Nonexpansive Retractions in Hyperconvex Spaces

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Submitted by William F. Ames

Received March 20, 2000

DEDICATED TO GEORGE LEITMANN

This paper is primarily concerned with the study of conditions on a hyperconvex subset $D$ of a hyperconvex metric space $M$ which assure that there exists a nonexpansive retraction $R$ of $M \setminus D$ onto $D$ which has the property that $R(M \setminus D) \subseteq \partial D$.

A related question we take up is, when is such a retraction $R$ proximinal, that is, when does $R$ have the property

$$d(x, R(x)) = \text{dist}(x, D)$$

$^1$Most of this research was conducted while the first author was visiting the University of Iowa. He thanks the faculty and staff of the Mathematics Department at Iowa for their kind hospitality. Also, the first and third authors acknowledge the support of DGICYT Research Project PB96-1335-C02-01.
for each \( x \in M \)? Among other things, we show that if a subset \( D \) of a hyperconvex metric space \( M \) has nonempty interior and is externally hyperconvex relative to \( M \) in a very weak sense, then there always exists a nonexpansive retraction of \( M \) onto \( D \) which maps \( M \setminus D \) onto \( \partial D \). We also show that any compact weakly externally hyperconvex subset of a hyperconvex \( M \) is a proximinal nonexpansive retract of \( M \).

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**Key Words:** hyperconvex spaces; nonexpansive retractions; nonexpansive mappings; condensing mappings fixed points.

1. INTRODUCTION

The geometry of hyperconvex spaces yields many nice facts. In [1] it is shown that a metric space is hyperconvex if and only if it is injective. (A metric space \( H \) is injective if given metric spaces \( M \) and \( N \) with \( M \) a subspace of \( N \) and a nonexpansive mapping \( f : M \to H \), there exists a nonexpansive extension \( \tilde{f} \) of \( f \) such that \( \tilde{f} : N \to H \).) Since it is known that every metric space has an injective hull [8], it follows that every metric space is isometric with a subspace of a (minimal) hyperconvex superspace. Also, it is known that a real Banach space \( X \) is hyperconvex if and only if it is isometrically isomorphic to a space \( C(K) \) of continuous real-valued functions defined on a stonian space \( K \). Thus the \( L^\infty \) spaces provide prototypical examples of classical hyperconvex spaces. (See, e.g., [14, 17] for classical results.)

It is known that every hyperconvex metric space is a nonexpansive retract of any metric space in which it is isometrically embedded. In this paper we are concerned with the following questions: Under what conditions is a hyperconvex subset \( D \) of a metric space \( M \) a proximinal nonexpansive retract of \( M \)? Precisely, when does there exist a nonexpansive retraction \( r \) of \( M \) onto \( D \) with the property

\[
d(x, r(x)) = \text{dist}(x, D) := \inf\{d(x, y) : y \in D\}
\]

for each \( x \in M \)? A related question is when does there exist a retraction of \( M \setminus D \) onto \( \partial D \)? Here we show (Theorem 3.2) that if a subset \( D \) of a hyperconvex metric space \( M \) is externally hyperconvex relative to \( M \) in a very weak sense, then given any \( \varepsilon > 0 \) there is a nonexpansive retraction \( R_\varepsilon \) of \( M \) onto \( D \) with the property that given any \( u \in M \setminus D \) there exists \( x \in M \) such that \( d(u, x) \leq \varepsilon \) and \( d(x, R_\varepsilon(x)) = \text{dist}(x, D) \). Moreover, if \( \text{int}(D) \neq \emptyset \) then \( R_0 \) may be chosen so that \( R_\varepsilon(M \setminus D) \subseteq \partial D \). We also show that if \( D \) is compact, then \( D \) is in fact a proximinal nonexpansive retract of \( M \). Some fixed-point theoretic implications are discussed as well.

It is shown in [20] (Theorem 10) that every nonempty ball intersection (admissible set) in a hyperconvex space \( M \) is a proximinal nonexpansive
retract of $M$. It is also shown in [7] that the same is true for order intervals in an AM lattice $X$. Here we deal with more general subsets.

To describe our results in detail we need some notation and terminology. For a subset $A$ of a metric space $M$ we use $N_\varepsilon(A)$ to denote the closed $\varepsilon$-neighborhood of $A$. Thus

$$N_\varepsilon(A) = \{ x \in M : \text{dist}(x, A) \leq \varepsilon \}.$$ 

An admissible subset of $M$ is a set of the form $\bigcap_{i} B(x_i; r_i)$ where $\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in M$ with respective radii $r_i$.

