a crossing change, we can reformulate the notion of Vassiliev invariant in terms of claspers.

Definition 8. Let A be an Abelian group, and $k \geqslant 0$ be an integer. An A-valued knot invariant v is a *Vassiliev invariant of degree* k if, for all knot K and all family $F = \{C_1, \ldots, C_{k+1}\}$ of $(k+1)$ disjoint C-degree 1 claspers for K, the following equality holds:

$$\sum_{F' \subseteq F} (-1)^{|F'|} v(K_{F'}) = 0.$$

3. Goussarov–Habiro theory for string links in homology balls

Here and throughout the paper, unless said otherwise, by homology we mean *integral* homology. Thus by homology ball we mean a compact oriented 3-manifold whose integral homology groups are isomorphic to those of the 3-ball.

3.1. String links in homology balls

3.1.1. Definition and properties

Let D^2 be the standard two-dimensional disk, and $x_1, \ldots, x_n$ be n marked points in the interior of D^2.

Definition 9. An *n-component string link* in a homology ball M, also called *n-string link*, is a proper, smooth embedding

$$\sigma : \bigsqcup_{i=1}^{n} I_i \to M$$

of n disjoint copies I_i of the unit interval such that, for each i, the image σ_i of I_i runs from $(x_i, 0)$ to $(x_i, 1)$ *via* the identification $\partial M = \partial(D^2 \times I)$.

σ_i is called the ith *string of* σ. It is equipped with an (upward) orientation induced by the natural orientation of I.

A *framed n-string link* in M is a string link equipped with an isotopy class of non-singular sections of its normal bundle, whose restriction to the boundary is fixed.

We denote by $\mathcal{SL}^{hb}(n)$ the set of framed n-string links in homology balls, considered up to diffeomorphisms relative to the boundary (that is, up to diffeomorphisms whose restriction to the boundary is the identity).

Given two elements (M, σ) and (M', σ') of $\mathcal{SL}^{hb}(n)$, we can define their *product* as follows. Denote by $M \cdot M'$ the homology ball obtained by identifying $\Sigma \times \{1\} \subset \partial M$ and $\Sigma \times \{0\} \subset \partial M'$. $(M, \sigma) \cdot (M', \sigma')$ is defined by stacking σ' over σ in $M \cdot M'$. An example is given in Fig. 3.

This product induces a monoid structure on $\mathcal{SL}^{hb}(n)$, with $(D^2 \times I, 1_n)$ as unit element. Here 1_n is the trivial n-string link.

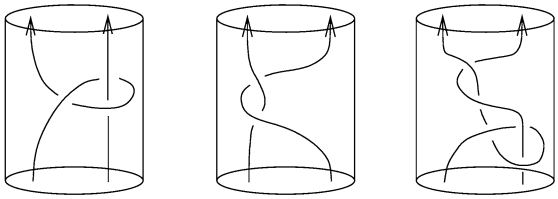

Fig. 3. Two 2-string links in $D^2 \times I$, and their product.

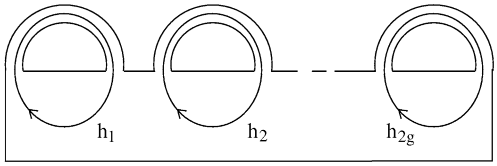

Fig. 4.

Notations 10. Throughout this paper, the notation 1_{D^2} will be often used for the product $D^2 \times I$.

D^2_n will denote the n-punctured disk $D^2 \setminus \{x_1, \ldots, x_n\}$. $H := H_1(D^2_n, \mathbf{Z})$ will denote its first integral homology group, and $H_{(2)} := H_1(D^2_n, \mathbf{Z_2})$.

$\mathcal{B} = \{e_1, \ldots, e_n\}$ denotes the basis of H induced by the n curves $h_1, h_2, \ldots, h_n$ of D^2_n shown in Fig. 4.

Similarly, $\mathcal{B}_{(2)} = \{\overline{e_1}, \ldots, \overline{e_n}\}$ is the associated basis of $H_{(2)}$.

Let $(M, \sigma) \in \mathcal{SL}^{hb}(n)$. We denote by $\widehat{M}$ the homology sphere obtained by pasting a copy of $(D^2 \times I)$ along its boundary, *via* the identification $\partial M = \partial(D^2 \times I)$. At the string links level, suitably pasting a copy of $(1_{D^2}, 1_n)$ along the boundary of M maps $\sigma \subset M$ to a framed oriented n-component link $\hat{\sigma} \subset \widehat{M}$. $(\widehat{M}, \hat{\sigma})$ is called the *closure* of (M, σ). In particular, for $M = 1_{D^2}$, it is the usual notion of closure for σ as defined in [10].

Given an element (M, σ) of $\mathcal{SL}^{hb}(n)$, let us denote by $T(\sigma)$ a tubular neighbourhood of σ. We denote by $M^\sigma := M \setminus T(\sigma)$ the exterior of the string link: the boundary of M^σ is identified with $\partial(D^2_n \times I)$. Let i_ε ($\varepsilon = 0, 1$) be the embeddings

$$i_\varepsilon : D^2_n \to D^2_n \times \{\varepsilon\} \subset M^\sigma.$$

We need the following classical result of Stallings.

Theorem 11. [26, Theorem 5.1] *Let $f : A \to B$ be a map between connected* CW*-complexes that induces an isomorphism on the first homology groups and a surjective homomorphism on the second homology groups. Then for all $k \geqslant 2$, f induces an isomorphism at the level of each nilpotent quotient of the fundamental group*

$$f_k : \frac{\pi_1(A)}{(\pi_1(A))_k} \xrightarrow{\simeq} \frac{\pi_1(B)}{(\pi_1(B))_k},$$

where, for any group G, G_k is the kth term of its lower central series.

So by a standard Mayer–Vietoris calculation and the above theorem, the map i_ε ($\varepsilon = 0, 1$) induces an isomorphism

$$(i_\varepsilon)_k : \frac{\pi_1(D^2_n)}{(\pi_1(D^2_n))_k} = \frac{F}{F_k} \xrightarrow{\simeq} \frac{\pi_1(M^\sigma)}{(\pi_1(M^\sigma))_k},$$

for each $k \geqslant 2$, where F stands for the free group on n generators. So any element σ of $\mathcal{SL}^{hb}(n)$ induces an automorphism of F/F_{k+1}, called its kth *Artin representation*, defined by $\mathcal{A}_k(\sigma) = (i_1)^{-1}_{k+1} \circ (i_0)_{k+1}$.

Actually, $\mathcal{A}_k(\sigma)$ conjugates each generator x_i of F/F_{k+1} by λ_i, the ith *longitude of* σ mod F_{k+1}: the framing on σ defines a curve in M^σ parallel to σ_i, which determines an element of $\pi_1(M^\sigma)$. The image in F/F_{k+1} of this element by $(i_1)^{-1}_{k+1}$ is λ_i.

Denote by $\mathcal{SL}^{hb}(n)[k] := \operatorname{Ker} \mathcal{A}_k$ the submonoid of all n-string links inducing the identity on F/F_{k+1}. Note that $\mathcal{SL}^{hb}(n) = \mathcal{SL}^{hb}(n)[1]$ and that $(M, \sigma) \in \mathcal{SL}^{hb}(n)[2]$ if and only if σ has null-homologous longitudes, that is, vanishing framings and linking numbers [11].

3.1.2. Goussarov–Habiro theory for framed string links in homology balls

Denote by $\mathcal{SL}_k^{hb}(n)$ the submonoid of all elements $(M, \sigma) \in \mathcal{SL}^{hb}(n)$ which are Y_k-equivalent to $(1_{D^2}, 1_n)$. There is a descending filtration of monoids

$$\mathcal{SL}^{hb}(n) \supset \mathcal{SL}_1^{hb}(n) \supset \mathcal{SL}_2^{hb}(n) \supset \cdots$$

and for all $k \geqslant 1$, the quotient

$$\overline{\mathcal{SL}}_k^{hb}(n) := \mathcal{SL}_k^{hb}(n)/Y_{k+1}$$

is an Abelian group (this follows from standard calculus of claspers). This section is devoted to the study of the case $k = 1$. First, we identify the monoid $\mathcal{SL}_1^{hb}(n)$.

Proposition 12. *The elements of $\mathcal{SL}_1^{hb}(n)$ are those n-string links in homology balls with vanishing framings and linking numbers*:

$$\mathcal{SL}_1^{hb}(n) = \mathcal{SL}^{hb}(n)[2].$$

(The proof is postponed to the end of this section.) The next result characterizes the degree 1 Goussarov–Habiro finite type invariants for string links in homology balls.

