Existence results and asymptotic behavior for nonlocal abstract Cauchy problems

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Abstract

The purpose of this paper is to study the existence and asymptotic behavior of solutions for Cauchy problems with nonlocal initial datum generated by accretive operators in Banach spaces.

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1. Introduction

In this article we discuss the existence of integral solutions for the nonlocal initial value problem

\[
\begin{aligned}
    u'(t) + A(u(t)) &\ni F(u)(t), \quad t \in (0, T), \\
    u(0) &= g(u),
\end{aligned}
\]

(1)

where \( A : D(A) \subseteq X \to 2^X \) is an \( m \)-accretive operator on \( X \), \( F : \mathcal{L}^1(0, T; X) \to \mathcal{L}^1(0, T; X) \), and \( g : C([0, T]; X) \to D(A) \) are given functions.

The motivation for this study is that nonlocal Cauchy problems have, in some cases, better effects in applications than the traditional Cauchy problem with a local initial datum. For example in [19], the author used this type of problems to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

The task concerning abstract nonlocal semilinear initial value problems (i.e., when \( A \) in Eq. (1) is linear) was initiated by Byszewski [15]. Since then, using different fixed point theorems, several authors have investigated the problem of nonlocal initial conditions for different classes of abstract differential equations in Banach spaces, among others, we refer to [12,13,28] and the references therein. In this context, we should mention that in [7], the authors use Sadovskii’s fixed point theorem to prove the existence of mild solutions to a semilinear abstract integrodifferential equation with a nonlocal initial condition under compactness and analyticity assumptions on an associated linear semigroup. In the first part of our paper we will improve some of these results.

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The nonlinear nonlocal problem (1) was first considered by Aizicovici and Gao [1]. There, both \( F \) and \( g \) satisfy Lipschitz conditions. Later, this study has been a subject of research in [2–4,27], among other papers (see references therein). In most of the above mentioned papers the Lipschitz conditions on \( F \) and \( g \) are relaxed at the expense of stronger restrictions on \( A \). In one of the results of [27] it is required that the semigroup generated by \(-A\) be equicontinuous. Although, in many articles it is assumed that \(-A\) generates a compact semigroup, which is quite a strong restriction on \( m\)-accretive operators, at least in the case of hyperbolic problems (see [26]). Here we will prove some existence results for integral solution for problem (1) without using the compactness of the semigroup generated by \(-A\), which still allows us to relax the Lipschitz conditions on \( F \) and \( g \) given in [1].

2. Preliminaries

Let \((X, \| \cdot \|)\) be a real Banach space and let \( X^* \) be its dual. Denote by \( C([0, T]; X) \) the space of \( X \)-valued continuous functions on \([0, T]\) with the norm \( \| u \|_\infty = \sup \{ \| u(t) \| : t \in [0, T] \} \), and denote by \( L^1(0, T; X) \) the space of \( X \)-valued Bochner integrable functions on \([0, T]\) with the norm \( \| u \|_1 = \int_0^T \| u(t) \| \, dt \).

A mapping \( A : X \to 2^X \) will be called an operator on \( X \). The domain of \( A \) is denoted by \( D(A) \) and its range by \( \mathcal{R}(A) \). An operator \( A \) on \( X \) is said to be accretive if the inequality \( \| x - y + \lambda(z - w) \| \geq \| x - y \| \) holds for all \( \lambda \geq 0 \), \( z \in Ax \), and \( w \in Ay \). If, in addition, \( \mathcal{R}(I + \lambda A) \) is for one, hence for all, \( \lambda > 0 \), precisely \( X \), then \( A \) is called \( m \)-accretive. Accretive operators were introduced by F.E. Browder [14] and T. Kato [20] independently. Those accretive operators which are \( m \)-accretive play an important role in the study of nonlinear partial differential equations.

We consider the Cauchy problem

\[
\begin{align*}
  u'(t) + A(u(t)) &\equiv f(t), \quad t \in (0, T), \\
  u(0) &= x_0 \in D(A),
\end{align*}
\]

where \( A \) is \( m \)-accretive on \( X \), \( f \in L^1(0, T; X) \). It is well known that (2) has a unique integral solution in the sense due to Bénilan [9], i.e., there exists a unique continuous function \( u : [0, T] \to \bar{D}(A) \) such that \( u(0) = x_0 \) and moreover for each \( (x, y) \in A \) and \( 0 \leq s \leq t \leq T \) we have that

\[
\| u(t) - x \|^2 - \| u(s) - x \|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle \, d\tau.
\]

Here the function \( \langle , \rangle_A : X \times X \to \mathbb{R} \) is defined by \( \langle y, x \rangle_A = \sup \{ x^*(y) : x^* \in J(x) \} \), where \( J : X \to 2^{X^*} \) is the duality mapping on \( X \), i.e., \( J(x) = \{ x^* \in X^* : x^*(x) = \| x \|^2, \| x^* \| = \| x \| \} \).

It is also well known that (3) yields the inequality

\[
\| u(t) - x \| \leq \| x_0 - x \| + \int_0^t \| f(s) - y \| \, ds,
\]

for all \( (x, y) \in A \) and \( 0 \leq t \leq T \).

If \( u, v \) are integral solutions of \( u'(t) + A(u(t)) \equiv f(t) \) and \( v'(t) + A(v(t)) \equiv g(t) \) respectively, with \( f, g \in L^1(0, T; X) \), then

\[
\| u(t) - v(t) \| \leq \| u(0) - v(0) \| + \int_0^t \| f(s) - g(s) \| \, ds.
\]

We refer the reader to [8,11,17], for background material on accretivity.

Let \( (E, d) \) be a complete metric space and \( B \) the family of bounded subsets of \( E \). For every \( B \in B \) define the Kuratowski measure of noncompactness \( \alpha(B) \) of the set \( B \) as the infimum of the numbers \( r \) such that \( B \) admits a finite covering by sets of diameter smaller than \( r \) (see [6]).