**Definition 1.1.** A metric space $M$ is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of $M$ and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\bigcap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$.

**Definition 1.2.** A subset $E$ of a metric space $M$ is said to be externally hyperconvex (relative to $M$) if given any family $\{x_\alpha\}$ of points in $M$ and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \quad \text{and} \quad \text{dist}(x_\alpha, E) \leq r_\alpha$$

it follows that $\bigcap_\alpha B(x_\alpha; r_\alpha) \cap E \neq \emptyset$.

It is shown in [1] that any admissible subset of a hyperconvex space $M$ is externally hyperconvex relative to $M$, and that the externally hyperconvex subsets of $M$ are proximinal in $M$ (thus if $H$ is externally hyperconvex in $M$ and if $x \in M$, then there exists $h \in H$ such that $d(x, h) = \text{dist}(x, H)$).

Externally hyperconvex subsets of hyperconvex spaces are studied in more detail in [12] where, for example, it is shown that a set-valued mapping $T^*$ of any set $S$ which takes values in the space of nonempty externally hyperconvex subsets of $M$ always has a point-valued selection $T$ which satisfies

$$d(T(x), T(y)) \leq D_H(T^*(x), T^*(y)),$$

where $D_H$ denotes the Hausdorff metric. In particular, if $S = M$ and $T^*$ is Lipschitzian, then the selection $T$ has the same Lipschitz constant.

Below we introduce a definition which is much weaker than external hyperconvexity, essentially retaining only what is needed for proximinality. The results of this paper focus on this weaker concept.

**Definition 1.3.** A subset $E$ of a metric space $M$ is said to be weakly externally hyperconvex (relative to $M$) if $E$ is externally hyperconvex relative to $E \cup \{z\}$ for each $z \in M$. Precisely, given any family $\{x_\alpha\}$ of points
in $M$ all but at most one of which lies in $E$, and any family $\{r_a\}$ of real numbers satisfying
\[ d(x_a, x_\beta) \leq r_a + r_\beta, \quad \text{with } \text{dist}(x_a, E) \leq r_a \text{ if } x_a \notin E, \]
it follows that $\cap_a B(x_a; r_a) \cap E \neq \emptyset$.

The diagonal of the unit square in $\mathbb{R}^2$ offers an easy example of a weakly externally hyperconvex subset of a hyperconvex space which is not externally hyperconvex (and hence not admissible). In fact, in [20] Sine anticipates that the results we obtain in Section 4 should apply to just such sets (see Observation (4), on p. 763 of [20]).

2. WEAK EXTERNAL HYPERCONVEXITY

Before proving our main results we take a closer look at the concept of weak external hyperconvexity. We begin with a fact that is immediate from the definition.

**Lemma 2.1.** Suppose $D$ is a weakly externally hyperconvex subset of a metric space $M \setminus E$. Then there exists a nonexpansive retraction $r$ of $D \cup \{z\}$ onto $D$ with $d(z, r(z)) = \text{dist}(z, D)$.

**Proof.** For each $x, y \in D$, $d(x, y) \leq d(x, z) + d(y, z)$. Therefore by the definition of weak external hyperconvexity,
\[ (\cap_{x \in D} B(x; d(x, z))) \cap B(z; \text{dist}(z, D)) \cap D \neq \emptyset. \]
It suffices to take $r(x) = x$ for $x \in D$ and $r(z)$ to be any point of the above intersection.

This leads quickly to the following.

**Lemma 2.2.** Suppose $D$ is a weakly externally hyperconvex subset of a metric space $M \setminus E$, and let $F$ be a finite subset of $M \setminus D$. Then there exists a nonexpansive retraction $r$ of $D \cup F$ onto $D$ for which $d(z, r(z)) = \text{dist}(z, D)$ for each $z \in F$.

**Proof.** Let $F = \{x_1, \ldots, x_n\}$ and suppose the points of $F$ are ordered so that
\[ i < j \Rightarrow \text{dist}(x_i, D) \leq \text{dist}(x_j, D). \]
By Lemma 2.1 there exists a nonexpansive retraction $r$ of $D \cup \{x_1\}$ onto $D$ with $d(x_1, r(x_1)) = \text{dist}(x_1, D)$. Let $\mu = \text{dist}(x_2, D)$ and set
\[ A(x_2) = (\cap_{z \in D} B(z; d(z, x_2))) \cap B(r(x_1); d(x_1, x_2)) \cap B(x_2; \mu) \cap D. \]
It is now easy to check that we have a family of closed balls, \textit{only one of which is not centered in} \( E \), and for which the conditions of Definition 1.3 hold. Since \( A(x_2) \neq \emptyset \) it is possible to choose \( r(x_2) \in A(x_2) \) and conclude that the lemma holds for \( n = 2 \). The full conclusion follows by finite induction. \( \blacksquare \)

We now show that if \( M \) is hyperconvex, then the converse of the above result holds. For this result we also need the following lemma due to Sine [19].