Theorem 13. *Let (M, σ) and (M', σ') be two n-string links in homology balls with vanishing framings and linking numbers (i.e. two elements of $\mathcal{SL}_1^{hb}(n)$). The following assertions are equivalent*:

(a) *(M, σ) and (M', σ') are Y_2-equivalent*;
(b) *(M, σ) and (M', σ') are not distinguished by degree* 1 *Goussarov–Habiro finite type invariants*;
(c) *(M, σ) and (M', σ') are not distinguished by Milnor's triple linking numbers, nor the* mod 2 *reduction of the Sato–Levine invariant, the Arf invariant and Rochlin's μ-invariant.*

See Section 3.2 for the definitions of the above-mentioned invariants.

Remark 14. When considering higher degrees, the implication (a) ⇒ (b) remains true (as well as for knots and links in homology spheres). The converse implication is also true when $n = 1$, that is for knots in homology spheres (see [8]), and it is conjectural for string links with $n > 1$ components.

This conjecture is to be compared with [8, Conjecture 6.13], for Vassiliev invariants of (classical) string links (see also Section 5).

The proof of the theorem is given in Section 3.3. It consists in computing the Abelian group $\overline{\mathcal{SL}}_1^{hb}(n)$, in a *graphical* way. More precisely, we will define in Section 3.3.1 a Y-shaped diagrams space $\mathcal{A}_1(P_n)$ and a surjective surgery map $\mathcal{A}_1(P_n) \overrightarrow{\varphi_1} \overline{\mathcal{SL}}_1^{hb}(n)$. We will see that ψ turns out to be an isomorphism, with inverse induced by the invariants listed in Theorem 13.

3.1.3. Y_1-equivalence for string links: proof of Proposition 12

We first prove the inclusion $\mathcal{SL}_1^{hb}(n) \subset \mathcal{SL}^{hb}(n)[2]$: any element of $\mathcal{SL}^{hb}(n)$ obtained from $(1_{D^2}, 1_n)$ by a finite sequence of Y_1-moves has null homologous longitudes. It suffices to show that, if (M_2, σ_2) is obtained from $(M_1, \sigma_1) \in \mathcal{SL}^{hb}(n)$ by surgery along a degree 1 clasper G, these elements have homologous longitudes. Denote by $M_i^{\sigma_i}$ the exterior of the string links ($i = 1, 2$). We have

$$M_2^{\sigma_2} \cong \left(M_1^{\sigma_1}\right) \setminus \operatorname{int}\bigl(N(G)\bigr) \cup_{j|_\partial \circ h} (H_3),$$

where $j : H_3 \hookrightarrow 1_{D^2} \setminus 1_n$ is the embedding of a genus 3 handlebody onto a regular neighbourhood $N(G)$ of G, and where h is an element of the Torelli group of $\Sigma_3 = \partial H_3$—see [17, Lemma 1] for an explicit description of this diffeomorphism. h induces the identity on $\pi_1(\Sigma_3)/\pi_1(\Sigma_3)_2$: it follows (by a Van Kampen type argument) that there is an isomorphism

$$\frac{\pi_1(M_1^\sigma)}{(\pi_1(M_1^\sigma))_2} \xrightarrow{\simeq} \frac{\pi_1(M_2^{\sigma'})}{(\pi_1(M_2^{\sigma'}))_2},$$

which is compatible with the maps i_ε; $\varepsilon = 0, 1$. The assertion follows.

The other inclusion is essentially due to N. Habegger [9]. First, recall that every homology sphere is Y_1-equivalent to the 3-sphere S^3 [20,8]; likewise every homology ball is Y_1-equivalent to $B^3 \cong D^2 \times I$. So it suffices to show that a framed string link σ in $D^2 \times I$ whose framings and linking numbers are all zero is Y_1-equivalent to $(1_{D^2}, 1_n)$. By a sequence of connected sums on σ with copies of the 0-framed Borromean link, we can furthermore suppose that all Milnor's triple linking numbers are zero: such connected sums are nothing else but Y_1-moves (each leaf of the clasper being a meridian of the string on which connected sum is performed). By [15, Theorem D], σ is thus *surgery equivalent* to the trivial string link, that is, σ is obtained from 1_n by a sequence of surgeries on trivial (± 1)-framed knots K_i in the exterior of σ, these knots having vanishing linking numbers with σ. The union $\bigcup_i K_i$ is a (± 1)-framed boundary link: surgery on such a link is known to be equivalent to a sequence of Y_1-surgeries [9, Corollary 6.2].

3.2. Classical invariants for string links in homology balls

3.2.1. Rochlin's μ-invariant

Let M be a closed 3-manifold endowed with a spin structure s, and let (W, S) be a compact spin 4-manifold spin-bounded by (M, s) (that is, $\partial W = M$ and S coincides with s on M). Then, the *modulo* 16 signature $\sigma(W)$ of W is a well-defined closed spin 3-manifolds invariant $R(M, s)$, called the *Rochlin invariant* of M. In the case of homology spheres, there is a unique spin structure s_0, and $R(M, s_0)$ is divisible by 8:

$$\mu(M) := \frac{R(M, s_0)}{8} \in \mathbf{Z}_2$$

is an invariant of homology spheres called *Rochlin's μ-invariant*.

For elements (M, σ) of $\mathcal{SL}^{hb}(n)$, we set

$$R(M, \sigma) := \mu(\widehat{M}),$$

where the homology sphere $\widehat{M}$ is the closure of M as defined in Section 3.1. The following result of G. Massuyeau implies that the restriction of R to $\mathcal{SL}_1^{hb}(n)$ factors to a homomorphism of Abelian groups

$$R : \overline{\mathcal{SL}}_1^{hb}(n) \to \mathbf{Z}_2.$$

Proposition 15. [17, Corollary 1] *Rochlin's invariant is a degree* 1 *finite type invariant* (*in the Goussarov–Habiro sense*) *of integral homology spheres.*

3.2.2. Milnor invariants

Let σ be an n-string link in a homology ball M. Recall from Section 3.1 that F is the free group on n generators, and that F_k is the kth term of its lower central series. Recall also that $\lambda_i \in F/F_{k+1}$ denotes the ith longitude of σ mod F_{k+1}.

Denote by $P(n)$ the ring of power series in the non-commutative variables $X_1, \dots, X_n$. The *Magnus expansion* [16] $F \to P(n)$ is a group homomorphism which maps each generator x_i of F to $1 + X_i$.

Definition 16. The *Milnor's μ-invariant of length l*, $\mu_{i_1 \dots i_l}$ of σ is the coefficient of the monomial $X_{i_1} \dots X_{i_{l-1}}$ in the Magnus expansion of the longitude $\lambda_{i_l} \in F/F_k$ for a certain $k \geqslant l$.

For example, Milnor's invariants of length 2 are just the linking numbers. Here, we deal with Milnor's invariants of length 3, also called *Milnor's triple linking number*. The following proposition-definition follows from Lemma 19 below.

Proposition 17. *For all $i < j < k \in \{1, \dots, n\}$, there is a well-defined homomorphism of Abelian groups*

$$\overline{\mathcal{SL}}_1^{hb}(n) \xrightarrow{\mu_{ijk}} \mathbf{Z}$$

induced by Milnor's triple linking number.

Remark 18. In general, Milnor's triple linking numbers are not additive on $\mathcal{SL}(n)$. The homomorphism defect is given by linking numbers, so it vanishes for elements of $\mathcal{SL}_1^{hb}(n)$.

Lemma 19. *Let (M, σ) be a framed string link in a homology ball. Let also G be a degree 2 clasper in M disjoint from σ and let (M_G, σ_G) be the result of the surgery along G. Then, there exists an isomorphism*

$$\frac{\pi_1(M^\sigma)}{(\pi_1(M^\sigma))_3} \xrightarrow{\simeq} \frac{\pi_1(M_G^\sigma)}{(\pi_1(M_G^\sigma))_3}$$

compatible with the inclusions i_ε; $\varepsilon = 0, 1$.

Proof. The reader is refered to the proof of [19, Lemma 3.13]. □

3.2.3. The Arf invariant

Let K be a knot in a homology sphere M, and S be a Seifert surface for K of genus g. Denote by $\centerdot$ the *mod* 2 *reduction* of the homological intersection form on $H_1(S, \mathbf{Z}_2)$. Let $\delta_2 : H_1(S, \mathbf{Z}_2) \to \mathbf{Z}_2$ be the map defined by

$$\delta_2(\alpha) = lk(\alpha, \alpha^+) \text{ (mod 2)},$$

where α^+ is a parallel copy of α in the positive normal sense of S (for a fixed orientation of M). δ_2 is a quadratic form with $\centerdot$ as associated bilinear form: the *Arf invariant of the knot* K [24] is the Arf invariant of δ_2, that is, for a given symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ for $\centerdot$

$$Arf(K) = Arf(\delta_2) = \sum_{i=1}^{g} \delta_2(a_i)\delta_2(b_i).$$

Remark 20. The fact that the Arf invariant is still well-defined for knots in homology spheres essentially follows from the following fact (see for example [5] for a proof): two Seifert surfaces S_0 and S_1 for a knot K in an homology sphere M are related by a sequence of isotopies, additions and removals of tubes $S^1 \times I$. Indeed, as we will see in the proof of Proposition 22, such tubes do not contribute to the Arf invariant.

