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J. Math. Anal. Appl. 325 (2007) 1240–1256

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Modified hierarchy basis for solving singular boundary value problems

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Received 18 July 2005

Available online 29 March 2006

Submitted by M. Milman

Abstract

In this paper, we develop an efficient preconditioning method on the basis of the modified hierarchy basis for solving the singular boundary value problem by the Galerkin method. After applying the preconditioning method, we show that the condition number of the linear system arising from the Galerkin method is uniformly bounded. In particular, the condition number of the preconditioned system will be bounded by 2 for the case $q(x) = 0$ (see Eq. (1) in the paper). Numerical results are presented to confirm our theoretical results.

Keywords: Modified hierarchy basis; Preconditioning; Stability; Singular boundary value problem

1. Introduction

Our investigations in this paper are concerned with the preconditioning method on the basis of the modified hierarchy basis for the numerical solution of the singular boundary value problem arising from the radically symmetric elliptic partial differential equations, a problem with numerous applications (see, e.g., [20]). When the Dirichlet problem

$$-\Delta u(\mathbf{x}) + q(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \text{in } \mathbf{B},$$

$$u = 0, \quad \text{on } \partial\mathbf{B},$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.02.043

is defined on a unit ball $\mathbf{B} := B_1(0)$ in $\mathbb{R}^d$ and the data depend only on the radical coordinate, then after a change of variables, the problem will reduce to a one-dimensional singular boundary value problem,

$$\begin{aligned} &-xu''(x) - (d-1)u'(x) + xq(x)u(x) = xf(x), \quad x \in (0,1), \\ &u'(0) = u(1) = 0, \end{aligned} \tag{1}$$

where $q(x) \geqslant 0$ and $q(x) \in L_\infty(0,1)$.

For the smooth data, it has been proven (see, e.g., [9,12,19,20]) that the (smooth) solution can be approximated with high order accuracy by the Galerkin method with a piecewise polynomial subspace. Therefore, no special functions are required in the basis. Convergence results of the Galerkin method for the singular boundary values problems have been studied for the case $q(x) = 0$ in detail in [12]. In [9], Eriksson and Thomee established the general optimal order error estimates and even generalized their results to the corresponding time dependent problems. It shows that the Galerkin method would give the same convergence results for the singular problems as for the nonsingular problems. For the solution with certain smoothness (such as in H^2), the simple piecewise linear nodal basis shall satisfy the approximation requirement.

However, it is an interesting problem to efficiently solve the large system of linear equations arising from the Galerkin method for the singular boundary value problems. Like its counterpart for the regular elliptic problems, the linear system arising from the Galerkin method for the singular boundary value problems is also ill conditioned. For the regular elliptic boundary value problem, multigrid methods (see, e.g., [1,3,4]), and numerous other preconditioning methods (see, e.g., [2,4,16,17,23,24]), were successfully developed. Nevertheless, to our best knowledge, presently there are few references about preconditioning methods of the Galerkin method for the singular problems. In the paper, we aim to design an easily implemented preconditioning method by making use of the modified hierarchy basis.

The hierarchy basis has been discussed extensively in [25,26], and has been proven to be an efficient preconditioning method for low-dimensional regular elliptic problems. In this paper, we construct a modified hierarchy basis based on the concept of "stability" (see, e.g., [13,15]), and the "norm equivalence" for the Sobolev space (see, e.g., [5–7,14,23,24]). Such basis is then adapted to the nodal basis introduced in Section 2 for the singular boundary value problem, and thus the preconditioning can be achieved. It will be shown later that after applying the preconditioning method based on the modified hierarchy basis, the condition number of the stiffness matrix arising from the Galerkin method will be uniformly bounded. In particular, the condition number is extremely small and bounded by 2 for the case $q(x) = 0$.