**Lemma 2.3.** If \( M \) is hyperconvex and if \( D = \bigcap_a B(z_a; r_a) \) for \( \{z_a\} \subseteq D \), then for any \( \rho > 0 \),

\[
N_\rho(D) = \bigcap_a B(z_a; r_a + \rho).
\]

**Theorem 2.1.** A subset \( D \) of a hyperconvex metric space \( M \) is weakly externally hyperconvex if and only if \( D \) is a proximinal nonexpansive retract of \( D \cup F \) for any finite \( F \subseteq M \setminus D \).

**Proof.** In view of Lemma 2.1 it suffices to show that if \( D \) is not weakly externally hyperconvex then there exists \( z \in M \setminus D \) such that

\[
B(z; \text{dist}(z, D)) \cap (\bigcap_{v \in D} B(v; d(v, z))) \cap D = \emptyset.
\]

This clearly implies that there does not exist a nonexpansive retraction of \( D \cup \{z\} \) onto \( D \) which satisfies the condition of Lemma 2.1. By definition, if \( D \) is not weakly externally hyperconvex there exist \( z \in M \), a family \( \{v_a : a \in A\} \) in \( D \), and a family \( \{r_a\} \) in \( \RR \) for which \( d(v_a, v_\beta) \leq r_a + r_\beta \), \( d(v_a, z) \leq r_a + \text{dist}(z, D) \), and for which

\[
B(z; \text{dist}(z, D)) \cap (\bigcap_a B(v_a; r_a)) \cap D = \emptyset.
\]

However, since \( M \) is hyperconvex it must be the case that

\[
D_1 = B(z; \text{dist}(z, D)) \cap (\bigcap_a B(v_a; r_a)) \neq \emptyset.
\]

If \( B(z; \text{dist}(z, D)) \cap (\bigcap_a B(v; d(v, z))) \cap D = \emptyset \) we are done. Otherwise we proceed as follows. Since \( D_1 \cap D = \emptyset \) we may assume

\[
\text{dist}(B(z; \text{dist}(z, D)) \cap D, D_1) = d > 0.
\]

It is possible to choose \( w_1 \in D_1 \) and \( w \in B(z; \text{dist}(z, D)) \cap D \) so that \( d(w_1, w) = d + \epsilon \) for sufficiently small \( \epsilon > 0 \). By the hyperconvexity of \( M \),

\[
B\left(z; \frac{d + \epsilon}{2}\right) \cap B\left(w_1; \frac{d + \epsilon}{2}\right) \cap B\left(w; \frac{d + \epsilon}{2}\right) \neq \emptyset.
\]
Let $z_1$ be any point in this intersection and observe that if $d(z_1, p) < \frac{d + \varepsilon}{2}$ for any $p \in D$, then
\[
d(z_1, p) < d(z, z_1) + d(z_1, p) < \text{dist}(z, D) - \frac{d + \varepsilon}{2} + \frac{d + \varepsilon}{2} = \text{dist}(z, D)
\]
which is a contradiction. Since $\text{dist}(z_1, D) \leq d(z_1, w) \leq (d + \varepsilon)/2$, we conclude
\[
\text{dist}(z_1, D) = \frac{d + \varepsilon}{2}.
\]
If the metric projection of $D \cup \{z_1\}$ onto $D$ were to have a nonexpansive selection, the set
\[
B(z_1; \text{dist}(z_1, D)) \cap (\bigcap_{\alpha} B(v_\alpha; d(v_\alpha, z_1))) \cap D
\]
would have to be nonempty. However, since
\[
d(z_1, v_\alpha) \leq r_\alpha + \frac{d + \varepsilon}{2},
\]
for each $\alpha$ we have
\[
B(z_1; d(z_1, D)) \cap (\bigcap_{\alpha} B(v_\alpha; d(v_\alpha, z_1))) \subseteq B(z; d(z, D)) \cap \left( \bigcap_{\alpha} B\left(v_\alpha; r_\alpha + \frac{d + \varepsilon}{2}\right) \right),
\]
and by Sine’s Lemma,
\[
B(z; d(z, D)) \cap \left( \bigcap_{\alpha} B\left(v_\alpha; r_\alpha + \frac{d + \varepsilon}{2}\right) \right) \subseteq N_{(d + \varepsilon)/2}(D_1).
\]
Clearly this neighborhood of $D_1$ cannot intersect $D \cap B(z, \text{dist}(z, D))$ for $\varepsilon \geq 0$ sufficiently small, so we conclude
\[
B(z_1; \text{dist}(z_1, D)) \cap (\bigcap_{\alpha} B(v_\alpha; d(v_\alpha, z_1))) \cap D = \emptyset.
\]