**Definition 1.** Let \( D \) be a nonempty subset of \( E \). A mapping \( T : D \to E \) is said to be an \( \alpha - k\)-set contraction, \( k \in (0, 1) \), if \( T \) is continuous and if for all bounded subsets \( C \) of \( D \), \( \alpha(T(C)) \leq k \alpha(C) \). \( T \) is said to be \( \alpha \)-condensing if \( T \) is continuous and \( \alpha(T(A)) < k \alpha(A) \) for every bounded subset \( A \) of \( D \) with \( \alpha(A) > 0 \).
Let \( K \) be a subset of a Banach space \( X \). Suppose \( T_1 : K \to X \) is a contraction mapping with Lipschitz constant \( k < 1 \), and suppose \( T_2 : K \to X \) is a continuous and compact mapping \((\overline{T(D)} \text{ is compact for every bounded subset } D \text{ of } K)\). Then it is not difficult to see that the mapping \( T = T_1 + T_2 : K \to X \) is an \( \alpha - k \)-set contraction.

The following theorems will be the key in the proof of our main results. The first one was proved by Sadovskii [25] in 1967. In 1955 Darbo [18] proved the same result for \( \alpha - k \)-set contractions, \( k < 1 \). Such mappings are obviously \( \alpha \)-condensing. The second one is a sharpening of the first one and it is due to W.V. Petryshyn [24].

**Theorem 2** (Darbo–Sadovskii). Suppose \( M \) is a nonempty bounded closed and convex subset of a Banach space \( X \) and suppose \( T : M \to M \) is \( \alpha \)-condensing. Then \( T \) has a fixed point.

**Theorem 3** (Petryshyn). Let \( X \) be a Banach space and let \( T : X \to X \) be an \( \alpha \)-condensing mapping. If there exists \( r > 0 \) such that \( Tx \neq \lambda x \) for any \( \lambda > 1 \) whenever \( \|x\| = r \), then \( T \) has a fixed point.

### 3. Semilinear case

In this section we discuss the nonlocal initial value problem

\[
\begin{aligned}
u'(t) &= A(u(t)) + f(t, u(t)), \quad t \in (0, T), \\
u(0) &= g(u),
\end{aligned}
\tag{4}
\]

where

(a) \( A \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e., \( C_0 \)-semigroup) \( R(t) \). Let \( N = \sup\{\|R(t)\|, t \in [0, T]\} \).

(b) \( f : [0, T] \times X \to X \) is a compact mapping such that there exist \( m, k \in L^1(0, T; \mathbb{R}^+) \) and an increasing continuous function \( \Omega : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \|f(t, x)\| \leq m(t)\Omega(\|x\|) + k(t) \) for any \( x \in X \) and a.e. \( t \in [0, T] \). Moreover, \( f(., x) \) is measurable for \( x \in X \), and \( f(t, .) \) is continuous for a.e. \( t \in [0, T] \).

(c) \( g : C([0, T]; X) \to D(A) \) is a continuous mapping such that there exist \( \beta \in (0, 1) \) and \( \gamma > 0 \) satisfying \( \|g(u)\| \leq \beta \|u\|_\infty + \gamma \). Moreover, for every \( C \subset C([0, T]; X) \) bounded and noncompact the inequality \( \alpha(g(C)) < \frac{1}{N} \alpha(C) \) holds.

(d) \( \lim \inf_{r \to \infty} \frac{N\|m\|_1\Omega(r) + N\beta r}{r} < 1 \).

**Theorem 4.** If (a), (b), (c) and (d) hold, then there exists at least one mild solution of problem (4).

Recall that a mild solution of problem (4) is a function \( u \in C([0, T]; X) \) such that

\[
u(t) = R(t)g(u) + \int_0^t R(t-s)f(s, u(s)) \, ds.
\]

This type of problem was studied for example in [12,28].

**Proof of Theorem 4.** It is clear that we will obtain the result if we show that the mapping \( J : C([0, T]; X) \to C([0, T]; X) \) defined by

\[
Ju(t) = R(t)g(u) + \int_0^t R(t-s)f(s, u(s)) \, ds
\]

has a fixed point.

Now, we can introduce the following two mappings:

\[
K : C([0, T]; X) \to C([0, T]; X) \quad \text{by} \quad K(u)(t) = \int_0^t R(t-s)f(s, u(s)) \, ds,
\]
and

\[ S : C([0, T]; X) \to C([0, T]; X) \quad \text{by} \quad S(u)(t) = R(t)g(u). \]

This allows us to write \( J = S + K \).

In Lemma 3.1 of [28], by using Ascoli’s theorem, it is proved that \( K \) is a compact and continuous mapping.

On the other hand, it is easy to see that \( S \) is a continuous function.

Since \( R(t) \) is a \( C_0 \)-semigroup, we have

\[ \| Su - Sv \|_\infty = \sup \{ \| Su(t) - Sv(t) \| : t \in [0, T] \} = \sup \{ \| R(t)g(u) - R(t)g(v) \| : t \in [0, T] \} \leq N \| g(u) - g(v) \| . \]

Now, we are going to show that \( S \) is \( \alpha \)-condensing. Indeed, let \( C \) be a bounded set of \( C([0, T]; X) \).

By hypothesis we know that \( \alpha(g(C)) < \frac{1}{N} \alpha(C) \), therefore \( \alpha(S(C)) \leq N\alpha(g(C)) < \alpha(C) \).

Since \( J = S + K \), then \( J \) is \( \alpha \)-condensing.

Finally, given \( r > 0 \), consider \( B_r(0) = \{ u \in C([0, T]; X) : \| u \|_\infty \leq r \} \). We will show that there exists \( r_0 > 0 \) such that \( B_{r_0}(0) \) is \( J \)-invariant.