For elements of $\mathcal{SL}_1^{hb}(n)$, the Arf invariant is defined in the obvious way: for an integer $1 \leqslant i \leqslant n$, denote by $a_i(M, \sigma)$ the *Arf* invariant of $\hat{\sigma}_i$, the ith component of the link $\hat{\sigma} \in \widehat{M}$. We clearly have the following proposition-definition:

Proposition 21. *For any integer $1 \leqslant i \leqslant n$, the map $a_i : \mathcal{SL}_1^{hb}(n) \to \mathbf{Z}_2$ is a homomorphism of monoids, called the i*th Arf invariant of (M, σ).

Further, this invariant happens to behave well under a Y_2-move.

Proposition 22. *The Arf invariant of knots in homology spheres is invariant under a Y_2-move.*

*As a consequence, for any $1 \leqslant i \leqslant n$, the i*th *Arf invariant of string links in homology balls factors through a homomorphism of Abelian groups*

$$a_i : \overline{\mathcal{SL}}_1^{hb}(n) \to \mathbf{Z}_2.$$

Proof. Let K be a knot in a homology sphere M, and let S be a Seifert surface for K. Let G be a degree 2 clasper for (M, K); thanks to Proposition 3(2), we can suppose that G is a H-graph. It suffices to show that $Arf(M, K) = Arf(M_G, K_G) \in \mathbf{Z}_2$.

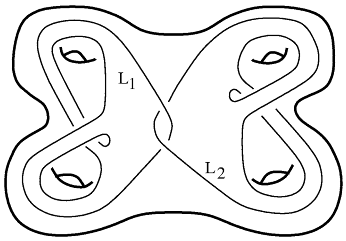

Fig. 5. The 2-component link L.

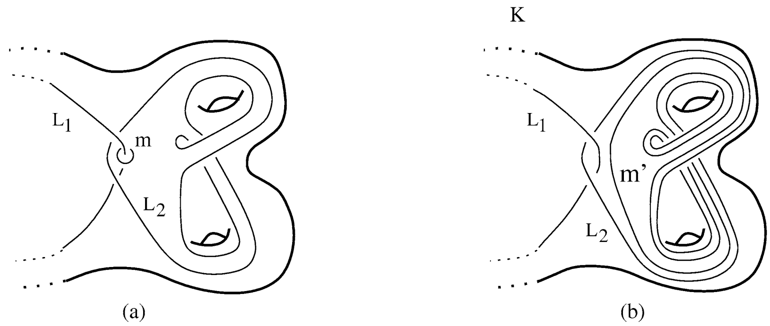

Fig. 6.

Denote by N a regular neighbourhood of G, which is a genus 4 handlebody. The 10-component surgery link associated to G, depicted in Fig. 1, is Kirby-equivalent to the 2-component link L depicted in Fig. 5. This can be checked by using moves 2, 9 and 1 of [8, Proposition 2.7] (see also [14, p. 254]).

K being disjoint from G, we can suppose that it is also disjoint from N. But S may intersect N and the knot K. We construct a new Seifert surface S' for L, satisfying $S' \cap L = \emptyset$, by adding tubes $S^1 \times I$ to S in N: these tubes are portions of (parallel copies of) a tubular neighbourhood of the link L. The general procedure for constructing S' is explained in Appendix A. S' can be seen in $M \setminus L$, and thus in the surgered manifold M_G.

Now observe that such an addition of tube does not affect the Arf invariant of K: if we denote by (m, l) a meridian/longitude pair for this tube, we have indeed $\delta_2(m) = 0$, such a meridian m having vanishing self-linking.

We must also show that this pair does not contribute to the Arf invariant of $(K)_G$. In other words, if we denote by (m', l') the image of (m, l) after surgery on L, we must show that $\delta_2(m')\delta_2(l') = 0$. Observe that the meridian m can be isotoped in a small ball B of N where the crossing between L_1 and L_2 occurs—see Fig. 6(a). Thus, surgery on L sends m to a curve m', which is a parallel copy of L_2 outside of B, as shown in Fig. 6(b): we have $\delta_2(m') = lk(m', (m')^+) = 0$. □

3.2.4. The Sato–Levine invariant

Let $L = L_1 \cup L_2$ be a 2-component oriented link such that $lk(L_1, L_2) = 0$. The components of L bound some Seifert surfaces S_1 and S_2 such that $L_1 \cap S_2 = L_2 \cap S_1 = \emptyset$. S_1 and S_2 intersect along circles $S_1 \cap S_2 = C_1 \cup \cdots \cup C_n = C$. The self-linking of C relative to any of both surfaces is called the *Sato–Levine invariant* of L [25]:

$$\beta(L) = lk\big(C, C^+\big).$$

The fact that β is still well-defined for links in homology spheres is again a consequence of the fact recalled in Remark 20. Indeed, if we add a tube t to (say) S_1, it will only intersect S_2 along copies of a meridian of t (up to isotopy): such a meridian has vanishing self-linking number and links no other component of $S_1 \cap S_2$.

The Sato–Levine invariant can also be defined for elements (M, σ) of $\mathcal{SL}_1^{hb}(n)$. For any pair of integers (i, j) such that $1 \leqslant i < j \leqslant n$, we denote by $\beta_{ij}(M, \sigma)$ the Sato–Levine invariant of the 2-component link of $\widehat{M}$ obtained by closing the ith and jth components of σ: $\beta_{ij}(M, \sigma) := \beta(\hat{\sigma_i} \cup \hat{\sigma_j})$.

Note that this makes sense by Proposition 12, as elements of $\mathcal{SL}_1^{hb}(n)$ have vanishing linking numbers. Moreover, β_{ij} is additive.

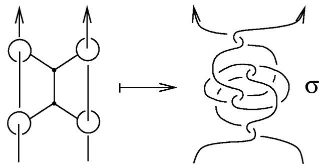

Fig. 7.

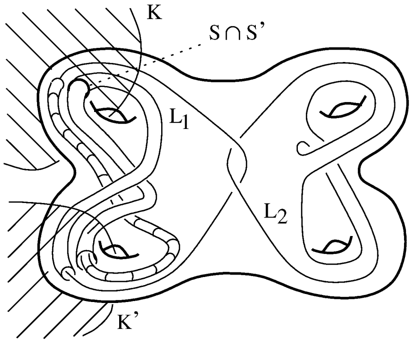

Fig. 8.

Proposition 23. $\forall\ 1 \leqslant i < j \leqslant n$, *the map* $\beta_{ij} : \mathcal{SL}_1^{hb}(n) \to \mathbf{Z}$ *is a homomorphism of monoids, called the* Sato–Levine invariant β_{ij}.

Note that the Sato–Levine invariant is not invariant under Y_2-moves: for example, it takes value 2 on the string link σ depicted below, obtained by surgery on $(1_{D^2}, 1_n)$ along a H graph whose leaves are meridians of 1_n as depicted in Fig. 7. But it turns out that it is the case for its mod 2 reduction.

Proposition 24. *The* mod 2 *reduction of the Sato–Levine invariant of links in homology spheres is invariant under a* Y_2*-move.*

In particular, for any $1 \leqslant i < j \leqslant n$*, the Sato–Levine invariant* $\beta_{i,j}$ *of string links in homology balls factors through a homomorphism of Abelian groups*

$$\beta_{ij}^{(2)} : \overline{\mathcal{SL}}_1^{hb}(n) \to \mathbf{Z}_2.$$

Proof. Let $K \cup K'$ be a 2-component oriented link with linking number 0 in a homology sphere M. Let G be a degree 2 clasper for $(M, K \cup K')$ (which, as in the preceding proof, can be supposed to be a H-graph), and N be a regular neighbourhood of G. We must show that

$$\beta^{(2)}(M, K \cup K') = \beta^{(2)}(M_G, K_G \cup K'_G) \in \mathbf{Z}_2.$$

We denote respectively by S and S' a Seifert surface for K and K': $S \cap S' = C_1 \cup \cdots \cup C_n = C$. Consider in N the 2-component surgery link $L = L_1 \cup L_2$ associated to G depicted in Fig. 5. K and K' are supposed to be disjoint from N, but S and S' may intersect N (and thus L).

When S (respectively S') intersects L, we add some tubes to built a new Seifert surface for K (respectively K'), which is disjoint from L. The procedure for such an addition of tube is the same as the procedure explained in Appendix A for a knot. We denote by $\widetilde{C}$ the set of elements of $S \cap S'$ which are possibly created (in N) under this addition of tube: $S \cap S' = C \cup \widetilde{C}$. A simple example of such a situation is given in Fig. 8.

Clearly, $\widetilde{C}$ is a finite number of copies of small meridians of L_1 and L_2. We clearly have $lk(\widetilde{C}, C^+) = lk(C, \widetilde{C}^+) = lk(\widetilde{C}, \widetilde{C}^+) = 0$. It remains to prove that, *after surgery* along L, the elements of $\widetilde{C} \subset S \cap S'$ do also not contribute to $\beta^{(2)}(K_G \cup K'_G)$.