Apart from handling the preconditioning for the above problem, the modified hierarchy basis maybe also useful for a category of singular perturbation problems [8,18], which received significant attention recently. They demonstrate the similar small coefficient as in (1) and exhibit the singularity at the boundary. The preconditioning even for the tensor product case of specially grided meshes in two-dimensional space is of particular interest. Therefore, a careful follow on study on modified hierarchy basis may generalize our method to this application.

This paper is divided into three parts. In Section 2, we propose the preconditioning method on the basis of the modified hierarchy basis for the singular boundary value problem, and show the connection between the concept of norm equivalence and stability of the modified hierarchy basis. The condition number of the preconditioned stiffness matrix is proven to be uniformly bounded. In Section 3, we provide basic error estimates for the Galerkin approximation from the piecewise linear nodal basis subspace V_h with its element v satisfying the boundary conditions

$v'(0) = v(1) = 0$. We will show such subspace provides the same approximation order as the linear nodal basis subspace without the condition $v'(0) = 0$. Numerical examples are computed to confirm our results in Section 4.

2. The Galerkin method and the modified hierarchy basis

We consider the boundary value problem of the form

$$-\big(x^\alpha u'(x)\big)' + x^\alpha q(x)u(x) = x^\alpha f(x), \quad x \in (0,1), \tag{2}$$

$$u'(0) = u(1) = 0, \tag{3}$$

where $\alpha = d - 1$.

Let v be a real-valued Lebesgue measurable function on $\mathbb{R}$. We define the $L_2(0,1)$ inner product by $\langle u, v\rangle := \int_0^1 u(x)v(x)\,dx$, and $L_2(0,1)$ space by $L_2(0,1) := \{v\colon \|v\|_{L_2(0,1)} < \infty\}$. The weighted L_2 space $\dot{L}_2(0,1)$ is defined by $\dot{L}_2(0,1) := \{v\colon \int_0^1 |x^{\alpha/2}v(x)|^2\,dx < \infty\}$. Moreover, the weighted Sobolev space $\dot{H}_0^1(0,1)$ is the closure of the set $\{v\colon v \in C([0,1]) \cap C^1(0,1),\ v(1) = 0\}$ with respect to the weighted Sobolev norm $\|v\|_{\dot{H}^1} := (\int_0^1 x^\alpha(|v(x)|^2 + |v'(x)|^2)\,dx)^{1/2}$. Define the symmetric bilinear form $a(\cdot,\cdot)$ as follows: For $u, v \in \dot{H}_0^1(0,1)$,

$$a(u,v) := \int_0^1 x^\alpha u'(x)v'(x)\,dx + \int_0^1 q(x)x^\alpha u(x)v(x)\,dx. \tag{4}$$

Then the solution u of the singular boundary value problem also solves the variational problem

$$a(u,v) = \big\langle x^\alpha f(x), v(x)\big\rangle, \quad \forall v \in \dot{H}_0^1(0,1). \tag{5}$$

Here, with some ambiguity, we also use x^α to denote function $x \mapsto x^\alpha$, $x \in (0,1)$, and we assume that $f \in \dot{L}_2(0,1)$ $(x^{\alpha/2} f(x) \in L_2(0,1))$.

We have the following Poincaré-type inequality [12].

Lemma 1.

$$\big\|x^{\alpha/2}v\big\|_{L_2} \leqslant \frac{2}{\alpha+1}\big\|x^{\alpha/2}v'\big\|_{L_2}, \quad v \in \dot{H}_0^1.$$

Proof. We have

$$\begin{aligned}\int_0^1 x^\alpha v^2(x)\,dx &= \int_0^1 \left(\frac{x^{\alpha+1}}{\alpha+1}\right)' v^2(x)\,dx \\ &= -\int_0^1 \left(\frac{x^{\alpha+1}}{\alpha+1}\right) 2v(x)v'(x)\,dx + \left(\frac{x^{\alpha+1}}{\alpha+1}\right) v^2(x)\bigg|_0^1 \\ &\leqslant \frac{2}{\alpha+1}\big\|x^{\alpha/2}v\big\|_{L_2(0,1)}\big\|x^{\alpha/2}v'\big\|_{L_2(0,1)}\|x\|_{L_\infty(0,1)}.\end{aligned}$$