Otherwise, for every \( r > 0 \) we can find \( u_r \in B_r(0) \) with \( \| Ju_r \|_\infty > r \). This means that \( \frac{1}{r} \| Ju_r \|_\infty > 1 \). Therefore we can assume that \( \liminf_{r \to \infty} \frac{1}{r} \| Ju_r \|_\infty \geq 1 \).

\[ \| Ju_r(t) \| \leq \| R(t)g(u_r) \| + \int_0^t \| R(t-s)f(s, u_r(s)) \| \, ds \]

\[ \leq N \left( \| g(u_r) \| + \int_0^t \| f(s, u_r(s)) \| \, ds \right) \]

\[ \leq N \left( \beta \| u_r \|_\infty + \gamma + \int_0^T m(s) \Omega \left( \| u_r(s) \| \right) ds + \| k \|_1 \right) \]

\[ \leq N \left( \beta r + \Omega(r) \| m \|_1 + \gamma + \| k \|_1 \right). \]

Consequently,

\[ \liminf_{r \to \infty} \frac{1}{r} \| Ju_r \|_\infty \leq \liminf_{r \to \infty} \frac{N \| m \|_1 \Omega(r) + N\beta r}{r} < 1. \]

Thus, by Darbo–Sadovkii’s theorem we obtain the result. \( \square \)

We shall finish with an extension of Theorems 2.1 and 2.2 of [28]. In order to do it we introduce the following hypothesis

\( (c') \quad g : C([0, T]; X) \to \overline{D(A)} \) is a continuous mapping such that there exists \( M > 0 \) satisfying \( \| g(u) \| \leq M \). Moreover, for every \( C \subset C([0, T]; X) \) bounded and noncompact the inequality \( \alpha(g(C)) < \frac{1}{N} \alpha(C) \) holds.

**Theorem 5.** Let (a), (b) and (c’) hold. Then problem (4) has a mild solution provided that, either

(i) \( \Omega \) is the identity mapping, or

(ii) \( \int_0^T m(s) \, ds < \int_0^\infty \frac{ds}{\Omega(s)} \).

**Proof.** Following the steps of the above result it is easy to see that \( J : X \to X \) is an \( \alpha \)-condensing mapping. Therefore, in order to get the result it will be enough to show that \( J \) is under the conditions of Theorem 3.

Indeed, let \( \lambda > 1 \), and suppose that \( Ju = \lambda u \), then

\[ \| u(t) \| \leq \lambda \| u(t) \| \leq NM + N \int_0^t m(s) \Omega \left( \| u(s) \| \right) ds + N\| k \|_1. \]
Define $x(t) := NM + N \int_0^t m(s)\Omega(\|u(s)\|)ds + N\|k\|_1$. Since $x(t)$ is differentiable, $\Omega$ is an increasing function and $\|u(t)\| \leq x(t)$, we have

$$x'(t) \leq Nm(t)\Omega(x(t)).$$

(5)

If $\Omega$ is the identity mapping the last inequality is

$$x'(t) \leq Nm(t)x(t),$$

hence, by using Gronwall’s inequality, we obtain

$$x(t) \leq e^{NMt}\Omega((NM + N\|k\|_1)).$$

Thus, if we take $r = e^{NM}\Omega((NM + N\|k\|_1) + 1$ we see that if $\|u\|_\infty = r$, then $Ju \neq \lambda u$ for every $\lambda > 1$. This implies that the conditions of Theorem 3 hold.

Otherwise, we have to suppose (ii), therefore from (5) we derive

$$\int_0^x(t) \frac{ds}{NM + N\|k\|_1} \leq \int_0^t m(s)ds < \int_0^T m(s) ds < \int_0^\infty \frac{ds}{NM + N\|k\|_1}.$$ 

Consequently there exists $r > 0$, independent of $\lambda$, such that $x(t) < r$ for all $t \in [0, T]$, therefore $\|u\|_\infty < r$. This allows us to apply Theorem 3 to obtain the result. ∎

4. Nonlinear case

**Definition 6.** A function $u : [0, T] \rightarrow \overline{D(A)}$ is said to be an integral solution of Eq. (1) if it is an integral solution of Eq. (2), with $F(u)$ and $g(u)$ in place of $f$ and $x_0$, respectively.

Consider the abstract Cauchy problem (1) in a Banach space $X$ under the following assumptions:

(H1) $A : D(A) \subseteq X \rightarrow 2^X$ is an $m$-accretive operator on $X$.

(H2) $F : L^1(0, T; X) \rightarrow L^1(0, T; X)$ is a continuous and compact mapping such that $\|F(v)\|_1 \leq a\|v\|_1 + b$ for every $v \in L^1(0, T; X)$, for some positive constants $a$ and $b$.

(H2) $F : L^1(0, T; X) \rightarrow L^1(0, T; X)$ is a continuous and compact mapping such that $\|F(v)(t)\|_X \leq a\|v(t)\|_X + k(t)$ for every $v \in L^1(0, T; X)$ and for all $t \in [0, T]$, where $a$ is a positive constant and $k \in L^1(0, T; \mathbb{R}^+)$ with $\|k\|_1 = b$.

(H3) $g : C([0, T]; X) \rightarrow \overline{D(A)}$ is a continuous mapping such that there exist $\beta \in (0, 1)$ and $\gamma > 0$ satisfying $\|g(u)\| \leq \beta\|u\|_\infty + \gamma$. Moreover, for every $C \subset C([0, T]; X)$, bounded and noncompact, the inequality $\alpha(g(C)) < \alpha(C)$ holds.

(H4) $1 - (\beta + Ta) > 0$.

(H5) $1 - \beta e^{aT} > 0$.

**Theorem 7.** Let (H1), (H2), (H3) and (H4) hold. Then problem (1) has at least one integral solution.

**Proof.** Let $v \in C([0, T]; X)$ be fixed and consider the Cauchy problem

$$\begin{cases}
u'(t) + A(u(t)) \ni F(v)(t), & 0 < t < T, \\
u(0) = g(v).
\end{cases}$$

(6)

Since $v \in C([0, T]; X) \subset L^1(0, T; X)$ and $F$ maps $L^1(0, T; X)$ into itself, it is clear that $F(v) \in L^1(0, T; X)$, moreover, $g(v) \in \overline{D(A)}$. Therefore (6) has a unique integral solution $u_v$.