- Suppose that $\widetilde{C} = \{m\}$, where m is a meridian of any of both components. Denote by c its image after surgery on G: as seen in the proof of Proposition 22, we have $lk(c, c^+) = 0$.

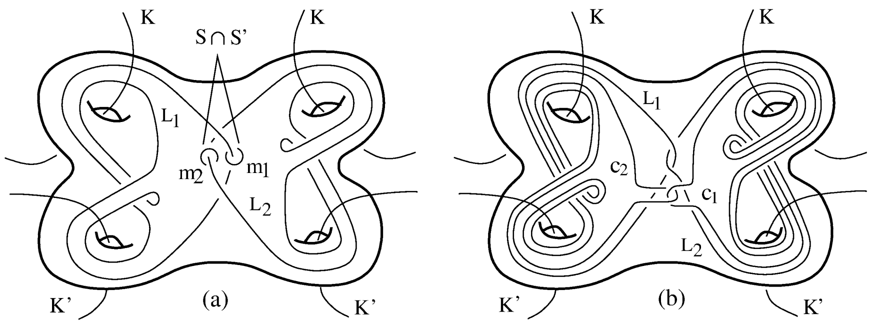

Fig. 9.

- Now, consider the case $\widetilde{C} = \{m_1, m_2\}$, a pair of meridians of L_1 and L_2. An example is given by the situation of Fig. 9(a).

Again, surgery on G sends (m_1, m_2) to a pair of curves (c_1, c_2), which are parallel copies of L_1 and L_2 outside of a ball of N where the crossing between L_1 and L_2 occurs—see Fig. 9(b). Thus, c_1 and c_2 satisfy

$$\begin{aligned} lk\big(c_1 \cup c_2, (c_1 \cup c_2)^+\big) &= lk\big(c_1, c_1^+\big) + lk\big(c_2, c_1^+\big) + lk\big(c_1, c_2^+\big) + lk\big(c_2, c_2^+\big) \\ &= 2.lk\big(c_1, c_2^+\big) = \pm 2. \end{aligned}$$

It follows that, in these two particular cases, the mod 2 reduction of β remains unchanged. The general case, where $\widetilde{C}$ consists in several copies of m_1 and m_2, is proven the same way. □

Remark 25. Note that a (less direct) proof of Proposition 24 can be given using a formula of K. Murasugi that expresses the modulo 2 reduction of the Sato–Levine invariant of a link in terms of its *Arf* invariants [23]. Indeed, the *Arf* invariant of a link can be expressed as the Arf invariant of a knot *related* to L, that is (roughly) obtained by performing a connected sum of its components along some band [24]. The result then follows from Proposition 22.

3.3. Degree 1 invariants for string links: proof of Theorem 13

As announced in Section 3.1.2, the proof of Theorem 13 consists in computing the Abelian group $\overline{\mathcal{SL}}_1^{hb}(n)$. This computation goes in two steps. First we will construct a combinatorial upper bound, by defining a surjective homomorphism $\varphi_1 : \mathcal{A}_1(P_n) \to \overline{\mathcal{SL}}_1^{hb}(n)$, where $\mathcal{A}_1(P_n)$ is a space of diagram. Second, we will show that ψ is actually an isomorphism, with inverse given by the invariants listed in Theorem 13.

The development of the proof, and the objects it involves, are similar to those used in the proof of [19, Theorem 1.4]. We will recall and use several material and facts presented in the latter, to which the reader is refered for more details.

3.3.1. Combinatorial upper bound

Let P_n denote the Abelian group $H \oplus \mathbf{Z}_2$. We denote by $\mathcal{A}_1(P_n)$ the free Abelian group generated by Y-shaped unitrivalent graphs, whose trivalent vertex is equipped with a cyclic order on the incident edges and whose univalent vertices are labelled by P_n, subject to the two following relations

Multilinearity: $\mathsf{Y}[z_0 \cdot z_1; z_2; z_3] = \mathsf{Y}[z_0; z_2; z_3] + \mathsf{Y}[z_1; z_2; z_3]$,

Slide: $\mathsf{Y}[z_1; z_1; z_2] = \mathsf{Y}[s; z_1; z_2]$,

where $z_0, z_1, z_2, z_3 \in P_n$. Here, the notation $\mathsf{Y}[z_1, z_2, z_3]$ stands for the graph whose univalent vertices are colored by z_1, z_2 and $z_3 \in P_n$ in accordance with the cyclic order. This notation is invariant under cyclic permutation of the z_i's.

Remark 26. Note that, as a consequence of the Multilinearity and Slide relations, the Antisymmetry relation

$$\mathsf{Y}[z_1; z_2; z_3] = -\mathsf{Y}[z_2; z_1; z_3]$$

holds in $\mathcal{A}_1(P_n)$—for example, apply the Slide relation to $\mathsf{Y}[z_1 + z_2; z_1 + z_2; z_3]$.

Consider the map

$$\rho : \mathcal{A}_1(P_n) \to \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$$

defined on the generators of $\mathcal{A}_1(P_n)$ by:

$$\rho\big(\mathsf{Y}\big[(e_i,0);(e_j,0);(e_k,0)\big]\big) = e_i \wedge e_j \wedge e_k \in \Lambda^3 H,$$
$$\rho\big(\mathsf{Y}\big[(e_i,0);(e_j,0);(0,1)\big]\big) = \overline{e_i} \wedge \overline{e_j} \in \Lambda^2 H_{(2)},$$
$$\rho\big(\mathsf{Y}\big[(e_i,0);(0,1);(0,1)\big]\big) = \overline{e_i} \in H_{(2)},$$
$$\rho\big(\mathsf{Y}\big[(0,1);(0,1);(0,1)\big]\big) = 1 \in \mathbf{Z}_2,$$

where $1 \leqslant i < j < k \leqslant n$, and where $(e_i)_i$ (respectively $(\overline{e_i})_i$) are the basis elements of H (respectively $H_{(2)}$) defined in Notations 10.

ρ is clearly well-defined and we actually have the following lemma.

Lemma 27. *The map ρ is an isomorphism.*

This is proved in the same way as [19, Lemma 4.24] (see also [18, Lemma 6.3]).

We now construct the surgery map

$$\varphi_1 : \mathcal{A}_1(P_n) \to \overline{\mathcal{SL}}_1^{hb}(n).$$

For each generator $\mathsf{Y} = \mathsf{Y}[z_1; z_2; z_3]$ of $\mathcal{A}_1(P_n)$, where $z_i := (h_i, \varepsilon_i) \in P_n$, we set

$$\varphi_1(\mathsf{Y}) := \big(D^2 \times I, 1_n\big)_{\phi(\mathsf{Y})},$$

where $\phi(\mathsf{Y})$ is a degree 1 connected clasper (a Y-graph) for $(D^2 \times I, 1_n)$ constructed from the informations contained in the diagram Y:

For $i \in \{1,2,3\}$, consider an oriented simple closed curve c_i in $D_n^2 \times \{1\} \subset D^2 \times I$ such that $[c_i] = h_i \in H$, framed along the surface. Then push this framed curves down in the interior of $(D^2 \times I) \setminus 1_n \cong (D_n^2 \times I)$, by adding a ε_i-twist. The resulting oriented framed knot is denoted by K_i. Next, pick an embedded 2-disk D in the interior of $D_n^2 \times I$ and disjoint from the K_i's, orient it in an arbitrary way, and connect it to the K_i's with some bands e_i. These band sums have to be compatible with the orientations, and to be coherent with the cyclic ordering $(1,2,3)$.

Proposition 28. *Let Y be a generator of $\mathcal{A}_1(P_n)$. The Y_2-equivalence class of $(D^2 \times I, 1_n)_{\phi(\mathsf{Y})}$ does not depend on the choice of $\phi(\mathsf{Y})$ (obtained by the above construction). Hence, we have a well-defined, surjective* surgery map

$$\mathcal{A}_1(P_n) \xrightarrow{\varphi_1} \overline{\mathcal{SL}}_1^{hb}(n).$$

The proof is strictly the same as the proof of [19, Theorem 2.11], and essentially uses the calculus of claspers. In particular, the independence on the choice of ϕ follows from facts similar to [3, Corollaries 4.2 and 4.3, Lemma 4.4].

3.3.2. Characterization of Y_2-equivalence for string links

Set $V := \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$, and let

$$\tau : \mathcal{SL}_1^{hb}(n) \to \Lambda^3 H_{(2)} \oplus V$$

be defined, for any $(M,\sigma) \in \mathcal{SL}_1^{hb}(n)$, by

$$\tau(M,\sigma) = \sum_{1 \leqslant i<j<k \leqslant n} \mu_{ijk}^{(2)}(M,\sigma).\overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k} + \sum_{1 \leqslant i<j \leqslant n} \beta_{ij}^{(2)}(M,\sigma).\overline{e_i} \wedge \overline{e_j}$$
$$+ \sum_{1 \leqslant i \leqslant n} a_i(M,\sigma).\overline{e_i} + R(M).$$

Here, $\mu_{ijk}^{(2)}$ denotes the mod 2 reduction of Milnor's triple linking number μ_{ijk}.