Note that $\|x\|_{L_\infty(0,1)} = 1$. This completes the lemma. □

Now we define another inner product for $\dot{H}_0^1(0,1)$ by

$$\langle u,v\rangle_E := \int_0^1 x^\alpha u'(x)v'(x)\,dx, \quad u,v\in \dot{H}_0^1. \tag{6}$$

By Lemma 1, we have the following inequalities:

$$\langle v,v\rangle_E \leqslant a(v,v) \leqslant \left(1+\left(\frac{2}{\alpha+1}\right)^2\|q\|_{L_\infty(0,1)}\right)\langle v,v\rangle_E, \quad v\in \dot{H}_0^1. \tag{7}$$

Hereafter, we fix $\alpha=1$ for simplicity. The case $\alpha>1$ can be handled in the same way without any extra difficulty.

For the uniform partition of $[0,1]$, $0=x_0<x_1<\cdots<x_{2^n}=1$, $x_j=2^{-n}j$, $j=0,\ldots,2^n$, let ϕ be the hat function $\phi(x):=\max\{0,1-|x|\}$, and

$$\phi_{n,1} := \big(\phi(2^n\cdot)+\phi(2^n\cdot-1)\big)\chi_{[0,1]}, \tag{8}$$

$$\phi_{n,j} := \phi\big(2^n\cdot-j\big), \quad j=2,\ldots,2^n-1, \tag{9}$$

where $\chi_{[a,b]}$, $a<b$, is the characteristic function on the interval $[a,b]$. Let

$$V_n := \operatorname{span}\big\{\phi_{n,j}\colon\ j=1,\ldots,2^n-1\big\}.$$

It is easily seen that $V_n\subset V_{n+1}$ for $n=1,2,\ldots.$

The Galerkin method is seeking an element $u_n\in V_n$ such that

$$a(u_n,v)=\langle xf,v\rangle, \quad v\in V_n. \tag{10}$$

Inequality (7) shows that $a(\cdot,\cdot)$ is elliptic. By the Lax–Milgram theorem, existence and uniqueness of the solution is guaranteed for both (5) and (10).

Taking

$$u_n=\sum_{j=1}^{2^n-1} c_{n,j}\phi_{n,j},$$

we rewrite (10) as

$$\sum_{j=1}^{2^n-1} a(\phi_{n,j},\phi_{n,l})c_{n,j}=\langle xf,\phi_{n,l}\rangle, \quad l=1,\ldots,2^n-1, \tag{11}$$

or more briefly,

$$A_nC_n=F_n, \tag{12}$$

where (j,l) entry of the 2^n-1 by 2^n-1 stiffness matrix A_n is $a(\phi_{n,j},\phi_{n,l})$, $C_n := (c_{n,1},\ldots,c_{n,2^n-1})^{\mathrm T}$, and $F_n := (\langle xf,\phi_{n,1}\rangle,\ldots,\langle f,\phi_{n,2^n-1}\rangle)^{\mathrm T}$. Here, the superscript 'T' denotes the transpose of a vector or a matrix.

The condition number of a nonsingular M by M matrix A is defined by $\kappa(A):=\|A\|\|A^{-1}\|$, where $\|A\|:=\sup_{\mathbf{x}\in\mathbb{R}^M}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$, $\mathbf{x}:=(x_1,\ldots,x_M)^{\mathrm T}$, and $\|\mathbf{x}\|:=(\sum_{i=1}^M x_i^2)^{1/2}$. When A is positive definite and symmetric, we have $\kappa(A)=\lambda_{\max,A}/\lambda_{\min,A}$, where $\lambda_{\max,A}$, $\lambda_{\min,A}$ are the maximum and the minimum eigenvalues of the matrix A, respectively.