Define the mapping $J : C([0, T]; X) \rightarrow C([0, T]; X)$ by $J(v) = u_v$. We wish to prove that $J$ has a fixed point. To this end, we will divide the proof into several steps:

**Step 1.** Let us show that $J$ is a continuous mapping.

Indeed, let $(v_n)$ be a sequence in $C([0, T]; X)$ which is convergent to $v \in C([0, T]; X)$. In order to show that $Jv_n \rightarrow Jv$ we proceed as follows:
Theorem 8. Let \( (H_1), (H_2'), (H_3) \) and \( (H_5) \) hold. Then problem (1) has at least one integral solution.

Proof. We follow the argument used in the proof of Theorem 7 and thus we may conclude that the mapping \( J : C([0, T]; X) \to C([0, T]; X) \) is \( \alpha \)-condensing. Therefore, in order to obtain the result we argue as follows:

Consider \( \lambda > 1 \), and suppose that \( J(v) = \lambda v \). Then, if \( x \in D(A) \) and \( y \in A(x) \), we have:

\[
\| v(t) \| \leq \| v(t) \| \leq 2\| x \| + \beta \| v \|_\infty + \gamma + a\| v \|_1 + b + T\| y \|.
\]

Since \( v \in C([0, T]; X) \) and \( \lambda > 1 \) we obtain

\[
\| v \|_\infty \leq \lambda^r \| v \|_\infty \leq (\beta + Ta)\| v \|_\infty + 2\| x \| + T\| y \| + \gamma + b.
\]

Thus, since by \( (H_4), 1 - (\beta + Ta) > 0 \), we have

\[
\| v \|_\infty \leq \frac{2\| x \| + T\| y \| + \gamma + b}{1 - (\beta + Ta)} = C.
\]

Clearly, \( C \) is a constant which is independent of \( v \) and \( \lambda \). Therefore, if we take \( r = C + 1 \) we infer that if \( \| v \|_\infty = r \), then \( Jv \neq \lambda v \) for every \( \lambda > 1 \). Thus, by Theorem 3 we may conclude that \( J \) has a fixed point. \( \square \)
Applying Gronwall’s inequality to (7) yields
\[ \| v(t) \| \leq (2\|x\| + \gamma + b + T\|y\| + \beta\|v\|_\infty)e^{aT}, \]
then
\[ \| v \|_\infty \leq (2\|x\| + \gamma + b + T\|y\| + \beta\|v\|_\infty)e^{aT}. \]
Therefore,
\[ (1 - \beta e^{aT})\| v \|_\infty \leq (2\|x\| + \gamma + b + T\|y\|)e^{aT}. \]
Thus, since by (H5), \( 1 - \beta e^{aT} > 0 \), we have
\[ \| v \|_\infty \leq \frac{(2\|x\| + T\|y\| + \gamma)e^{aT}}{1 - \beta e^{aT}} = K. \]
Clearly, \( K \) is a constant which is independent of \( v \) and \( \lambda \). Therefore if we take \( r = K + 1 \) we see that if \( \| v \|_\infty = r \), then \( Jv \neq \lambda v \) for every \( \lambda > 1 \). Thus, by Theorem 3 we may conclude that \( J \) has a fixed point. \( \square \)

**Remark 9.** If we have either in (H3) \( \beta = 0 \) or in (H'2) \( a = 0 \), i.e., either \( g \) or \( F \) is a bounded function, then by Theorem 8, problem (1) has a global integral solution.

Now, consider again the abstract Cauchy problem (1) under the following assumptions:

(H''2) \( F : C([0, T]; X) \rightarrow L^1([0, T]; X) \) is a continuous mapping such that there exist \( a \in (0, 1) \) and \( b > 0 \) satisfying
\[ \| F(u) \|_1 \leq a\|u\|_\infty + b. \]
Moreover, there exists \( k_1 \in (0, 1) \) such that \( \alpha(F(D)) < k_1\alpha(D) \) for all bounded subset \( D \) of \( C([0, T]; X) \).

(H'3) \( g : C([0, T], X) \rightarrow \overline{D(A)} \) is a continuous map such that there exist \( \beta \in (0, 1) \) and \( \gamma > 0 \) with \( \| g(u) \| \leq \beta\|u\|_\infty + \gamma. \)
Moreover, there exists \( k_2 \in (0, 1) \) such that \( \alpha(g(D)) < k_2\alpha(D) \) for all bounded subset \( D \) of \( C([0, T]; X) \).

**Theorem 10.** If (H1), (H''2), (H'3) hold and \( \max\{k_1 + k_2, a + \beta\} < 1 \), then there exists at least one global integral solution of problem (1).

**Proof.** Let \( v \in C([0, T]; X) \) be fixed and consider the Cauchy problem (6).
Since \( F(v) \in L^1([0, T]; X) \), and moreover, \( g(v) \in \overline{D(A)} \), then (6) has a unique integral solution \( u_v \).
Define the mapping \( J : C([0, T]; X) \rightarrow C([0, T]; X) \) by \( J(v) = u_v \). We wish to show that \( J \) has a fixed point. To see this, we proceed as follows.