Our next corollary follows from the proof of Theorem 2.1 and Lemma 2.1.

**Corollary 2.1.** A subset $D$ of a hyperconvex metric space is weakly externally hyperconvex if and only if $D$ is a proximinal nonexpansive retract of $D \cup \{z\}$ for any $z \in M \setminus D$. 
3. NONEXPANSIVE RETRACTIONS

We begin with the following fact.

**Theorem 3.1.** Suppose $D$ is a weakly externally hyperconvex subset of a metrically convex metric space $M$. Then given any $\varepsilon > 0$ there exists a nonexpansive retraction $R : M \to D$ with the property that if $u \in M \setminus D$ there exists $v \in M \setminus D$ such that $d(v, R(u)) = \text{dist}(v, D)$ and $d(u, v) \leq \varepsilon$. In particular, if $\text{int}(D) \neq \emptyset$, then $R(M \setminus D) \subset N_\varepsilon(\partial D)$.

**Proof.** Let $\varepsilon > 0$ be fixed and set

$$S_\varepsilon = \{ u \in M : \text{dist}(u, D) = \varepsilon \}.$$ 

Let $\mathcal{S}_\varepsilon$ denote the family of all ordered pairs $(H_\varepsilon, r)$ where $H_\varepsilon \subseteq S_\varepsilon$ and $r$ is a proximinal nonexpansive retraction of $D \cup H_\varepsilon$ onto $D$. Lemma 2.1 assures that $\mathcal{S}_\varepsilon \neq \emptyset$. We order $\mathcal{S}_\varepsilon$ in the usual way, setting $(H_\varepsilon, r_1) \leq (K_\varepsilon, r_2)$ if and only if $H_\varepsilon \subseteq K_\varepsilon$ and $r_2$ is an extension of $r_1$. By Zorn’s Lemma ($\mathcal{S}_\varepsilon, \leq$) contains a maximal element, say $(H_\varepsilon, r)$. Suppose there exists $v \in S_\varepsilon \setminus H_\varepsilon$ and let

$$P(v) = (\bigcap_{x \in D} B(x; d(x, v))) \cap (\bigcap_{u \in H_\varepsilon} B(r(u); d(u, v)))$$

$$\cap B(v; \varepsilon) \cap D.$$

Since $d(r(u), u) = \varepsilon$, $d(r(u), v) \leq d(u, v) + \varepsilon$. This fact and a case-by-case check of the remaining possibilities shows that any two respective centers and radii in the above family satisfy the condition $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. Therefore, since $D$ is weakly externally hyperconvex, $P(v) \neq \emptyset$. Selecting $r'(v) \in P(v)$ and setting $r'(z) = r(z)$ for $z \in H_\varepsilon \cup D$, we conclude that $(H_\varepsilon, r) \leq (H_\varepsilon \cup \{ v \}, r')$, contradicting the maximality of $(H_\varepsilon, r)$. We therefore conclude that there is a retraction $r_\varepsilon$ of $H_\varepsilon$ onto $D$ with the property $r_\varepsilon(x) \in P(x)$ for each $x \in S_\varepsilon$. Next we observe that if $v \in S_{2\varepsilon}$, then the set

$$P'(v) = (\bigcap_{x \in D} B(x; d(x, v))) \cap (\bigcap_{u \in S_\varepsilon} B(r(u); d(u, v)))$$

$$\cap B(v; 2\varepsilon) \cap D$$

is nonempty. The key step is noting that, for any $u \in S_\varepsilon$,

$$d(v, r(u)) \leq d(u, v) + d(r(u), u) = d(u, v) + \varepsilon < d(u, v) + 2\varepsilon.$$

By selecting a point in $P'(v)$ it is possible to extend $r_\varepsilon$ nonexpansively from $S_\varepsilon$ to $S_\varepsilon \cup \{ v \}$. The argument given just now shows how to extend $r_\varepsilon$ to a nonexpansive retraction $r_{2\varepsilon}$ of $D \cup S_\varepsilon \cup S_{2\varepsilon}$ onto $D$ with the property

$$r_{2\varepsilon}(x) \in B(x; \text{dist}(x, D)) \cap D$$

for each $x \in S_\varepsilon \cup S_{2\varepsilon}$. 