The following error estimate will be established in the next section:

$$\big\|x^{1/2}(u-u_n)\big\|_{L_2}\leqslant C\big(2^{-n}\big)^2\big\|x^{1/2}u''\big\|_{L_2}.$$

Consequently, the subspace V_n has to be large enough to guarantee that the error $u - u_n$ is sufficient small. However, increasing the number n will dramatically increase the condition number of the associated stiffness matrix A_n (see, e.g., [1]), which makes solving u_n numerically difficult. It is well known in the literature that for an ill-conditioned large linear system, without any preconditioning, it is impossible to find an efficient solver. Therefore, seeking a suitable preconditioning method will be important for solving the discretized system numerically. There is an abundance of literature contributed to this topic for the regular elliptic boundary problems, such as [2,25,26]. Recently, wavelet methods have been introduced to serve as new preconditioning methods (see, e.g., [10,21,22]). Stability plays the key role in the wavelet preconditioning method. In other words, if one is able to find a basis which is stable in the corresponding Sobolev space, then the condition number of the associated stiffness matrix is uniformly bounded. A basis, say $\{\psi_i\}_{i=1}^{\infty}$, is stable if it satisfies,

$$C_0\left(\sum_{i=1}^{\infty} c_i^2\right) \leqslant \left\|\sum_{i=1}^{\infty} c_i\psi_i\right\|^2 \leqslant C_1\left(\sum_{i=1}^{\infty} c_i^2\right),$$

where C_0, C_1 are two positive constants independent of $\{c_i\}_{i=1}^{\infty}$, and $\|\cdot\|$ refers to the norm for the space in which we are interested. Stability of the shift invariant space has been studied extensively in [13,15].

To find a proper preconditioning matrix for A_n in (12), we introduce the following lemma.

Lemma 2. *If two positive definite symmetric $M \times M$ matrices A, B satisfy the following condition*:

$$C_0\mathbf{x}^{\mathrm{T}}B\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}}A\mathbf{x} \leqslant C_1\mathbf{x}^{\mathrm{T}}B\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^M,$$

then for any $M \times M$ matrix S,

$$\kappa\big(SAS^{\mathrm{T}}\big) \leqslant \frac{C_1}{C_0}\kappa\big(SBS^{\mathrm{T}}\big).$$

Lemma 2 tells that once one finds a good preconditioning matrix for B, then it is also a good preconditioning matrix for A provided that the ratio C_1/C_0 is not large. Basic properties of positive definite matrices and their condition numbers maybe found in [11, Chapter 7].

Lemma 3. *For $n = 1, 2, \ldots$, let $\chi_n := \sum_{k=1}^{n} 2^{-k/2}\chi_{[2^{-k},2^{-k+1}]}$, and $g_{n,j} := \chi_n\phi'_{n,j}$, $j = 1, \ldots, 2^n - 1$. Let $u = \sum_{j=1}^{2^n-1} c_{n,j}\phi_{n,j}$. Then*

$$\int_0^1 \left|\sum_{j=1}^{2^n-1} c_{n,j}g_{n,j}(x)\right|^2 dx \leqslant \langle u, u\rangle_E \leqslant 2\int_0^1 \left|\sum_{j=1}^{2^n-1} c_{n,j}g_{n,j}(x)\right|^2 dx, \tag{13}$$

where $\langle\cdot,\cdot\rangle_E$ is defined in (6) *with $\alpha = 1$.*

We may think of $g_{n,j}$ as the weighted derivative of $\phi_{n,j}$, and the weights are 2^{-k}, $k = 1, \ldots, n$, on the subintervals $(2^{-k}, 2^{-k+1})$, $k = n, \ldots, 1$. In other words, we discretize the weight x in the inner product form $\langle\cdot,\cdot\rangle_E$ through χ_n.

