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Note

A note on a nonlinear functional equation and its application

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ABSTRACT

This paper treats the following type of nonlinear functional equations

$$\varphi(x) = m\frac{H(x, \varphi[g(x)])}{H_\varphi(x, \varphi[g(x)])},$$

where m is a real number, $H(x, y)$ and $g(x)$ are given functions, and $\varphi(x)$ is an unknown function. Under certain conditions, we prove that such type of equations admits a unique continuous solution.

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1. Introduction

In this paper, we consider a new class of nonlinear functional equations arising in a two-dimensional optimal stopping problem. Under certain conditions, we prove that the present class of nonlinear functional equations admits a unique continuous solution.

Let U and V be topological and complete metric spaces, respectively. Let $H(x, y): U \times V \to V$ and $g: U \to U$ be given real-valued functions satisfying the following conditions:

(A_0) $H: U \times V \to V$ and $g: U \to U$ are continuous functions and $H(x, y)$ has a continuous partial derivative $H_y \neq 0$ in $U \times V$;

(A_1) there exist a nondecreasing function $\alpha: [0, \infty) \to [0, \infty)$ and a positive real number $m > 1$ such that

$$\rho\left(m\frac{H(x, y)}{H_y(x, y)}, m\frac{H(x, \bar{z})}{H_{\bar{z}}(x, \bar{z})}\right) \leqslant \alpha\big(m\rho(y, \bar{z})\big),$$

for $x \in U$ and $y, \bar{z} \in V$, where ρ is a metric on V.

Our main concern in this paper is to prove that the functional equation

$$\varphi(x) = m\frac{H(x, \varphi[g(x)])}{H_\varphi(x, \varphi[g(x)])} \tag{1.1}$$

admits a unique continuous solution $\varphi: U \to V$ assuming that conditions (A_0) and (A_1) hold.

The original motivation of the present paper is a class of nonlinear functional equations treated earlier by Kuczma (see [3,4], etc.), which is a special case of Eq. (1.1).

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2. Main result

Our main result is stated and proved in the next theorem.

Theorem 2.1. *If U is a compact space and conditions (A_0), (A_1) and*

$$\alpha(0) = 0, \qquad \lim_{s\to t^+} \sup \alpha(s) < t \quad \text{for all } t \in (0,\infty),$$

hold, then the nonlinear functional equation (1.1) has a unique continuous solution $\varphi : U \to V$.

Proof. The proof follows similarly as in Baron [1], and using a well-known fixed point theorem due to Boyd and Wong [2]. □

Remark 2.1. It must be stressed that, even though the proof of the above result follows similarly as in [1], the present class of nonlinear functional equations (1.1) is not dealt with by Baron [1].

3. Application to optimal stopping problems

In this section we consider one example to illustrate the main result of this paper. We consider the problem of characterizing the optimal stopping boundary of a two-dimensional optimal stopping problem. The present optimal stopping problem can be regarded as an extension of the one-dimensional case in Salminen [5].

Let $Q_t = (x_t, y_t)$ be a two-dimensional diffusion process given by

$$\begin{aligned} dx_t &= \theta(x_t)\,dt + \beta(x_t)\,dB_t^1; \qquad x(0) = x, \\ dy_t &= \mu y_t\,dt + \sigma y_t\,dB_t^2; \qquad y(0) = y, \end{aligned} \tag{3.1}$$

initially starting at (x, y) in the positive quadrant, where $\sigma > 0$, μ are fixed constants, $\beta(\cdot) > 0$, $\theta(\cdot)$ are bounded measurable functions and $B_t = (B_t^1, B_t^2)$ is a two-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Consider the following optimal stopping problem: Find a stopping time τ^* such that

$$\sup_\tau \mathbf{E}^{x,y}\big[H(x_\tau, y_\tau)e^{-\int_0^\tau p(x_s)\,ds}\big], \tag{3.2}$$

where the sup is taken over all finite stopping times τ, $\mathbf{E}^{x,y}$ denotes the expectation with respect to the law $\mathbf{P}^{x,y}$ of the process Q_t starting at (x, y) in the positive quadrant, and $p : \mathbf{R}_+ \to \mathbf{R}$ is a bounded Borel function.

It turns out that, using the classical smooth-fit principle [6], the optimal stopping boundary $y = \varphi(x)$ of the optimal stopping problem (3.2) is characterized by a nonlinear functional equation of the form (1.1). We shall now give one simple example where $\varphi(x)$ is given explicitly.

Example 3.1. Let $m > 1$ be a fixed positive constant, $H(x, \varphi(x)) = x - \varphi(x)$ and let $U = \{x\colon\ 0 \leqslant x \leqslant a\}$ and $V = [0,\infty)$, where $0 < a < \infty$. Let $\alpha(mt)$ be given by

$$\alpha(mt) = \begin{cases} bt & \text{for } 0 < t \leqslant 1, \\ c & \text{for } t \geqslant 1 \end{cases}$$

where $b = 1/m$ and $c = b$.

Clearly, all the conditions in Theorem 2.1 are satisfied. In this particular case, $\varphi(x) = \frac{mx}{m-1}$ is a unique continuous solution of the functional equation (1.1).

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