Now let $S = \bigcup_{n=1}^{\infty} S_{\delta_n}$. By proceeding as above and using induction it is easy to see that there exists a nonexpansive retraction $r$ of $D \cup S$ onto $D$ with the property
\[ r(x) \in B(x; \text{dist}(x, D)) \cap D \quad \text{for each } x \in S. \]
Now, since $D$ is itself hyperconvex it is possible to extend $r$ to a nonexpansive mapping $R : M \to D$ (by [1]). The conclusion now follows from the fact that if $x \in M \setminus D$ then there exists $u \in S$ such that $d(x, u) \leq \varepsilon$; hence $d(R(x), R(u)) \leq \varepsilon$.

We now take up the questions of what more can be said if $M$ is hyperconvex. In the next section we consider the case when $D$ is compact.

At this point we need more information about weak external hyperconvexity.

**Lemma 3.1.** Let $D$ be a subset of a hyperconvex space $M$. Suppose that for any $x \in M \setminus D$ and any $\delta \in (0, \text{dist}(x, D))$ there exists a nonexpansive mapping $R$ from $D \cup \{x\}$ into $N_{\delta}(D)$ which leaves each point of $D$ fixed and for which
\[ R(x) \in B(x; \text{dist}(x, N_{\delta}(D))) \cap N_{\delta}(D). \]
Then $D$ is weakly externally hyperconvex.

**Proof.** Let $x \in M \setminus D$. In view of Corollary 2.1 we need only to show that there exists a nonexpansive retraction $r : D \cup \{x\} \to D$ for which $d(r(x), x) = \text{dist}(x, D)$. To obtain such a retraction, choose $\{\delta_n\} \subset (0, \text{dist}(x, D))$ so that $\{\delta_n\}$ is decreasing and $\sum_n \delta_n < \infty$. By assumption there exists a nonexpansive mapping $R_1$ from $D \cup \{x\}$ into $N_{\delta_1}(D)$ which leaves each point of $D$ fixed and for which
\[ R_1(x) \in B(x; \text{dist}(x, N_{\delta_1}(D))) \cap N_{\delta_1}(D). \]
Similarly, there exists a nonexpansive mapping from $D \cup \{R_1(x)\}$ into $N_{\delta_2}(D)$ which leaves each point of $D$ fixed and for which
\[ R_2(R_1(x)) \in B(x; \text{dist}(R_1(x), N_{\delta_2}(D))) \cap N_{\delta_2}(D). \]
Set $\hat{R}_2(x) = R_2 \circ R_1(x)$. Now proceed by induction to define for each $n \in \mathbb{N}$ a nonexpansive mapping $R_n$ from $D \cup \{\hat{R}_{n-1}(x)\}$ into $N_{\delta_n}(D)$ which leaves each point of $D$ fixed and for which
\[ R_n(\hat{R}_{n-1}(x)) \in B(\hat{R}_{n-1}(x); \text{dist}(\hat{R}_{n-1}(x), N_{\delta_n}(D))) \cap N_{\delta_n}(D). \]
It is straightforward to show that $\{\hat{R}_n(x) := R_n(\hat{R}_{n-1}(x))\}$ is a Cauchy sequence and that the desired retraction is obtained by taking $r(x) = \lim_n \hat{R}_n(x)$ and $r(u) = u$ for all $u \in D$.  \qed
**Theorem 3.2.** If $D$ is a weakly externally hyperconvex subset of a hyperconvex space $M$, then for any $\varepsilon > 0$, $N_\varepsilon(D)$ is also weakly externally hyperconvex.

**Proof.** Let $x$ be any point of $M \setminus N_\varepsilon(D)$ and let $\delta$ be any number in the interval 

$$(0, \text{dist}(x, N_\varepsilon(D))).$$

In view of Lemma 3.1 it suffices to show that there exists a nonexpansive mapping $R'$ of $N_\varepsilon(D) \cup \{x\}$ into $N_\delta(N_\varepsilon(D))$ which leaves each point of $N_\varepsilon(D)$ fixed and for which $d(x, R'(x)) = \text{dist}(x, N_{\varepsilon+\delta}(D))$. By Theorem 3.1, there exists a retraction $R$ from $M$ onto $D$ for which

$$d(u, R(u)) \leq \text{dist}(u, D) + \delta$$

for any $u \in M$. Moreover, this retraction may be defined so that $d(x, R(x)) = \text{dist}(x, D)$.