Proof. Noting that $g_{n,j} = 0$, $j = 1, \dots, 2^n - 1$ on $(0, 2^{-n})$, we get

$$\langle u, u\rangle_E = \sum_{k=1}^{n} \int_{2^{-k}}^{2^{-k+1}} x\left|u'(x)\right|^2 dx.$$

Accordingly,

$$\langle u, u\rangle_E = \sum_{k=1}^{n} \int_{2^{-k}}^{2^{-k+1}} x\left|\sum_{j\in I_k} c_{n,j}\phi'_{n,j}(x)\right|^2 dx,$$

where $I_k := \{2^{n-k}, \dots, 2^{n-k+1}\}$. Now we have

$$\langle u, u\rangle_E \geqslant \sum_{k=1}^{n} \int_{2^{-k}}^{2^{-k+1}} 2^{-k}\left|\sum_{j\in I_k} c_{n,j}\phi'_{n,j}(x)\right|^2 dx \tag{14}$$

$$= \sum_{k=1}^{n} \int_{2^{-k}}^{2^{-k+1}} \left|\sum_{j\in I_k} \bigl(c_{n,j}2^{-k/2}\phi'_{n,j}(x)\bigr)\right|^2 dx. \tag{15}$$

By the definition of $\phi_{n,j}$ in (8), (9), we have

$$\phi'_{n,j}(x) = \begin{cases} 2^n, & (j-1)2^{-n} < x < j2^{-n}, \\ -2^n, & j2^{-n} < x < (j+1)2^{-n}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, on each subinterval $(2^{-k}, 2^{-k+1})$, $k = n, n-1, \dots, 1$, it follows that

$$\sum_{j\in I_k} c_{n,j}2^{-k/2}\phi'_{n,j} = \sum_{j\in I_k} c_{n,j}\chi_n\phi'_{n,j} = \sum_{j\in I_k} c_{n,j}g_{n,j}.$$

This together with (15) yields

$$\langle u, u\rangle_E \geqslant \int_0^1 \left|\sum_{j=1}^{2^n-1} \bigl(c_{n,j}g_{n,j}(x)\bigr)\right|^2 dx.$$

The proof of the right inequality of (13) is similar and is omitted. □

Combining Lemma 3 with inequality (7), we have:

Lemma 4. *Denote by A_n the matrix $(a(\phi_{n,j}, \phi_{n,l}))_{j,l=1,\dots,n}$, $A_{E,n}$ the matrix $(\langle\phi_{n,j}, \phi_{n,l}\rangle_E)_{j,l=1,\dots,n}$ and by $\tilde{A}_n$ the matrix $(\langle g_{n,j}, g_{n,l}\rangle)_{j,l=1,\dots,n}$. Then the inequalities*

$$\mathbf{x}^{\mathrm{T}} A_{E,n}\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}} A_n\mathbf{x} \leqslant \bigl(1 + \|q\|_{L_\infty(0,1)}\bigr)\mathbf{x}^{\mathrm{T}} A_{E,n}\mathbf{x}, \quad \forall\mathbf{x} \in \mathbb{R}^{2^n-1}, \tag{16}$$

and

$$\mathbf{x}^{\mathrm{T}} \tilde{A}_n\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}} A_{E,n}\mathbf{x} \leqslant 2\mathbf{x}^{\mathrm{T}} \tilde{A}_n\mathbf{x}, \quad \forall\mathbf{x} \in \mathbb{R}^{2^n-1}, \tag{17}$$

hold true.

The following theorem is a simple consequence of Lemmas 2 and 4.

Theorem 1. *For any matrix S of the same size as A_n,*

$$\kappa\big(SA_nS^{\mathrm{T}}\big) \leqslant 2\big(1+\|q\|_{L_\infty(0,1)}\big)\kappa\big(S\tilde{A}_nS^{\mathrm{T}}\big).$$

By Theorem 1, we reduce the problem to preconditioning the much simpler matrix $\tilde{A}_n$ instead of A_n. Due to the similarity between the basis $\{g_{l,j}\}$ and the derivative of the basis $\{\phi_{n,i}\}$, it is natural to construct another orthogonal basis similar to the hierarchy basis to preconditioning $\tilde{A}_n$ (see, e.g., [25,26]). We will construct such basis in the rest of this section.