It follows from Propositions 15, 17, 22 and 24 that this well-defined map factors through a homomorphism of Abelian groups

$$\overline{\mathcal{SL}}_1^{hb}(n) \xrightarrow{\tau} \Lambda^3 H_{(2)} \oplus V.$$

Denote by T the composition

$$T : \mathcal{A}_1(P_n) \xrightarrow{\rho} \Lambda^3 H \oplus V \xrightarrow{-\otimes \mathbf{Z}_2} \Lambda^3 H_{(2)} \oplus V.$$

Lemma 29. *The following diagram commutes*

$$\begin{tikzcd} \mathcal{A}_1(P_n) \arrow[r, two heads, "\varphi_1"] \arrow[dr, "T"'] & \overline{\mathcal{SL}}_1^{hb}(n) \arrow[d, "\tau"] \\ & \Lambda^3 H_{(2)} \oplus V \end{tikzcd}$$

Proof. P is generated by $(0,1)$ and $(e_i, 0)$, $i = 1, \dots, n$. So, thanks to the Slide relation, there are four distinct types of generators Y for $\mathcal{A}_1(P_n)$, listed below ($1 \leqslant i < j < k \leqslant n$): we prove that, in these four cases, $\tau(\varphi_1(\mathsf{Y})) = T(\mathsf{Y})$.

(1) $\mathsf{Y} = \mathsf{Y}[(0,1); (0,1); (0,1)]$.

In this case, $T(\mathsf{Y}) = 1 \in \mathbf{Z}_2$. On the other hand, a representative for $\varphi_1(\mathsf{Y}) \in \overline{\mathcal{SL}}_1^{hb}(n)$ is $(1_{D^2}, 1_n)_G$, where G is contained in a ball disjoint from 1_n and its leaves are three copies of the (-1)-framed unknot. It follows that $(1_{D^2}, 1_n)_G \cong (P, 1_n)$, where the closure of P is the Poincaré sphere: $R(P, 1_n) = 1$. Moreover,

$$\mu_{rst}(P, 1_n) = \beta_{rs}^{(2)}(P, 1_n) = a_s(P, 1_n) = 0,$$

$\forall\, r \neq s \neq t \in \{1, \dots, n\}$. It follows that $\tau(P, 1_n) = 1 \in \mathbf{Z}_2$.

(2) $\mathsf{Y} = \mathsf{Y}[(e_i, 0); (e_i, 0); (e_i, 0)]$.

A representative for $\varphi_1(\mathsf{Y})$ is $(1_{D^2}, 1_n)_G$, where the three leaves of G are small meridians of the ith string $(1_n)_i$ of 1_n. Thus $(1_{D^2}, 1_n)_G \cong (1_{D^2}, T_i)$, where T_i only differs from 1_n by a copy of the trefoil on the ith string—see the Fig. 10(a). We have $a_r(1_{D^2}, T_i) = \delta_{r,i}$, and

$$\mu_{rst}(1_{D^2}, T_i) = \beta_{rs}^{(2)}(1_{D^2}, T_i) = R(1_{D^2}, T_i) = 0, \quad \forall (r, s, t).$$

It follows that $\tau \circ \varphi_1(\mathsf{Y}) = \overline{e_i} = T(\mathsf{Y})$.

(3) $\mathsf{Y} = \mathsf{Y}[(e_i, 0); (e_i, 0); (e_j, 0)]$.

A representative for $\varphi_1(\mathsf{Y})$ is obtained from $(1_{D^2}, 1_n)$ by surgery along a Y-graph G having two copies of a meridian of $(1_n)_i$ and one copy of a meridian of $(1_n)_j$ as leaves: $(1_{D^2}, 1_n)_G \cong (1_{D^2}, w_{ij})$, where the ith and jth strings of w_{ij} form a Whitehead link, see Fig. 10(b). The Sato–Levine invariant of the Whitehead link being 1, we obtain $\beta_{rs}^{(2)}(1_{D^2}, w_{ij}) = \delta_{(r,s),(i,j)}$, and

$$\mu_{rst}(1_{D^2}, w_{ij}) = a_r(1_{D^2}, w_{ij}) = R(1_{D^2}, w_{ij}) = 0, \quad \forall (r, s, t).$$

It follows that $\tau \circ \varphi_1(\mathsf{Y}) = \overline{e_i} \wedge \overline{e_j} \in \Lambda^2 H_{(2)}$, which coincides with $T(\mathsf{Y})$.

(4) $\mathsf{Y} = \mathsf{Y}[(e_i, 0); (e_j, 0); (e_k, 0)]$.

A representative for $\varphi_1(\mathsf{Y})$ is $(1_{D^2}, \sigma_{ijk})$, obtained from 1_n by performing a connected sum on strings σ_i, σ_j and σ_k with the three components of a Borromean ring, see Fig. 10(c). It follows that $\mu_{abc}(\sigma_{ijk}) = 1$ for $(a, b, c) = (i, j, k)$, and 0 otherwise. Moreover,

$$\beta_{rs}^{(2)}(1_{D^2}, \sigma_{ijk}) = a_r(1_{D^2}, \sigma_{ijk}) = R(1_{D^2}, \sigma_{ijk}) = 0, \quad \forall (r, s).$$

We thus obtain $\tau(\varphi_1(\mathsf{Y})) = \overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k} = T(\mathsf{Y})$, which completes the proof. $\square$

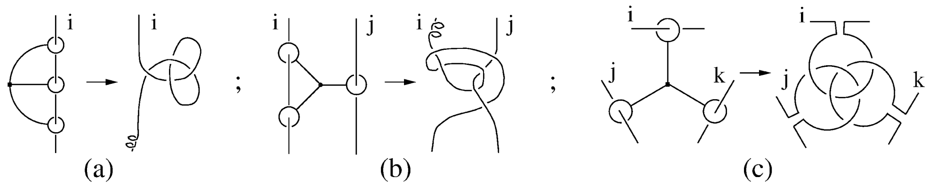

Fig. 10.

Furthermore, we can define by Proposition 17 a homomorphism of Abelian groups

$$\overline{\mathcal{SL}}_1^{hb}(n) \xrightarrow{\mu_3} \Lambda^3 H$$

by setting $\mu_3(M,\sigma) = \sum_{1\leqslant i<j<k\leqslant n} \mu_{ijk}(M,\sigma).e_i \wedge e_j \wedge e_k$.

The following lemma is a direct consequence of computations contained in the preceding proof (Case 4).

Lemma 30. *The following diagram commutes*

$$\begin{tikzcd} \mathcal{A}_1(P_n) \arrow[r, two heads, "\varphi_1"] \arrow[dr] & \overline{\mathcal{SL}}_1^{hb}(n) \arrow[d, "\mu_3"] \\ & \Lambda^3 H \end{tikzcd}$$

Lemmas 30 and 29 can then be summarized as follows.

Proposition 31. *The diagram*

$$\begin{tikzcd} \mathcal{A}_1(P_n) \arrow[r, "\varphi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{SL}}_1^{hb}(n) \arrow[d, "(\mu_3,\tau)"] \\ & \Lambda^3 H \oplus V \end{tikzcd}$$

commutes, and all of its arrows are isomorphisms.

More precisely, Lemmas 30 and 29 imply the commutativity. The fact that φ_1 (and thus (μ_3,τ)) is an isomorphism follows.

We are now ready to prove Theorem 13. Assertion (c) $\Rightarrow$ (a) is indeed a direct consequence of Proposition 31. As outlined in Remark 14, assertion (a) $\Rightarrow$ (b) is a general fact, which follows from the definition of a finite type invariant. Let us prove that (b) implies (c) by showing that in fact any homomorphism of Abelian groups $\overline{\mathcal{SL}}_1^{(hb)}(n) \xrightarrow{f} A$ gives a degree 1 invariant. Let (M,σ) be a n-string link in a homology ball and let G_1, G_2 be some disjoint Y-graphs for (M,σ). We aim to show that:

$$f(M,\sigma) - f\big((M,\sigma)_{G_1}\big) - f\big((M,\sigma)_{G_2}\big) + f\big((M,\sigma)_{G_1\cup G_2}\big) = 0. \tag{1}$$

Let G be a collection of disjoint Y-graphs for $(1_{D^2}, 1_n)$ such that $(M,\sigma) = (1_{D^2}, 1_n)_G$ (up to Y_2-equivalence). By possibly isotoping G_1 and G_2 in $M \setminus \sigma$, they are disjoint from G. We then put $(M_i,\sigma_i) = ((1_{D^2}, 1_n))_{G_i}$. Up to Y_2-equivalence, $(M,\sigma)_{G_i} = (M,\sigma)\cdot(M_i,\sigma_i)$ and $(M,\sigma)_{G_1\cup G_2} = (M,\sigma)\cdot(M_1,\sigma_1)\cdot(M_2,\sigma_2)$. Eq. (1) follows then from the additivity of f.