**Step 1.** Let us show that \( J \) is a continuous mapping.
Indeed, let \( (v_n) \) be a sequence in \( C([0, T]; X) \) which is convergent to \( v \in C([0, T]; X) \). In order to show that \( Jv_n \rightarrow Jv \) we proceed as follows:
For each \( t \in [0, T] \), by the properties of the integral solutions, we have
\[ \| Jv_n(t) - Jv(t) \| \leq \| g(v_n) - g(v) \| + \int_0^t \| F(v_n)(\tau) - F(v)(\tau) \| d\tau \leq \| g(v_n) - g(v) \| + \| F(v_n) - F(v) \|_1. \]
Since \( g \) and \( F \) are continuous maps on \( C([0, T]; X) \) hence we derive that \( \| Jv_n - Jv \|_\infty \rightarrow 0. \)

**Step 2.** We show that \( J \) is \( \alpha \)-condensing.
Let \( C \) be a bounded, noncompact subset of \( C([0, T]; X) \).
By hypothesis (H'3), we have that \( \alpha(g(C)) < k_2\alpha(C) \). This means that there exist \( \{D_i\} : i = 1, 2, \ldots, n \), subsets of \( C \) with \( C = \bigcup D_i \) and for every \( i \in \{1, 2, \ldots, n\} \),
\[ \text{diam}(g(D_i)) \leq k_2\alpha(C). \]
On the other hand, by hypothesis (H''2), we have that \( \alpha(F(C)) < k_1\alpha(C) \). Thus, there exist \( \{V_j\} : j = 1, \ldots, m \), subsets of \( C \) with \( C = \bigcup V_j \) satisfying:
\[ \text{diam}_{\|\cdot\|_1}(F(V_j)) \leq k_1\alpha(C). \]
Consider \( \{ Z_{ij} = D_i \cap V_j : i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \} \); it is clear that \( C = \bigcup Z_{ij} \). Moreover, given \( u, v \in Z_{ij} \), we have:

\[
\| J u(t) - J v(t) \| \leq \| g(u) - g(v) \| + \int_0^t \| (F u)(\tau) - (F v)(\tau) \| \, d\tau \leq \| g(u) - g(v) \| + \| F(u) - F(v) \|_1 \\
\leq \text{diam}_{\| \cdot \|_1} (g(Z_{ij})) + \text{diam}_{\| \cdot \|_1} (F(Z_{ij})) = k_2 \alpha(C) + k_1 \alpha(C) = (k_1 + k_2) \alpha(C).
\]

Hence, \( \text{diam}_{\| \cdot \|_\infty} (J(Z_{ij})) \leq (k_1 + k_2) \alpha(C) \). Consequently, since by hypothesis \( k_1 + k_2 < 1 \), we obtain \( \alpha(J(C)) < \alpha(C) \).

**Step 3.** We verify that we are under the conditions of Theorem 3. Consider \( \lambda > 1 \), and suppose that \( J(v) = \lambda v \); then if \( x \in D(A) \) and \( y \in A(x) \), we have:

\[
\| \lambda v(t) - x \| \leq \| g(v) - x \| + \int_0^t \| F(v)(s) - y \| \, ds \leq \| g(v) - x \| + \int_0^t \| F(v)(s) \| \, ds + T \| y \|.
\]

Therefore, by \( (H'_2) \) and \( (H'_3) \)

\[
\lambda \| v(t) \| \leq 2 \| x \| + \beta \| v \|_\infty + \gamma + a \| v \|_\infty + b + T \| y \|.
\]

Since \( v \in C([0, T]; X) \) and \( \lambda > 1 \), we obtain

\[
\| v \|_\infty \leq \lambda \| v \|_\infty \leq (\beta + a) \| v \|_\infty + 2 \| x \| + T \| y \| + \gamma + b.
\]

Thus, since \( a + \beta < 1 \) we have

\[
\| v \|_\infty \leq \frac{2 \| x \| + T \| y \| + \gamma + b}{1 - (\beta + a)} = C.
\]

Clearly, \( C \) is a constant which is independent of \( v \) and \( \lambda \). Therefore, if we take \( r = C + 1 \) we see that if \( \| v \|_\infty = r \), then \( J v \neq \lambda v \) for every \( \lambda > 1 \). Thus, by Theorem 3 we may conclude that \( J \) has a fixed point. \( \square \)

**Remark 11.** In [1,16] there are uniqueness results for this type of problems. However in the last result, although we cannot say anything about the uniqueness of the solution, it is easy to see that it is a generalization (with respect to existence) of Theorem 3.3 of [1].

**Remark 12.** In [4], S. Aizicovici and V. Staicu prove that the set of integral solutions of the problem

\[
u'(t) = -A u(t) + F(t, u(t)), \quad 0 < t < T; \quad u(0) = g(u),
\]

is a nonempty compact subset of \( C([0, T]; X) \). They assume that \( X \) is a separable Banach space with uniformly convex dual, \( A \) is an \( m \)-accretive operator on \( X \) such that \( -A \) generates a compact semigroup, \( g : C([0, T]; \overline{D(A)}) \to \overline{D(A)} \) is a contraction, but \( F : [0, T] \times X \to 2^X \) has nonempty closed convex values, and is strongly-weakly upper semicontinuous with respect to its second variable (see [4] to check the exact assumptions on \( g \) and \( F \) in the above problem).

In this sense, if we assume the hypotheses of any of our above theorems and we set \( K := \{ u \in C([0, T]; X) : u \) is solution of problem (1)\}, clearly \( K \) is nonempty bounded and closed. It is also clear that under these assumptions it is not possible to guarantee that \( K \) becomes a singleton. However, by the definition of the mapping \( J \) in the above theorems we know that \( K = J(K) \); moreover in each of the above cases \( J \) is \( \alpha \)-condensing. Therefore, \( K \) is a compact set, otherwise,

\[\alpha(K) = \alpha(J(K)) < \alpha(K),\]

which is a contradiction.
Example 13. Consider the nonlinear partial differential equation of the form:

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) - \Delta \rho(u(x, t)) = f(t, u(x, t)), & (x, t) \in \Omega \times [0, T], \\
u(x, 0) - \sum_{i=1}^{p} c_i u(x, t_i) = \int_{0}^{T} \int_{\Omega} h(t, x, z, u(z, t)) \, dt \, dz + \varphi(x), & x \in \Omega, \\
u(x, t) = 0, & (x, t) \in \partial \Omega \times [0, T],
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n\) is an open, bounded set with smooth boundary \(\partial \Omega\). We assume that

- \(f : [0, T] \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory type function such that
  \[f(., 0) \in L^1(0, T)\] and \(|f(t, x) - f(t, y)| \leq k(t)|x - y|\) for all \(x, y \in \mathbb{R}\)
  and \(k \in L^1(0, T)\) with \(c_0 = \int_{0}^{T} k(t) \, dt\);

- the function \(\rho : \mathbb{R} \to \mathbb{R}\) is continuous, nondecreasing and \(\rho(0) = 0\);

- \(c_0, c_1, \ldots, c_p, 0 < t_1, \ldots, t_p \leq T\) are constants such that \(\sum_{i=0}^{p} |c_i| < 1\) and \(\varphi \in L^1(\Omega)\);

- \(h : [0, T] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}\) is a function satisfying the Carathéodory conditions.