Proposition 1. *Let $\tilde{V}_n$ be the linear span of $g_{n,j}$, $j=1,\dots,2^{n-1}$. The sequence $\{\tilde{V}_n\}_{n=1,2,\dots}$ of subspaces is nested, that is, $\tilde{V}_n \subset \tilde{V}_{n+1}$ for all n.*

Proof. We shall show that the following relation is valid almost everywhere:

$$g_{n-1,j} = \begin{cases} g_{n,1}+g_{n,2}+\frac{1}{2}g_{n,3}, & j=1,\\ \frac{1}{2}g_{n,2j-1}+g_{n,2j}+\frac{1}{2}g_{n,2j+1}, & j=2,\dots,2^{n-1}-1. \end{cases} \tag{18}$$

For brevity, we define $\eta := 2^{-n}$. From the definition of $g_{n-1,1}$, we have

$$g_{n-1,1} = \begin{cases} -2^{(n-1)/2}, & 2\eta < x < 4\eta,\\ 0, & \text{otherwise}. \end{cases}$$

Note that

$$g_{n,1}(x) = \begin{cases} -2^{n/2}, & \eta < x < 2\eta,\\ 0, & \text{otherwise}, \end{cases}$$

$$g_{n,2}(x) = \begin{cases} 2^{n/2}, & \eta < x < 2\eta,\\ -2^{(n+1)/2}, & 2\eta < x < 3\eta,\\ 0, & \text{otherwise}, \end{cases}$$

and

$$g_{n,3}(x) = \begin{cases} 2^{(n+1)/2}, & 2\eta < x < 3\eta,\\ -2^{(n+1)/2}, & 3\eta < x < 4\eta,\\ 0, & \text{otherwise}. \end{cases}$$

Hence, for $x \in (0,4\eta)\setminus\{\eta,2\eta,3\eta\}$, we have

$$g_{n-1,1}(x) = g_{n,1}(x)+g_{n,2}(x)+\frac{1}{2}g_{n,3}(x).$$

To verify the second equation in (18), first we recall that

$$\phi'_{n-1,j} = \frac{1}{2}\phi'_{n,2j-1}+\phi'_{n,2j}+\frac{1}{2}\phi'_{n,2j+1}, \quad \text{a.e. } j=2,\dots,2^{n-1}-1.$$

Moreover, χ_{n-1} and χ_n agree on the interval $[2^{-n+1},1]$ and, for $j=2,\dots,2^{n-1}-1$, $\phi_{n-1,j}$ is supported in $[2^{-n+1},1]$. Therefore,

$$g_{n-1,j} = \chi_{n-1}\phi'_{n-1,j} = \chi_n\phi'_{n-1,j} = \chi_n\left(\frac{1}{2}\phi'_{n,2j-1} + \phi'_{n,2j} + \frac{1}{2}\phi'_{n,2j+1}\right)$$
$$= \frac{1}{2}g_{n,2j-1} + g_{n,2j} + \frac{1}{2}g_{n,2j+1}.$$

This proves the proposition. □

Similar to the construction of the hierarchy basis, let

$$\tilde{\psi}_{l-1,j} := g_{l,2j-1}, \quad j = 1, \dots, 2^{l-1},\ l = n, n-1, \dots, 1, \tag{19}$$

and

$$\tilde{W}_{l-1} := \operatorname{span}\{\tilde{\psi}_{l-1,j}\colon\ j = 1, \dots, 2^{l-1}\}.$$

Then we have:

Proposition 2. $\{\tilde{\psi}_{l,j}\colon\ l = 1, \dots, n,\ j = 1, \dots, 2^{l-1}\}$ *is an orthogonal basis for* $\tilde{V}_n$.