Now consider the following intersection of balls:

$$S = B(R(x); \varepsilon + \delta) \cap B(x; \text{dist}(x, D) - (\varepsilon + \delta)) \cap (\cap_{u \in N_\varepsilon(D)} B(u; d(u, x))).$$

To show that $S$ is nonempty it suffices (by hyperconvexity) to show that any two balls in the family intersect. Clearly,

$$B(R(x); \varepsilon + \delta) \cap B(x; \text{dist}(x, D) - (\varepsilon + \delta)) \neq \emptyset$$

since this intersection contains a point of the metric segment joining $x$ and $R(x)$. Now suppose $u \in D$. Then $d(R(x), u) \leq d(x, u)$ because $R$ is nonexpansive. On the other hand, if $u \in N_\varepsilon(D) \setminus D$, then

$$d(R(x), u) \leq d(R(x), R(u)) + d(R(u), u) \leq d(x, u) + \text{dist}(u, D) + \delta \leq d(x, u) + \varepsilon + \delta.$$

Thus

$$B(R(x); \varepsilon + \delta) \cap (\cap_{u \in N_\varepsilon(D)} B(u; d(u, x))) \neq \emptyset.$$ 

Since obviously $x \in B(x; \text{dist}(x, D) - (\varepsilon + \delta)) \cap (\cap_{u \in N_\varepsilon(D)} B(u; d(u, x)))$ we conclude that $S \neq \emptyset$.

Next note that if $v \in S$ then $d(v, R(x)) \leq \varepsilon + \delta$ so $v \in N_{\varepsilon+\delta}(D) = N_\delta(N_\varepsilon(D))$.

We now define $R'(x)$ to be any point of $S$ and $R'(u) = u$ for all $u \in N_\varepsilon(D)$. This gives us a nonexpansive mapping of $N_\varepsilon(D) \cup \{x\}$ into $N_\delta(N_\varepsilon(D))$ which leaves each point of $N_\varepsilon(D)$ fixed. The conclusion now follows from Lemma 3.1.  

We now show that the conclusion of Theorem 3.1 can be strengthened if the underlying space $M$ is also hyperconvex. In the proof of this theorem we use the fact that neighborhoods of weakly externally hyperconvex sets are themselves weakly externally hyperconvex.

**Theorem 3.3.** Let $D$ be a weakly externally hyperconvex subset of a hyperconvex metric space $M$. Then given any $\epsilon > 0$ there is a nonexpansive retraction $R_\epsilon$ of $M$ onto $D$ with the property that given any $u \in M \setminus D$ there exists $x \in M$ such that $d(u, x) \leq \epsilon$ and $d(x, R_\epsilon(x)) = \text{dist}(x, D)$. Moreover, if $\text{int}(D) \neq \emptyset$ then $R_\epsilon$ may be chosen so that $R_\epsilon(M \setminus D) \subset \partial D$.

**Proof.** Fix $\rho > 0$. Let $S_0 = \partial D$, $D_n = N_{\rho/n}(D)$, and $S_{\rho/n} = \{x \in M : \text{dist}(x, D) = \rho/n\}$, $n = 1, 2, \ldots$ By Theorem 3.1 there exists a nonexpansive retraction $r_0$ of $M$ onto $D_1$ for which $r_0(M \setminus D_\rho) \subset N_{\rho/2}(S_\rho)$.

Proceeding inductively, for each $n \geq 1$ there exists a nonexpansive retraction $r_n$ of $N_{\rho/2^{n-1}}(D)$ onto $N_{\rho/2n}(D)$ such that the restriction of $r_n$ to $S_{\rho/2^{n-1}}$ is a selection of the metric projection of $S_{\rho/2^{n-1}}$ onto $D_{\rho/2^n}$ and for which $r_n(N_{\rho/2^{n-1}}(D) \setminus N_{\rho/2n}(D)) \subset N_{\rho/2^n}(S_{\rho/2n})$.

Now if we set $\tilde{r}_n(x) = r_n \circ \cdots \circ r_0(x)$ for $x \in M$ and $n \geq 0$, then it follows that

$$\sum_{i=1}^{\infty} d(\tilde{r}_i(x), \tilde{r}_{i+1}(x)) \leq 2\rho \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$ 

Thus $\{r_n(x)\}$ is a Cauchy sequence. Define $R : M \to D$ by setting $R(x) = \lim_{n \to \infty} \tilde{r}_n(x)$.

