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A Remark on Integral Representations of $GL_{\mathbb{Z}}(n)$

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Let G be the general linear group $GL_{\mathbb{Z}}(n)$ with integral coefficients. A (integral) representation of G is an abelian group $\mathbb{W} \approx \mathbb{Z}^{\oplus g}$ with a G-action which is given by a polynomial with integral coefficients in the coordinates in G and the inverse of the determinant and the coordinates in $\mathbb{W}$.

We have the tautological representation of G on $\mathbb{V} = \mathbb{Z}^{\oplus n}$. Also we have the exterior powers $\Lambda^i\mathbb{V}$, the symmetric powers $S^i\mathbb{V}$, and the divided powers $\Gamma^i\mathbb{V} = \widehat{S^i(\hat{\mathbb{V}})}$, where $\hat{}$ denotes duality. Buchsbaum and Akin [1] have raised the question of writing any representation of G as the quotient of standard representations. I will prove a result in this direction.

THEOREM. *Any representation $\mathbb{W}$ is the quotient of direct sums of the diagonal representation of G on*

$$\Gamma^{i_1}\mathbb{V} \otimes \cdots \otimes \Gamma^{i_n}\mathbb{V} \otimes (\Lambda^n\mathbb{V})^{\otimes j}.$$

This result is the best possible I think. I will show that $\Gamma^i\mathbb{V}$ is not the quotient of $\mathbb{V} \otimes \cdots \otimes \mathbb{V} \otimes (\Lambda^n\mathbb{V})^{\otimes j}$ when $i = 2$ and $n = 2$. For the general case one reduces to the case $n = 2$ and does a calculation similar to that of the case of binary quadratic forms.

To prove the theorem we will translate the problem geometrically, and then algebraically a representation $\mathbb{W}$ of G can be thought of as the integral points of the scheme $\mathbb{W}' = \operatorname{Spec}(\operatorname{Sym}_{\mathbb{Z}} W)$, where W is the module of $\mathbb{Z}$-valued linear functions on $\mathbb{W}$. Thus we have a $GL_{\mathbb{Z}}(n)$-action on $\mathbb{W}'$, which corresponds to a comodule structure α: $W \to A \otimes_{\mathbb{Z}} W$, where A is the Hopf algebra $\Gamma(GL_{\mathbb{Z}}(n)_1 \mathcal{O}_{GL_{\mathbb{Z}(n)}}) = \operatorname{Sym}_{\mathbb{Z}}(\widehat{\operatorname{Hom}(\mathbb{V}, \mathbb{V})})_{\det}$, where det as a subscript denotes localization by the determinant.

By the coassociative law α is an A-comodule homomorphism where A coacts on $A \otimes_{\mathbb{Z}} W$ through the factor A(i.e., $\alpha'(a \otimes w) = \mu\alpha \otimes w$, where μ is the comultiplication in A). As coidentity $\times\, 1d_W$ is the inverse of α, α is an A-comodule injection $W \hookrightarrow A \otimes \cdots \otimes A$. From the localization we have an

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A-comodule injection $W \subsetneq (\oplus \operatorname{Sym}_{\mathbb{Z}}(\operatorname{Hom}(\mathbb{V}, \mathbb{V})) \otimes L^i))$, where $L = \mathbb{Z}$ with the comodule structure $\alpha(1) = (\det)^i \otimes 1$. As a (left)-comodule $\operatorname{Sym}_{\mathbb{Z}}(\widehat{\operatorname{Hom}(\mathbb{V}, \mathbb{V})}) = \operatorname{Sym}_{\mathbb{Z}}(\oplus^n \mathbb{V}) = \oplus_{(i_*)} \operatorname{Sym}^{i_1}(V) \otimes \cdots \otimes \operatorname{Sym}^{i_n}(V)$. Translating back to representations we get a homomorphism

$$\alpha^*: \left(\bigoplus_{(i_*)} \Gamma^{i_1}\mathbb{V} \otimes \cdots \otimes \Gamma^{i_n}\mathbb{V} \otimes (\Lambda^n \mathbb{V})^j \right) \to \mathbb{W}.$$

We only need to remark that α^* is surjective because $\alpha \otimes \mathbb{Z}/_{p\mathbb{Z}}$ is injective for any prime p for the same reason that α is injective. This proves the theorem.

Next we will see that $\Gamma^2\mathbb{V}$ is not a quotient of $\mathbb{V}^{\otimes n} \otimes (\Lambda^n \mathbb{V})^{\otimes i}$ when $n = 2$. If such quotient exists then $r + 2i = 2$ which can be seen by the action of the diagonal of G. Dually we will show that if $S^2\mathbb{V} \otimes (\Lambda^2\mathbb{V})^{\otimes i}$ is contained in $\mathbb{V}^n$ then the relative quotient has torsion.

First of all, $\mathbb{V}^{\otimes n}$ has a natural filtration with factors $S^k\mathbb{V} \otimes (\Lambda^2\mathbb{V})^l$, where $m = k + 2l$. This filtration is defined by induction on m. By induction one needs to filter the factor $\mathbb{V} \otimes S^{k^1}\mathbb{V} \otimes (\Lambda^2\mathbb{V})^{l^1}$ of $\mathbb{V} \otimes$ (filtration of $\mathbb{V}^{\otimes m-1}$) but we have the exact Kozul complex

$$0 \to \Lambda^2\mathbb{V} \otimes S^{p-1}\mathbb{V} \to \mathbb{V} \otimes S^p\mathbb{V} \to S^{p+1}\mathbb{V} \to 0.$$

Thus we have the filtration.

Let e and f be a basis of $\mathbb{V}$. Let U_λ for integral λ be the unipotent transformation in G given by $U_\lambda(e) = e$ and $U_\lambda(f) = f + \lambda$. The interesting fact is the U_λ-invariants for all λ is exact on the filtration. In fact we can write the invariants in $\mathbb{V}^{\otimes n}$ which are compatible with the filtration.

Let V_λ be the opposite unipotent transformation $V_\lambda(f) = f$ and $V_\lambda(e) = e + \lambda f$. Consider

$$V_\lambda(\psi) = \psi + \lambda\psi_1 + \lambda^2\psi_2 \qquad \text{for all } \lambda.$$

If ψ is generating invariant of $S^2(\mathbb{V}) \otimes (\Lambda^2\mathbb{V})^{\otimes i}$, then ψ_1 is divisible by 2 in $\mathbb{V}^{\otimes n}$, but this is clearly not true. Then the point is proven.

References

1. D. Buchsbaum and Akin, Characteristic-free representation theory of the general linear group, *Adv. in Math.* **58** (1985), 149–200.