(i) For almost every \((t, x, z) \in (0, T) \times \Omega \times \Omega\), \(h(t, x, z, .)\) is continuous;
(ii) For each fixed \(r \in \mathbb{R}\), \(h(., ., ., r)\) is a measurable function on \((0, T) \times \Omega \times \Omega\).

Moreover, \(|h(t, x, z, r) - h(t, y, z, r)| \leq \alpha_k(t, x, y, z)\) for \((t, x, z, r), (t, y, z, r) \in (0, T) \times \Omega \times \Omega \times \mathbb{R}\) with \(|r| \leq k\), where \(\alpha_k \in L^1((0, T] \times \Omega \times \Omega \times \Omega; \mathbb{R}^{+})\) is such that \(\lim_{x \to y} \int_{\Omega} \int_{0}^{T} \alpha_k(t, x, y, z) \, dt \, dz = 0\) uniformly in \(y \in \Omega\). In addition,

\[|h(t, x, z, r)| \leq \sigma(t, x, z)\] for all \(r \in \mathbb{R}\),

where \(\sigma \in L^1((0, T) \times \Omega \times \Omega)\).

Define the operator \(A\) by

\[D(A) = \{f \in L^1(\Omega) : \rho(f) \in W^{1,1}_0(\Omega), \Delta \rho(f) \in L^1(\Omega)\}\]

and \(A(f) = -\Delta \rho(f)\) for \(f \in D(A)\).

In [9] it was proved that \(A\) is an \(m\)-accretive operator on \(L^1(\Omega)\) with dense domain. Furthermore, we may choose \(\rho\) such that the semigroup generated by \(-A\) is not compact (for instance, see [26]). Hence these cases are out of the scope of the results given in [2,3].

Thus, problem (8) may be rewritten as the following nonlocal Cauchy problem:

\[
\begin{cases}
u'(t) + A(u(t)) = F(u)(t), & t \in (0, T), \\
u(0) = g(u),
\end{cases}
\]

where \(X\) is the Banach space \(L^1(\Omega)\).

\(g : C([0, T]; X) \to X\) is defined by \(g(u) = \sum_{i=1}^{p} c_i u(t_i) + S(u) + \varphi\), while

\[S(u)(x) = \int_{\Omega} \int_{0}^{T} h(t, x, z, u(z, t)) \, dt \, dz \quad \forall u \in C([0, T]; X),\]

and \(F : C([0, T]; X) \to L^1(0, T, X)\) is given by

\[F(u)(t)(x) := f(t, u(x, t)).\]
It is clear that
\[
\| F(u)(t) - F(v)(t) \|_X = \| f(t, u(., t)) - f(t, v(., t)) \|_X \\
= \int_\Omega |(t, u(x, t)) - f(t, x)| \, dx \\
\leq k(t) \int_\Omega |u(x, t) - v(x, t)| \, dx \\
= k(t) \| u(t) - v(t) \|_X \leq k(t) \| u - v \|_\infty.
\]

Therefore, we conclude that
\[
\| F(u) - F(v) \|_1 \leq k_1 \| u - v \|_\infty,
\]
with \( k_1 = c_0 \).

On the other hand, for each \( u \in C([0, T]; X) \) we have
\[
\| F(u)(t) \|_X = \int_\Omega |f(t, u(t, x))| \, dx \leq \int_\Omega |f(t, 0)| \, dx + \int_\Omega k(t)|u(t, x)| \, dx = |f(t, 0)|m(\Omega) + k(t)\| u(t) \|_X.
\]

Consequently
\[
\| F(u) \|_1 = \int_0^T \| F(u)(t) \|_X \leq b + k_1 \| u \|_\infty
\]
where \( m(\Omega) \) means the Lebesgue measure of \( \Omega \) and \( b = m(\Omega) \int_0^T |f(t, 0)| \, dt \). Hence \( F \) satisfies (H' \( H_2' \)).

It is also clear that if we set \( R(u) = \sum_{i=1}^p c_i u(t_i) \), then
\[
\| R(u) - R(v) \|_X = \left\| \sum_{i=1}^p c_i (u(t_i) - v(t_i)) \right\|_X \leq \sum_{i=1}^p |c_i| \| u(t_i) - v(t_i) \|_X \leq \sum_{i=1}^p |c_i| \| u - v \|_\infty.
\]

On the other hand, the properties of \( h \) allow us to adapt the results of Section V.4 of [23] and thus, it is easy to see that \( S \) is a compact and continuous map. Therefore, for any bounded subset \( C \) of \( C([0, T]; X) \) the mapping
\[
g = R + S + \varphi
\]
satisfies
\[
\alpha(g(C)) = \alpha(R(C) + S(C) + \varphi) \leq \alpha(R(C)) + \alpha(S(C)) = \alpha(R(C)) \leq k_2 \alpha(C),
\]
where \( k_2 = \sum_{i=1}^p |c_i| \).

Finally,
\[
\| g(u) \|_X \leq k_2 \| u \|_\infty + \gamma,
\]
where \( \gamma := \int_\Omega \int_\Omega \int_0^T \sigma(t, x, z) \, dt \, dz \, dx + \| \varphi \|_X < \infty \) since \( \sigma \in L^1((0, T) \times \Omega \times \Omega) \).

Finally, if we call \( a = k_1 \) and \( \beta = k_2 \), since \( k_1 + k_2 = \sum_{i=1}^p |c_i| < 1 \), then \( \max\{k_1 + k_2, a + \beta\} < 1 \). Consequently, every hypothesis of Theorem 10 is satisfied and thus we may conclude the existence of an integral solution of problem (8).