Proof. We shall verify the following properties:

(i) $\langle\tilde{\psi}_{l-1,j}, g_{l-1,j'}\rangle = 0$, $j = 1, \dots, 2^{l-1}$, $j' = 1, \dots, 2^{l-1} - 1$;
(ii) $\langle\tilde{\psi}_{l-1,j}, \tilde{\psi}_{l-1,j'}\rangle = 0$, $j \neq j'$;
(iii) $\tilde{V}_n = \tilde{W}_0 + \tilde{W}_1 + \cdots + \tilde{W}_{n-1}$.

First we prove (i). For $j \neq 1$, there exists k such that $2j - 1 \in \{2^{l-k} + 1, \dots, 2^{l-k+1} - 1\}$, and

$$\tilde{\psi}_{l-1,j} = g_{l,2j-1} = 2^{l-k/2}\begin{cases} 1, & (2j-2)2^{-l} < x < (2j-1)2^{-l}, \\ -1, & (2j-1)2^{-l} < x < (2j)2^{-l}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $g_{l-1,j'}$ is a constant on $\operatorname{supp}\{\tilde{\psi}_{l-1,j}\} = [(2j-2)2^{-l}, (2j)2^{-l}]$ for $j' = 1, \dots, 2^{l-1} - 1$, (i) is true. For the case $j = 1$, we obtain that $\tilde{\psi}_{l-1,1}$ $(= g_{l,1})$ is orthogonal to $\tilde{V}_{l-1}$ because $g_{l-1,j'}$, $j' = 1, \dots, 2^{l-1} - 1$, have no overlapped support with $\tilde{\psi}_{l-1,1}$.

(ii) Follows from

$$\operatorname{supp}\{\tilde{\psi}_{l-1,j}\} \cap \operatorname{supp}\{\tilde{\psi}_{l-1,j'}\} = \emptyset.$$

Finally, we turn to (iii). First, $\{g_{l,j}\}_{j=1}^{2^l-1}$ is defined to be a basis for $\tilde{V}_l$. Second, by (i) and (ii), we have

$$\tilde{V}_n = \tilde{V}_1 + \tilde{W}_1 + \tilde{W}_2 + \cdots + \tilde{W}_{n-1}. \tag{20}$$

According to definitions, $\tilde{\psi}_{0,1} = g_{1,1}$ by (19), $\tilde{W}_0 = \operatorname{span}\{\tilde{\psi}_{0,1}\}$, and $\tilde{V}_1 = \operatorname{span}\{g_{1,1}\}$. Therefore, $\tilde{V}_1$ can be replaced by $\tilde{W}_0$ in (20). This completes the proof. □

In what follows we shall provide the preconditioning method for $\tilde{A}_n$ in (17). More precisely, we can find two sparse matrices P and H based on the change of bases from $\{g_{n,j}\}_j$ to $\{\tilde{\psi}_{l,j}\}_{l,j}$ such that $(PH)\tilde{A}_n(PH)^{\mathrm{T}}$ is an identity matrix. By Theorem 1, it is clear that (PH) is also a good preconditioner for the stiffness matrix A_n. To find the matrices P and H, we shall write (18), (19) into the matrix form for the convenience of explanation.

Denote by G_l, $\tilde{\Psi}_l$ the vectors of functions $(g_{l,1}, \ldots, g_{l,2^l-1})^{\mathrm{T}}$, $(\tilde{\psi}_{l,1}, \ldots, \tilde{\psi}_{l,2^l})^{\mathrm{T}}$, respectively. Let $\tilde{\Psi} := (\tilde{\Psi}_0^{\mathrm{T}}, \ldots, \tilde{\Psi}_{n-1}^{\mathrm{T}})^{\mathrm{T}}$, and denote by $\tilde{A}_{\tilde{\Psi},n}$ the matrix $\langle \tilde{\Psi}, (\tilde{\Psi})^{\mathrm{T}} \rangle$. Then $\tilde{A}_{\tilde{\Psi},n}$ is a diagonal matrix by Proposition 2. Furthermore, one can find a diagonal matrix P such that