4. On the Milnor–Johnson correspondence

In this section, we study the relation between the Goussarov–Habiro theory for framed string links in homology balls and this theory for homology cylinders. Let us start with a short reminder on the latter.

4.1. Homology cylinders

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with 1 boundary component.

A *homology cylinder* M over $\Sigma_{g,1}$ is a homology cobordism with an extra homological triviality condition [4,8,14]. Alternatively, it can be defined as follows: a homology cylinder M over $\Sigma_{g,1}$ is a 3-manifold obtained from $\Sigma_{g,1}\times I$ by surgery along some claspers, that is, $M \sim_{Y_1} \Sigma_{g,1}\times I$.

The set of homology cylinders up to orientation-preserving diffeomorphisms is denoted by $\mathcal{HC}(\Sigma_{g,1})$. It is equipped with a structure of monoid, with product given by the stacking product and with $\Sigma_{g,1}\times I$ as unit element.

There is a descending filtration of monoids

$$\mathcal{HC}(\Sigma_{g,1}) = \mathcal{C}_1(\Sigma_{g,1}) \supset \mathcal{C}_2(\Sigma_{g,1}) \supset \cdots \supset \mathcal{C}_k(\Sigma_{g,1}) \supset \cdots$$

where $\mathcal{C}_k(\Sigma_{g,1})$ is the submonoid of all homology cylinders which are Y_k-equivalent to $1_{\Sigma_{g,1}}$. Moreover, as in the string link case, the quotient monoid $\bar{\mathcal{C}}_k(\Sigma_{g,1}) := \mathcal{C}_k(\Sigma_{g,1})/Y_{k+1}$ is an Abelian group for every $k \geqslant 1$.

As mentioned in [4,8], the Torelli group $\mathcal{T}_{g,1}$ of $\Sigma_{g,1}$ (the isotopy classes of self-diffeomorphisms of $\Sigma_{g,1}$ inducing an isomorphism in homology) naturally imbeds in $\mathcal{HC}(\Sigma_{g,1})$ via the mapping cylinder construction, and we can extend classical applications on the Torelli group to the realms of homology cylinders. In particular, we can extend the *first Johnson homomorphism* η_1 and the *Birman–Craggs homomorphism* β, originally used by D. Johnson in [12,13] for the computation of the Abelianized Torelli group. In [19], it is shown that these extensions actually are the degree 1 Goussarov–Habiro finite type invariants for homology cylinders.

Theorem 32. [19] *Let M and M' be two homology cylinders over $\Sigma_{g,1}$. The following assertions are equivalent*:

(a) *M and M' are Y_2-equivalent*;
(b) *M and M' are not distinguished by degree* 1 *Goussarov–Habiro finite type invariants*;
(c) *M and M' are not distinguished by the first Johnson homomorphism nor the Birman–Craggs homomorphism.*

This is proved, as in Section 3.3, by computing the Abelian group $\bar{\mathcal{C}}_1(\Sigma_{g,1})$ in a graphical way. More precisely, the authors define (in a strictly similar way) a space of diagrams $\mathcal{A}_1(P_{g,1})$ and a surjective surgery map $\mathcal{A}_1(P_{g,1}) \xrightarrow{\psi_1} \bar{\mathcal{C}}_1(\Sigma_{g,1})$, which actually is an isomorphism, with inverse given by η_1 and β.

4.2. From homology cylinders to string links

This result on homology cylinders over $\Sigma_{g,1}$ looks quite similar to Theorem 13 on framed n-string links in homology balls, and suggests a strong analogy between these objects.

This correspondence homology cylinders/string links has been studied by N. Habegger [9]: *via* a certain geometric construction relating these objects, Johnson homomorphisms coincides with Milnor's numbers. This result is refered to as the *Milnor–Johnson correspondence*. More precisely, Habegger shows that there exists a bijection between the sets $\mathcal{HC}(\Sigma_{g,1})$ and $\mathcal{SL}_1^{hb}(2g)$ which produces an isomorphism of Abelian groups

$$b : \bar{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\simeq} \overline{\mathcal{SL}}_1^{hb}(2g)$$

such that the Johnson homomorphism η_1 corresponds to Milnor's invariant μ_3 trough b. Proposition 31 allows us to go a bit further.

Theorem 33. *The homomorphism τ of Proposition* 31, *given by the Milnor, Sato–Levine, Arf and Rochlin invariants, is the analogue of the Birman–Craggs homomorphism β for the Milnor–Johnson correspondence.*

In other words, β and τ correspond through the isomorphism b.

The proof is given in the next subsection. Actually, we will also give an alternative proof for (part of) Habegger's result, based on the theory of claspers.

4.3. Birman–Craggs homomorphism for string links: proof of Theorem 33

Let us recall from [9] the construction on which the Milnor–Johnson correspondence lies. Consider the handle decomposition $A_1, B_1, \ldots, A_g, B_g$ of $\Sigma_{g,1}$ as in the left part of Fig. 11. Likewise, for the $2g$-punctured disk $D^2_{2g} \cong \Sigma_{0,2g+1}$, consider the handle decomposition $\{A'_i, B'_i\}_{i=1}^g$ given in the right part of the figure.

We identify $\Sigma_{g,1} \times I$ with $\Sigma_{0,2g+1} \times I$ using the diffeomorphism F defined by the g isotopies exchanging, in $\Sigma_{g,1} \times I$, the second attaching region of the handle $A_i \times I$ and the first attaching region of the handle $B_i \times I$.

Now, the product $\Sigma_{0,2g+1} \times I$ can be thought of as (the closure of) the complementary of the 0-framed trivial $2g$-string link 1_{2g} in $D^2 \times I$. This defines a bijection between the sets $\mathcal{C}_1(\Sigma_{g,1})$ and $\mathcal{SL}_1^{hb}(2g)$.

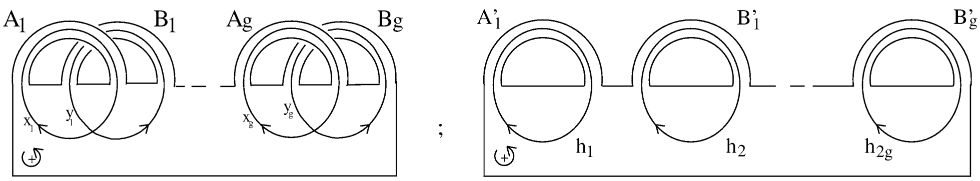

Fig. 11.

Indeed, let G be a degree 1 clasper for $\Sigma_{g,1} \times I$: the pair $(\Sigma_{g,1} \times I; G)$ defines an element of $\mathcal{C}_1(\Sigma_{g,1})$. By applying F to this pair, we obtain a clasper G' *of the same degree* for $(\Sigma_{0,2g+1} \times I) \cong 1_{D^2} \setminus 1_{2g}$: the triple $((1_{D^2}, 1_n); G')$ defines an element of $\mathcal{SL}_1^{hb}(2g)$.

Moreover, though this bijection is not a homomorphism, it produces an isomorphism of Abelian groups

$$\bar{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{b} \overline{\mathcal{SL}}_1^{hb}(2g).$$

This follows from the following observation. Let M_i $(i = 1, 2)$ be an element of $\bar{\mathcal{C}}_1(\Sigma_{g,1})$ obtained from $\Sigma_{g,1} \times I$ by surgery on the degree 1 clasper G_i. The product $M_1 \cdot M_2$ is mapped by b to an element which is obtained from $(1_{D^2}, 1_{2g})$ by surgery on the union $G_1' \cup G_2'$, where G_i' is the image of G_i under the diffeomorphism F (in particular, $\deg(G_i') = 1$). Up to Y_2-equivalence, we can suppose that these two claspers lie in *disjoint portions of the product* $D^2 \times I$; it follows that

$$(1_{D^2}, 1_{2g})_{G_1' \cup G_2'} \sim_{Y_2} (1_{D^2}, 1_{2g})_{G_1'} \cdot (1_{D^2}, 1_{2g})_{G_2'} = b(M_1) \cdot b(M_2).$$

Similar arguments show that we actually have an isomorphism of Abelian groups $\bar{\mathcal{C}}_k(\Sigma_{g,1}) \simeq \overline{\mathcal{SL}}_k^{hb}(2g)$, $\forall k \geqslant 1$.