Clearly, $R$ is a nonexpansive retraction of $M$ onto $D$, and since $x \in M \setminus D \Rightarrow \tilde{r}_n(x) \in M \setminus D$ for each $n$, it follows that if $\text{int}(D) \neq \emptyset$ then $R(M \setminus D) \subset \partial D$.

Also, if $x \in S_\rho$ then

$$d(x, R(x)) \leq \sum_{i=1}^{\infty} d(\tilde{r}_i(x), \tilde{r}_{i+1}(x)) = \rho \sum_{i=1}^{\infty} \frac{1}{2^i} = \rho.$$ 

Since $\rho > 0$ is arbitrary, the full conclusion of the theorem follows by the method of Theorem 3.1. □
4. NONEXPANSIVE SELECTIONS

If $D$ is compact in Theorem 3.1 the in general set-valued metric projection (best approximation map) always has a nonexpansive selection.

**Theorem 4.1.** Suppose $D$ is a compact weakly externally hyperconvex subset of a metric space $M$. Then $D$ is a proximinal nonexpansive retract of $M$.

Combined with Theorem 2.1, Theorem 4.1 gives the following.

**Theorem 4.2.** A compact subset $D$ of a hyperconvex metric space $M$ is weakly externally hyperconvex if and only if it is a proximinal nonexpansive retract of $M$.

**Lemma 4.1.** Let $M$ be a metric space and let $N$ be a compact subset of $M \setminus N$ such that for every finite subset $F$ of $M \setminus N$ there exists a proximinal nonexpansive retraction $\phi_F: N \cup F \to N$. Then $N$ is a proximinal nonexpansive retract of $M$.

**Proof.** Let $\mathcal{F}$ be the family of nonempty finite subsets of $M \setminus N$ ordered by the set inclusion $\subseteq$. For $F \in \mathcal{F}$, set

$$ U_F = \{ G \in \mathcal{F} : G \supseteq F \}. $$

Since $U_F \cap U_G = U_{F \cup G}$ it is easy to see that $\{ U_F \}_{F \in \mathcal{F}}$ is a filter base on $2^M$. Let $\mathcal{U}$ be an ultrafilter on $2^M$ which extends this base.

Fix $p \in N$, and for $F \in \mathcal{F}$ and $x \in M$ define

$$ r_F(x) = \begin{cases} \phi_F(x) & \text{if } x \in N \cup F, \\ p & \text{otherwise}. \end{cases} $$

Now define $r: M \to N$ by setting

$$ r(x) = \lim_{U \in \mathcal{U}} r_F(x), \quad x \in M. $$

Since $N$ is compact and $\mathcal{U}$ is an ultrafilter, the mapping $r$ is well defined. Moreover,

$$ U_{\{1\}} = \{ F \in \mathcal{F} : x \in F \} \subset \{ F \in \mathcal{F} : r_F(x) = \phi_F(x) \} \in \mathcal{U}. $$

This proves that

$$ d(r(x), r(y)) = \lim_{U \in \mathcal{U}} d(r_F(x), r_F(y)) = \lim_{U \in \mathcal{U}} d(\phi_F(x), \phi_F(y)) \leq d(x, y) $$

for $x, y \in M$, and also that $r(x) \in \{ y \in N : d(x, y) = \text{dist}(x, N) \}$.  

**Proof of Theorem 4.1** Lemma 2.2 assures that for each finite subset $F$ of $M \setminus D$ there is a nonexpansive retraction $r$ of $F \cup D$ onto $D$ with the property $d(z, r(z)) = \text{dist}(z, D)$ for each $z \in F$. Theorem 4.1 is a direct consequence of this fact and Lemma 4.1.
Remark 4.1. Theorem 4.1 holds if $D$ is compact relative to any topology for which the metric $d : M \times M \to \mathbb{R}$ is lower semicontinuous. Thus, if $M$ is a Banach space it suffices to assume that $D$ is weakly compact (or weak$^*$ compact if $M$ is a conjugate space).

Remark 4.2. It seems plausible that Theorem 4.2 holds without the compactness assumption. However, our methods do not quite establish this fact.

5. APPLICATIONS

For an application of Theorem 3.1 let $\gamma$ denote the usual Kuratowski measure of noncompactness. Recall that a mapping $T : M \to M$ is said to be condensing if $T$ is continuous and if $\gamma(T(A)) < \gamma(A)$ for every bounded subset $A$ of $M$ for which $\gamma(A) > 0$. It has been noted earlier (see [5, 13]) that if $M$ is a bounded hyperconvex space then every condensing mapping $T : M \to M$ has at least one fixed point.

