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Orthogonal Decomposition Defined by a Pair of Skew-Symmetric Forms

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1. INTRODUCTION

In this note, we examine a property of pairs of skew-symmetric forms. The main theorem is given in Sec. 2 below, with a matrix formulation in Sec. 3. An application of the result to passive network synthesis appears in [1] and [2], and the matrix formulation also allows immediate rederivation of a result of [3] on the characteristic polynomial of the product of two skew matrices.

2. MAIN RESULT

THEOREM 1. *Let ϕ_i $(i = 1, 2): X \times X \to R$ be bilinear skew-symmetric forms on an n-dimensional real vector space X possessing a positive definite inner product $\langle \cdot, \cdot \rangle$. Then there exists an orthogonal direct decomposition $X_1 \oplus X_2$ with* $\dim X_2 = [n/2]$ *and with ϕ_i zero on $X_i \times X_i$ $(i = 1, 2)$. (Here, $[n/2]$ is the greatest integer s for which $s \leqslant n/2$).*

Proof. We use induction on n. For $n = 1$, the result is immediate. First, observe there exist linear transformations U_i $(i = 1, 2): X \to X$ such that $\langle U_i x, y \rangle = \phi_i(x, y)$. For define $L_{ix}: X \to R$ by $L_{ix}(y) = \phi_i(x, y)$. Then L_{ix} is a linear functional and since X is an inner product space, there exists $z_i \in X$ with $L_{ix}(y) = \langle z_i, y \rangle$ by the canonical isomorphism between X and its dual. Define the transformation U_i by $z_i = U_i x$; it is easily checked that U_i is linear. The skew-symmetric property of $\phi_i(\cdot, \cdot)$ also shows that $U_i = - U_i^*$, with U_i^* the adjoint of U_i, for $\langle U_i x, y \rangle = \phi_i(x, y) = - \phi_i(y, x) = - \langle U_i y, x \rangle = - \langle y, U_i^* x \rangle = \langle - U_i^* x, y \rangle$.

Now let w be an arbitrary nonzero vector in X, and let Y_1 be the subspace generated by $(U_2U_1)^k w$, $k = 0, 1, 2, \ldots$. Set $Y_2 = U_1(Y_1)$. Then the skew property of the U_i shows that Y_1 and Y_2 are orthogonal. Further, if $Y = Y_1 \oplus Y_2$ and $m = \dim Y$, then $\dim Y_2 = [m/2]$. To see this, observe that $\dim Y_2 \leqslant \dim Y_1$ (from the definition of Y_2) and $\dim Y_1 \leqslant \dim Y_2 + 1$ (because Y_1 is generated by U_2Y_2 and by w). The two inequalities on $\dim Y_1$ and $\dim Y_2$ then imply $\dim Y_2 = [m/2]$.

Note further that $Y_1 \perp Y_2$ and $Y_2 = U_1(Y_1)$ imply that $\phi_1(\cdot, \cdot)$ is identically zero on $Y_1 \times Y_1$. Likewise, because $U_2Y_2 \subset Y_1$, $\phi_2(\cdot, \cdot)$ is identically zero on $Y_2 \times Y_2$.

Provided that simultaneously, m is not odd and n is not even, apply the induction hypothesis to Z, the orthogonal complement of Y in X, to obtain $Z = Z_1 \oplus Z_2$, $Z_1 \perp Z_2$, $\dim Z_2 = [(n - m)/2]$ and $\phi_i(\cdot, \cdot)$ zero on $Z_i \times Z_i$. Then take $X_i = Y_i \oplus Z_i$ $(i = 1, 2)$. It is readily checked that $\dim X_2 = [m/2] + [(n - m)/2] = [n/2]$. Further $\phi_i(\cdot, \cdot)$ is zero on $X_i \times X_i$, for it is obviously zero on $Y_i \times Y_i$, $Z_i \times Z_i$, while with $y_i \in Y_i$, $z_i \in Z_i$, one has $\phi_i(y_i, z_i) = \langle U_i y_i, z_i \rangle = 0$ since $U_i y_i \in Y$, $z_i \in Z$ and $Y \perp Z$.

In case m is odd and n is even, minor adjustment is required to make $\dim X_2 = [n/2]$. Apply a variant of the induction hypothesis to Z, still the orthogonal complement of Y in X, to obtain as above $Z = Z_1 \oplus Z_2$, $Z_1 \perp Z_2$ and $\phi_i(\cdot, \cdot)$ identically zero on $Z_i \times Z_i$, but now $\dim Z_1 = [(n - m)/2]$. Then proceed as before.

3. MATRIX STATEMENT AND MISCELLANEOUS POINTS

The matrix statement of the main result is immediately obtainable. (The superscript prime denotes matrix transposition.)

THEOREM 2. *Let S_i $(i = 1, 2)$ be $n \times n$ real skew-symmetric matrices. Then there exists a real orthogonal matrix V such that*

$$V'S_1V = \begin{bmatrix} 0 & S_{1b} \\ -S_{1b}' & S_{1c} \end{bmatrix}, \qquad V'S_2V = \begin{bmatrix} S_{2a} & S_{2b} \\ -S_{2b}' & 0 \end{bmatrix}, \tag{1}$$

with the zero blocks of size $n - [n/2]$ and $[n/2]$ respectively.

There is an obvious generalization to skew-Hermitian matrices S_i transformed by a unitary matrices V, and indeed a corresponding generalization of Theorem 1.

From (1), there follows a quick proof of the result of [3], to the effect that all nonzero eigenvalues of S_1S_2 have even multiplicity. For convenience, assume n is even; then

$$V'S_1S_2V = \begin{bmatrix} -S_{1b}S'_{2b} & 0 \\ 0 & -S'_{1b}S_{2b} \end{bmatrix}$$

and

$$\begin{aligned} \{\lambda_i(S_1S_2)\} &= \{\lambda_i(-S_{1b}S'_{2b})\} \cup \{\lambda_i(-S'_{1b}S_{2b})\}, \\ &= \{\lambda_i(-S_{1b}S'_{2b})\} \cup \{\lambda_i(-S_{2b}S'_{1b})\}, \\ &= \{\lambda_i(-S_{1b}S'_{2b})\} \cup \{\lambda_i(-S_{1b}S'_{2b})\}. \end{aligned}$$

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