At the level of homology, there is an obvious isomorphism between $H_1(\Sigma_{g,1}; \mathbf{Z})$ and $H_1(\Sigma_{0,2g+1}; \mathbf{Z})$ induced by the diffeomorphism F. We denote by H these homology groups. This isomorphism allows to identify the diagram spaces $\mathcal{A}_1(P_{g,1})$ and $\mathcal{A}_1(P_{2g})$. We thus have a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_1(P_{g,1}) \arrow[r] \arrow[d, "\psi_1"'] & \mathcal{A}_1(P_{2g}) \arrow[d, "\varphi_1"] \\
\bar{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[r, "b"] & \overline{\mathcal{SL}}_1^{hb}(2g)
\end{tikzcd} \tag{D}$$

whose arrows are isomorphisms.

Following Notations 10, set $H_{(2)} = H \otimes \mathbf{Z}_2$, and $V = \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. By considering the inverse maps (in the sense of [19, Theorem 1.4] and Proposition 31) of the vertical arrows of (D), we easily deduce the following commutative diagram

$$\begin{tikzcd}
\bar{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[r, "b"] \arrow[dr, "\simeq", "(\eta_1, \beta)"'] & \overline{\mathcal{SL}}_1^{hb}(2g) \arrow[d, "\simeq"', "(\mu_3, \tau)"] \\
 & \Lambda^3 H \oplus V
\end{tikzcd}$$

which shows that, *via* the isomorphism b, degree 1 invariants for homology cylinders over $\Sigma_{g,1}$ correspond to those of $2g$-string links in homology balls. More precisely, we deduce from diagram (D) the following result.

Lemma 34. *The two following diagrams commute*:

$$\begin{tikzcd}
\bar{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[r, "b"] \arrow[dr, "\eta_1"'] & \overline{\mathcal{SL}}_1^{hb}(2g) \arrow[d, "\mu_3"] \\
 & \Lambda^3 H
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[r, "b"] \arrow[dr, "\beta"'] & \overline{\mathcal{SL}}_1^{hb}(2g) \arrow[d, "\tau"] \\
 & \Lambda^3 H_{(2)} \oplus V
\end{tikzcd}$$

The first diagram recovers Habegger's Milnor–Johnson correspondence (at the lowest level). The second one proves Theorem 33.

Proof of Lemma 34. Consider in diagram (D) the projections $p : \mathcal{A}_1(P) \twoheadrightarrow \Lambda^3 H$, on the one hand, and $T : \mathcal{A}_1(P) \twoheadrightarrow \Lambda^3 H_{(2)} \oplus V$ on the other hand, where $\mathcal{A}_1(P)$ denotes either $\mathcal{A}_1(P_{g,1})$ or $\mathcal{A}_1(P_{2g})$. Recall from [19, Lemma 4.22] that the diagram

$$\begin{tikzcd}
\mathcal{A}_1(P_{g,1}) \arrow[r, two heads, "\psi_1"] \arrow[dr, "p"'] & \bar{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "\eta_1"] \\
& \Lambda^3 H
\end{tikzcd}$$

is commutative. This, together with Lemma 30, implies the commutativity of the first diagram. The second half of the result follows similarly from [19, Lemma 4.23] and Lemma 29. □

5. Comparing Goussarov–Habiro and Vassiliev theories

For several reasons, it is tempting to compare the results of Section 3 with Vassiliev theory. First, as seen in Section 2, both Goussarov–Habiro and Vassiliev theories can be defined using claspers (with some slight differences). Second, some results in the literature on Vassiliev invariants have strong similarities with Theorem 13, namely K. Taniyama and A. Yasuhara's characterization of clasp-pass equivalence for algebraically split links in the 3-sphere [27], and its analogue for string links [21].

Recall that the clasp-pass equivalence is the equivalence relation on links generated by isotopies and *clasp-pass moves*, which are local moves as illustrated in Fig. 12. As outlined in Remark 7, the clasp-pass equivalence is actually the same as C_3-equivalence, which implies Y_2-equivalence.

5.1. Goussarov–Habiro and Vassiliev invariants of string links

Let us first consider the string link case. Recall that the Casson knot invariant $\varphi(K)$ of a knot K is defined as the z^2 coefficient of the Alexander–Conway polynomial of K, and that its reduction modulo 2 coincides with the Arf invariant α studied in Section 3.2.3.

Recall also from [21] the definition of the 2-string link invariant V_2. Let $\sigma = \sigma_1 \cup \sigma_2$ be a 2-string link. Then

$$V_2(\sigma) := \varphi\big(p(\sigma)\big) - \varphi(\sigma_1) - \varphi(\sigma_2),$$

where $p(\sigma)$ denotes the plat-closure of σ: it is the knot obtained by identifying the upper (respectively lower) endpoints of σ_1 and σ_2. Clearly, V_2 is a $\mathbf{Z}$-valued Vassiliev invariant of degree two.

We want to relate Theorem 13 to the following:

Theorem 35. [21] *Let σ and σ' be two n-component algebraically split string links in $D^2 \times I$ (that is, with all linking numbers zero). Then, the following assertions are equivalent*:

(a) *σ and σ' are clasp-pass equivalent*;
(b) *σ and σ' are not distinguished by degree 2 Vassiliev invariants*;
(c) *σ and σ' are not distinguished by Milnor's triple linking numbers, nor the invariant V_2 and the Casson knot invariant.*

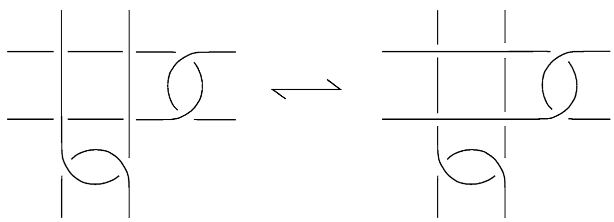

Fig. 12. A clasp-pass move.

We denote by $SL(n)$ the monoid of n-string links in $D^2 \times I$ up to isotopy (with fixed endpoints), and by $SL^{as}(n)$ the submonoid of algebraically split n-string links. When considered up to C_3-equivalence, the elements of $SL^{as}(n)$ form an Abelian group, denoted by $\overline{SL}^{as}(n)$.

Theorem 36. *The Abelian group $\overline{SL}^{as}(n)$ is surjectively mapped onto the subgroup $\overline{\mathcal{SL}}_1^{(0)}(n) \subset \overline{\mathcal{SL}}_1(n)$ of string links in homology balls having vanishing Rochlin's μ-invariant.*

Proof. Recall from [21] the isomorphism

$$(\mu_3, V_2, \varphi): \overline{SL}^{as}(n) \xrightarrow{\simeq} \Lambda^3 H \oplus S^2 H$$

given by the formula

$$\sum_{1 \leqslant i<j<k \leqslant n} \mu_{ijk}(\sigma).e_i \wedge e_j \wedge e_k + \sum_{1 \leqslant i<j \leqslant n} V_2(\sigma_i \cup \sigma_j).e_i \otimes e_j + \sum_{1 \leqslant i \leqslant n} \varphi(\sigma_i).e_i.$$

Here, $S^2 H$ is the degree two part of the symmetric algebra of H (we still make use of Notations 10).

On the other hand, we saw in Section 3.3 the isomorphism of Abelian groups

$$(\mu_3, \tau): \overline{\mathcal{SL}}_1^{hb}(n) \xrightarrow{\simeq} \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2,$$

where the $\mathbf{Z}_2$ part is detected by Rochlin's μ-invariant. We thus have the decomposition $\overline{\mathcal{SL}}_1^{hb}(n) = \overline{\mathcal{SL}}_1^{(0)}(n) \cup \overline{\mathcal{SL}}_1^{(1)}(n)$, where $\overline{\mathcal{SL}}_1^{(\varepsilon)}(n)$ ($\varepsilon = 0, 1$) is the subset of $\overline{\mathcal{SL}}_1^{hb}(n)$ consisting of elements (M, σ) such that $R(M) = \varepsilon$.

In particular, $\overline{\mathcal{SL}}_1^{(0)}(n)$ is an Abelian subgroup and we clearly have an isomorphism

$$\big(\mu_3, \beta^{(2)}, \alpha\big): \overline{\mathcal{SL}}_1^{(0)}(n) \xrightarrow{\simeq} \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)},$$

given by the formula

$$\sum_{1 \leqslant i<j<k \leqslant n} \mu_{ijk}(M, \sigma).e_i \wedge e_j \wedge e_k + \sum_{1 \leqslant i<j \leqslant n} \beta_{ij}^{(2)}(M, \sigma).\overline{e_i} \wedge \overline{e_j} + \sum_{1 \leqslant i \leqslant n} a_i(M, \sigma).\overline{e_i}.$$

Now, recall that the C_2-equivalence is the same as the Δ-equivalence: as in the link case [22], a n-string link σ is C_2-equivalent to 1_n if and only if it has vanishing linking numbers. So $\overline{SL}^{as}(n)$ is just the set of C_3-equivalence classes of n-string links which are C_2-equivalent to 1_n: given a generator σ of $\overline{SL}^{as}(n)$, there is a connected C-degree 2 clasper G_σ for $1_n \in 1_{D^2}$ such that $\sigma = (1_n)_{G_\sigma}$. We define a map

$$\overline{SL}^{as}(n) \xrightarrow{\eta} \overline{\mathcal{SL}}_1^{(0)}(n)$$

which consists in *puncturing* each disk-leaf of G_σ, that is removing a small disk d such that 1_n intersects the disk-leaf at the interior of d; further, equip 1_n with 0-framing. As Fig. 13 shows, puncturing a disk-leaf of G_σ produces a leaf. G_σ becomes a Y-graph $\widetilde{G}_\sigma$, and

$$T(\sigma) := (1_{D^2}, 1_n)_{\widetilde{G}_\sigma}.$$

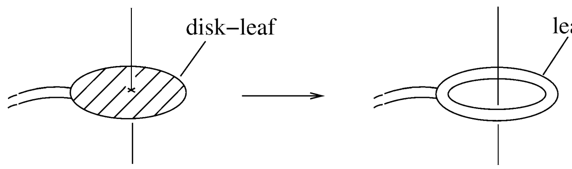

Fig. 13. The η map.