We are now able to extend this fact as follows.

**Theorem 5.1.** Suppose $D$ is a bounded weakly externally hyperconvex subset of a metrically convex metric space $M$, and let $T : D \to M$ be a uniformly continuous condensing mapping for which $T(\partial D) \subset D$. Then $T$ has a fixed point.

**Proof.** Let $\varepsilon > 0$ and choose $\varepsilon' \leq \varepsilon$ so that $d(u, v) \leq \varepsilon' \Rightarrow d(T(u), T(v)) \leq \varepsilon$. Now let $R_\varepsilon$ be the nonexpansive retraction assured by Theorem 3.1. It is easy to see that the mapping $R_\varepsilon \circ T : D \to D$ is condensing, and since $D$ is hyperconvex $R_\varepsilon \circ T$ has a fixed point, say, $x_\varepsilon \in D$. If $T(x_\varepsilon) \in D$, then $R_\varepsilon \circ T(x_\varepsilon) = T(x_\varepsilon) = x_\varepsilon$. If $T(x_\varepsilon) \notin D$, then there exists $y \in \partial D$ such that $d(x_\varepsilon, y) \leq \varepsilon'$. In this case (since $T(y) \in D$) we have

$$d(y, T(y)) \leq d(y, x_\varepsilon) + d(x_\varepsilon, T(y))$$
$$\leq \varepsilon + d(R_\varepsilon \circ T(x_\varepsilon), R_\varepsilon \circ T(y))$$
$$\leq \varepsilon + d(T(x_\varepsilon), T(y))$$
$$\leq 2\varepsilon.$$

This proves that $\inf\{d(y, T(y)) : y \in D\} = 0$. Since $T$ is condensing it easily follows that $T$ has a fixed point in $D$. 

The following is an easy consequence of Theorem 3.3. In [13] it was observed that this fact holds for admissible sets $D$. 

Theorem 5.2. Suppose $D$ is a bounded weakly externally hyperconvex subset of a hyperconvex metric space $M$, and suppose $T : D \to M$ is a nonexpansive mapping for which $T(\partial D) \subset D$. Then $T$ has a fixed point.

Proof. Let $R$ be a nonexpansive retraction of $M$ onto $D$ for which $R(M \setminus D) \subset \partial D$. Then $R \circ T : D \to D$ is nonexpansive and has the same fixed point set as $T$.

Another consequence of Theorem 3.3 is

Theorem 5.3. Suppose $D$ is a bounded weakly externally hyperconvex subset of a hyperconvex metric space $M$, and suppose $T : D \to M$ is a condensing mapping for which $T(\partial D) \subset D$. Then $T$ has a fixed point.

Finally, since compact hyperconvex spaces have the fixed point property for continuous mappings (e.g., [10, 16]), Theorem 4.2 yields Fan's approximation principle [6] for compact weakly externally hyperconvex sets.

Theorem 5.4. Let $D$ and $M$ satisfy the assumptions of Theorem 4.2 and suppose $T : D \to M$ is a continuous mapping. Then there exists $x \in D$ such that $d(x, T(x)) = \inf \{d(y, T(x)) : y \in D\}$.

Note added in proof. The assertion that we may assume $d > 0$ in the proof of Theorem 2.1 is likely not obvious. To see this, assume $d = 0$ and let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers for which $\sum \varepsilon_n < \infty$. Select $u_1 \in B(z; \text{dist}(z, D)) \cap D \cap (\cap B(v; B(\varepsilon_n + \varepsilon_1))$ and consider $B(z; \text{dist}(z, D)) \cap D \cap B(u; \varepsilon_1) \cap (\cap B(v; B(\varepsilon_n + \varepsilon_1))$. If this intersection is empty add $B(u; \varepsilon_1)$ to the family $\{B(v; \varepsilon_n)\}$ and obtain a new family for which $d > 0$. Otherwise, select $u_1$ in this intersection, consider $B(z; \text{dist}(z, D)) \cap D \cap B(u; \varepsilon_1) \cap (\cap B(v; B(\varepsilon_n + \varepsilon_1))$, and repeat the previous step. Either this process terminates after a finite number of steps, providing a new family for which $d > 0$, or we obtain a Cauchy sequence $\{u_n\}$ whose limit lies in $D_1 \cap D$, which is a contradiction.

REFERENCES