5. Asymptotic behavior

In this section we consider the following nonlocal Cauchy problem
\[
\begin{align*}
\{ & u'(t) + A(u(t)) \ni F(u)(t), \quad t \in (0, \infty), \\
& u(0) = g(u).
\end{align*}
\]
An integral solution of problem (10) is a continuous function $u : [0, \infty[ \to X$ satisfying Definition 6 for any $T \in (0, \infty)$. We shall use $BC(0, \infty; X)$ to denote the Banach space of all bounded and continuous functions from $[0, \infty]$ into $X$ equipped with the standard sup-norm.

We assume throughout the following assumptions:

(A) $A : D(A) \subseteq X \to 2^X$ is an $m$-accretive operator on $X$.

(B) $0 \in \mathcal{R}(A)$.

(C) $F : BC(0, \infty; X) \to L^1(0, \infty; X)$ is a continuous mapping, and there exists $k_1 \in (0, 1)$, such that for all bounded subsets $D$ of $BC((0, \infty); X)$, the inequality $\alpha(F(D)) \leq k_1 \alpha(D)$ holds. Moreover, there exist $a \in (0, 1)$ and $b > 0$ with $\|F(u)\|_1 \leq a\|u\|_\infty + b$.

(D) $g : BC(0, \infty; X) \to \overline{D(A)}$ is a continuous function such that there exist $\beta \in (0, 1)$ and $\gamma > 0$ satisfying $\|g(u)\| \leq \beta\|u\|_\infty + \gamma$. Moreover, there exists $k_2 \in (0, 1)$ such that for every $C \subseteq BC(0, \infty; X)$ bounded the inequality $\alpha(g(C)) \leq k_2 \alpha(C)$ holds.

Recall that in the particular case of the homogeneous Cauchy problem, i.e.,

$$\begin{cases}
u'(t) + A(u(t)) \ni 0, & t \geq 0, \\
u(0) = x_0 \in \overline{D(A)},
\end{cases}$$

(11)

the unique integral solution is given by the Crandall–Liggett exponential formula

$$u(t) := \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^n (x_0).$$

Moreover, the family $\mathcal{F} := \{ S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0 \}$, where $S(t)x := \lim_{n \to \infty} (I + \frac{t}{n} A)^n(x)$ is a nonexpansive semigroup, which is called the semigroup generated by $-A$ via the Crandall–Liggett exponential formula.

A continuous function $u : [0, \infty[ \to \overline{D(A)}$ is said to be an almost-orbit of $\mathcal{F}$ (see [22]) if

$$\lim_{t \to \infty} \sup_{s \in [0, \infty[} \| u(t + s) - S(s)u(t) \| = 0.$$

**Theorem 14.** If the assumptions (A), (B), (C) and (D) hold, then problem (10) has at least an integral solution whenever $\max\{k_1 + k_2, a + \beta\} < 1$ and such solutions are almost-orbits of semigroup generated by $-A$ via the Crandall–Liggett exponential formula.

**Proof.** Let $v \in BC(0, \infty, X)$ be fixed and consider the Cauchy problem

$$\begin{cases}
u'(t) + A(u(t)) \ni F(v)(t), & 0 < t < \infty, \\
u(0) = g(v).
\end{cases}$$

(13)

Since by assumption (C), $F(v) \in L^1(0, \infty; X)$, and by assumption (D), $g(v) \in \overline{D(A)}$, it is well known that (13) has a unique integral solution $u_v : [0, \infty[ \to \overline{D(A)}$.

Let us show that $u_v \in BC(0, \infty, \overline{D(A)})$.

Indeed, since by (B), $0 \in \mathcal{R}(A)$ there is $z \in D(A)$ such that $0 \in Az$. This means, among other things, that $z$ is a solution of the homogeneous problem

$$\begin{cases}
u'(t) + A(u(t)) \ni 0, & 0 < t < \infty, \\
u(0) = z.
\end{cases}$$

(14)

Therefore for every $0 < t < \infty$

$$\| u_v(t) - z \| \leq \| g(v) - z \| + \int_0^t \| F(v)(s) \| \, ds.$$

Consequently for all $0 < t < \infty$

$$\| u_v(t) \| \leq 2\| z \| + \| g(v) \| + \| F(v) \|_1,$$

i.e., $u_v \in BC(0, \infty, \overline{D(A)})$. 

The above fact allows us to define the mapping $J : BC(0, \infty; X) \to BC(0, \infty; X)$ given by $J(v) = uv$. To see that $J$ admits a fixed point (which would be an integral solution of problem (10)) it is enough to follow the steps given in Theorem 10.

In particular, in the third step since $0 \in A(z)$, if $Jv = \lambda v$ for some $\lambda > 1$ we conclude

$$
\|v\|_\infty \leq \frac{2\|z\| + \gamma + b}{1 - (\beta + a)},
$$

which implies that $J$ is under the hypotheses of Theorem 3.

Finally, let us show that if $u \in BC(0, \infty; D(A))$ is an integral solution of problem (10), then $u$ is any almost-orbit. Indeed, consider an fixed and arbitrary $t > 0$.

Hence

$$
\|u(t + s) - S(s)u(t)\| = \|u_t(s) - v(s)\| \leq \int_0^s \|f_t(r)\| dr = \int_0^s \|F(u)(t + r)\| dr \leq \int_t^\infty \|F(u)(\tau)\| d\tau.
$$

Since $F(u) \in L^1(0, \infty; X)$, we have

$$
\lim_{t \to \infty} \sup_{s \in [0, \infty]} \|u(t + s) - S(s)u(t)\| \leq \lim_{t \to \infty} \int_t^\infty \|F(u)(\tau)\| d\tau = 0.
$$

As a consequence of Theorem 14 along with Lemma 3.1 of [22], we have

**Proposition 15.** Let $u$ be an integral solution of problem (10). Then

$$
\lim_{t \to \infty} \|u(t) - z\| \text{ exists for each } z \in A^{-1}(0).
$$

Proposition 15 works for general $m$-accretive operators and in any Banach space. In order to obtain a better result than in Proposition 15 concerning asymptotic behavior we recall the following definition.