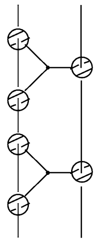

Fig. 14. An element of $\mathrm{Ker}(\eta)$ for $n = 2$.

Note that η has a non-trivial kernel; an example is given in Fig. 14. It follows from the proofs of Theorems 13 and 35 that we have a commutative diagram

$$\begin{tikzcd} \overline{SL}^{as}(n) \arrow[r, "\eta"] \arrow[d, "(\mu_3, V_2, \varphi)"', "\simeq"] & \overline{S\mathcal{L}}_1^{(0)}(n) \arrow[d, "\simeq"', "(\mu_3, \beta^{(2)}, \alpha)"] \\ \Lambda^3 H \oplus S^2 H \arrow[r, two heads, "t"] & \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \end{tikzcd}$$

where f is the surjective map given by the identity on $\Lambda^3 H$, and by

$$f(e_i \otimes e_j) = \overline{e_i} \wedge \overline{e_j} \text{if } i \neq j, \text{ and } f(e_i \otimes e_i) = \overline{e_i} \text{ otherwise}$$

on $S^2 H$. It follows that η is also surjective. $\square$

Moreover, the maps (μ_3, V_2, c_2) and $(\mu_3, \beta^{(2)}, a)$ coincide *via* the surjective map η (and t). In particular, it follows that

$$V_2 \equiv \beta \pmod 2.$$

However, these invariants are distinct over $\mathbf{Z}$, as mentioned in [21, Remark 2.7].

5.2. *The case of links*

In the case of links, we know the following on clasp-pass equivalence.

Theorem 37. [27, Theorem 1.4] *Let L and L' be two n-component algebraically split links in S^3. The following assertions are equivalent*:

(a) *L and L' are clasp-pass equivalent*;
(b) *L and L' are not distinguished by Milnor's triple linking numbers, nor the* mod 2 *reduction of the Sato–Levine invariant and the Casson knot invariant.*

As for Y_2-equivalence, one can check (using Theorem 13 and its proof) the following corollary, characterizing Y_2-equivalence for algebraically split links in homology spheres.

Corollary 38. *Let (M, L) and (M', L') be two n-component algebraically split links in homology spheres. Then, the following assertions are equivalent*:

(a) *(M, L) and (M', L') are Y_2-equivalent*;
(b) *(M, L) and (M', L') are not distinguished by Milnor's triple linking numbers, nor the mod 2 reduction of the Sato–Levine invariant, the Arf invariant and Rochlin's μ-invariant.*

This result is related to Theorem 37 in a similar way as Theorem 13 is related to Theorem 35. However, unlike in the string link case, there is no natural group or monoid structure on the sets of C_k or Y_k-equivalence classes of links.

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Appendix A. Tubing Seifert surfaces

Let us consider the 2-component link $L = L_1 \cup L_2$ in a genus 4 handlebody N depicted in Fig. 5. We fix an orientation on N and embed it in S^3. Let K be an oriented knot in S^3 disjoint from N, and let S be a Seifert surface for K: in general, S may intersect K. In this appendix we explain the general procedure to construct, starting from S, a new Seifert surface for K which is disjoint from L.

First, we fix some more notations. The handlebody N can be regarded as a ball B with 4 handles $D^2 \times I$ attached. The two handles intersecting L_1 are denoted by H_1 and H_2, and we denote by H_3 and H_4 the other two; the handles are numbered clockwise in Fig. 5, so that H_1 is in the lower left corner of the figure. Up to isotopy, we can suppose that S is disjoint from B, that is S only intersects N at its handles, along copies of $D^2 \times \{t\}$; $t \in I$. When the orientation of S is compatible with the orientation of N along the intersection disk, we call it a *positive intersection*. Otherwise, we call it a *negative intersection*. For $1 \leqslant i \leqslant 4$, we denote respectively by p_i and n_i the number of positive and negative intersections between the surface S and the handle H_i.

In view of the symmetry of the link L, we only have to deal with (say) the handles H_1 and H_2 (the handles H_3 and H_4 can be treated independently, in a similar way).

First, observe that if S intersects H_1 twice, with the opposite orientation, we can add two tubes to S as shown in Fig. 15(a), so that the new Seifert surface $\widetilde{S}$ satisfies $|L \cap S| = |L \cap \widetilde{S}| + 4$.

Likewise, we can always add $|p_i - n_i|$ such pairs of tubes to S in H_i $(i = 1, 2)$, by eventually nesting them, so that in each handle the remaining intersections all have the same sign. So we can suppose that $p_1.n_1 = p_2.n_2 = 0$. Suppose further that $n_1 = 0$ (the case $p_1 = 0$ is equivalent, due to the symmetry of L_1).

If $p_1 = p_2 = n_2 = 0$, we are done. Otherwise, there are essentially 4 different cases to study.

(1) Suppose that $p_2 = n_2 = 0$. In this case S is disjoint from the handle H_1. We can thus remove all the elements of $S \cap L_1$ by successively attaching and nesting p_1 tubes as depicted in Fig. 15(b). These tubes will be called tubes

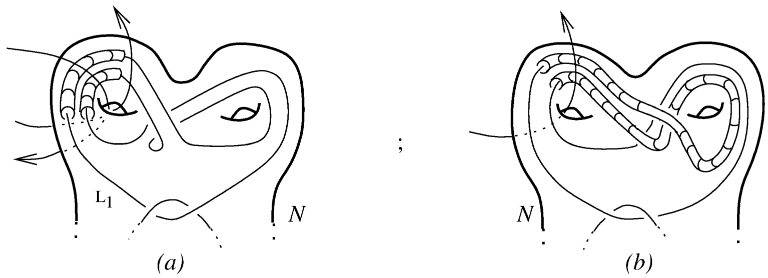

Fig. 15.

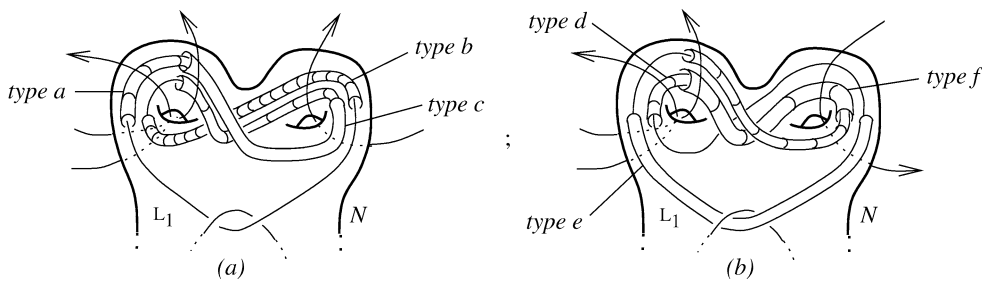

Fig. 16.

of type 1. When the two attaching circles of a tube t of type 1 lie in a disk $D^2 \times \{t\}$ of the handle H_i, we simply say that t is *attached in* H_i $(i = 1, 2)$.

(2) Suppose that $p_1 = 0$. This case is equivalent to the first one: S is disjoint from the handle H_2, so we can freely attach $p_2 + n_2$ tubes of type 1 in H_1.

(3) Suppose that p_1 and p_2 are non-zero. In this case, S always intersects N with the same sign. Fig. 16(a) illustrates the case $p_1 = 2$ and $p_2 = 1$. In general, we attach in a similar way p tubes of type a, m tubes of type b and m tubes of type c (following the notations of the figure), where $p := |p_1 - p_2|$ and $m := \max(p_1, p_2) - p$.

(4) Suppose that p_1 and n_2 are non-zero. Fig. 16(b) illustrates the case $p_1 = 2$ and $n_2 = 1$. As for the previous case, we deal with the general situation by attaching and nesting the same three types of tubes. Namely, we attach $|p_1 - n_2|$ tubes of type d and $(\max(p_1, n_2) - |p_1 - n_2|)$ tubes of type e and f.

The obtained surface is the required new Seifert surface for K.

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