**Definition 16.** (See [21].) Let $X$ be a Banach space, let $\phi : X \to [0, \infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ and which satisfies the following condition:

For every sequence $(x_n)$ in $X$ such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$.

An accretive operator $A : D(A) \to 2^X$ with $0 \in A(z)$ is said to be $\phi$-accretive at zero whenever the inequality

$$
\langle u, x - z \rangle_s \geq \phi(x - z), \quad \text{for all } (x, u) \in A,
$$

holds.

Now, having in mind the results concerning asymptotic behavior for $\phi$-accretive at zero operators which appear in [21], it is not difficult to obtain

**Proposition 17.** Consider problem (10) under the assumptions (A), (C) and (D) where $A \subseteq X \times X$ satisfies condition (15) for some $z \in X$. Also assume that either $X$ enjoys the Radon–Nikodym property or $X = L^1(\Omega)$, where $\Omega$ is a bounded subset in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and $A$ is an $m$-completely accretive operator (see [10]). Then $u(t) \to z$ as $t \to \infty$ for every integral solution $u$ of this problem.

**Example 18.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. It will be further assumed that $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:
(a) For almost all \( x \in \Omega \), \( r \rightarrow \varphi(x, r) \) is continuous and nondecreasing.
(b) For every \( r \in \mathbb{R} \), \( x \rightarrow \varphi(x, r) \) is in \( L^1(\Omega) \).
(c) \( \varphi(x, 0) = 0 \), \( \varphi(x, r) \neq 0 \) whenever \( r \neq 0 \) and there exist \( \lambda > 0 \) and \( \alpha \geq 2 \) such that \( \varphi(x, r)r \geq \lambda |r|^{\alpha} \).

The function \( \varphi(x, r) = |r|^{\gamma-1}r \) satisfies the above conditions whenever \( \gamma \geq 1 \).

Consider the following nonlinear boundary value problem:

\[
\begin{cases}
  u_t - \text{div}(|Du|^{p-2}Du) + \varphi(x, u) = f(t, u(t, x)), & \text{on } [0, \infty[ \times \Omega, \\
  \frac{\partial u}{\partial \eta} \in \beta(u), & \text{on } [0, \infty[ \times \partial \Omega, \\
  u(0, x) + \sum_{i=1}^{p} c_i u(t_i, x) = \int_{0}^{\infty} \int_{\Omega} h(t, x, z, u(t, z)) \, dt \, dz + \phi(x), & x \in \Omega,
\end{cases}
\]

where

(i) \( f : [0, \infty[ \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory type function such that

\[
f(., 0) \in L^1(0, \infty) \quad \text{and} \quad \left| f(t, x) - f(t, y) \right| \leq k(t)|x - y| \quad \text{for all } x, y \in \mathbb{R},
\]

and \( k \in L^1(0, \infty) \) with \( c_0 = \int_{0}^{\infty} k(t) \, dt \);

(ii) \( 1 < p < \infty, \frac{\partial}{\partial \eta} \) is the associated Neumann boundary operator, i.e., \( \frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta \rangle \), with \( \eta \) the unit outward normal on \( \partial \Omega \);

(iii) \( \beta \) is a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0) \);

(iv) \( \varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies conditions (a), (b) and (c) as above;

(v) \( c_1, \ldots, c_p, 0 < t_1, \ldots, t_p < \infty \), are constants such that \( \sum_{i=1}^{p} c_i < 1 \) and \( \phi \in L^1(\Omega) \);

(vi) \( h : [0, \infty[ \times \Omega \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a function satisfying the Carathéodory conditions. In addition

(a) for almost every \( (t, x, z) \in (0, \infty) \times \Omega \times \Omega \), \( h(t, x, z, .) \) is continuous,

(b) for each fixed \( r \in \mathbb{R}, h(., \ldots, r) \) is a measurable function on \( (0, \infty) \times \Omega \times \Omega \);

(vii) \( |h(t, x, z, r) - h(t, y, z, r)| \leq \alpha_k(t, x, y, z) \) for \( (t, x, z, r), (t, y, z, r) \in [0, \infty) \times \Omega \times \Omega \times \mathbb{R} \) with \( |r| \leq k \), where \( \alpha_k \in L^1([0, \infty) \times \Omega \times \Omega \times \Omega; \mathbb{R}^+) \) is such that \( \lim_{y \to x} \int_{\Omega} \int_{0}^{\infty} \alpha_k(t, x, y, z) \, dt \, dz = 0 \) uniformly in \( y \in \Omega \).

Following the arguments given in [21] it is possible to rewrite problem (16) as follows:

\[
\begin{cases}
  u' + Bu(t) = F(u), & 0 < t < \infty, \\
  u(0) = g(u),
\end{cases}
\]

where \( F(u)(t) := f(t, u(t, x)) \), and \( B \) is a densely defined, \( m \)-completely accretive and \( \phi \)-accretive at zero operator in \( L^1(\Omega) \) such that \( 0 \in B(0) \) (we refer the reader to [5], for an explicit description of the operator \( B \)).

On the other hand, \( g : BC([0, \infty), L^1(\Omega)) \rightarrow L^1(\Omega) \) is defined for every \( x \in \Omega \) by

\[
G(u)(x) = \phi(x) - \sum_{i=1}^{p} c_i u(t_i)(x) + \int_{0}^{\infty} \int_{\Omega} h(t, x, z, u(t)(z)) \, dt \, dz.
\]

Adapting the results of Section V.4 of [23] it is not difficult to see that the function \( g \) satisfies the above property (D) and then problem (16) is under the hypotheses of Proposition 17 and therefore if \( u \) is an integral solution of such problem we have that \( u(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

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