$$I_{2^n-1} \equiv P\tilde{A}_{\tilde{\Psi},n}P^{\mathrm{T}}, \tag{21}$$

where (l,l) entry of the matrix P is $\|\tilde{\Psi}(l)\|_{L_2}^{-1}$, and $\tilde{\Psi}(l)$ denotes the lth entry of the vector $\tilde{\Psi}$. Clearly, $P\tilde{\Psi}$ is a stable (orthonormal) basis for $\tilde{V}_n$, and due to the simple transformation from the basis G_n to the basis $P\tilde{\Psi}$ (see (18), (19)), $\tilde{A}_n$ can be preconditioned through a basis transformation from G_n to $P\tilde{\Psi}$. By (18), we have

$$G_{l-1} = B_{g,l-1}G_l, \tag{22}$$

where $B_{g,l-1}$ is a $2^{l-1} - 1$ by $2^l - 1$ matrix (only nonzero entries are listed).

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & & & \ddots & & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Denote by $B_{\tilde{\psi},l-1}$ the 2^{l-1} by $2^l - 1$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Then, (19) becomes

$$\tilde{\Psi}_{l-1} = B_{\tilde{\psi},l-1}G_l. \tag{23}$$

Thus, (22) and (23) together yield $\binom{G_{l-1}}{\tilde{\Psi}_{l-1}} = \binom{B_{g,l-1}}{B_{\tilde{\psi},l-1}}G_l$. By H_{l-1} we denote the $2^n - 1$ by $2^n - 1$ transformation matrix

$$H_{l-1} := \begin{pmatrix} B_{g,l-1} & 0 \\ B_{\tilde{\psi},l-1} & 0 \\ 0 & I_{2^n-2^l} \end{pmatrix}.$$

Then we have

$$\tilde{\Psi} = H_1 \cdots H_{n-1}G_n.$$

For brevity, let $H := H_1 \cdots H_{n-1}$, and thus we have the transformation between two bases $\tilde{\Psi} = HG_n$. Note that $\tilde{A}_{\tilde{\Psi},n} = H\tilde{A}_nH^{\mathrm{T}}$. By (21), we have

$$I_{2^n-1} \equiv (PH)\tilde{A}_n(PH)^{\mathrm{T}}. \tag{24}$$

Let S in Theorem 1 be PH in (24). Then

$$\kappa\big(SA_nS^{\mathrm{T}}\big) \leqslant 2\big(1 + \|q\|_{L_\infty(0,1)}\big).$$

Consequently, (PH) is a suitable preconditioner for A_n. Furthermore, it is easily seen that (PH) has $O(N)$ nonzero entries, where $N = 2^n - 1$ is the size of the basis functions for V_n. Therefore, implementation of the preconditioning shall be efficient. Detail discussion may be found in [22, Proposition 4.6].

Corollary. *For the case $q(x) = 0$, the condition number of the matrix $(PH)A_n(PH)^{\mathrm{T}}$ is bounded by 2 for all n.*

Now we provide a preconditioning algorithm for solving (12). Note that

$$A_nC_n = F_n \quad \Leftrightarrow \quad (PH)A_n(PH)^{\mathrm{T}}\big((PH)^{\mathrm{T}}\big)^{-1}C_n = (PH)F_n.$$

Then (12) is equivalent to the following linear equations with $\mathbf{x} = ((PH)^{\mathrm{T}})^{-1}C_n$,

$$\big[(PH)A_n(PH)^{\mathrm{T}}\big]\mathbf{x} = (PH)F_n. \tag{25}$$

To solve (12) for C_n, we first solve (25) for $\mathbf{x}$, and the solution of (12) is

$$C_n = (PH)^{\mathrm{T}}\mathbf{x}.$$

Note that the matrix $[(PH)A_n(PH)^{\mathrm{T}}]$ is well conditioned. Therefore it is efficient to solve (25) for $\mathbf{x}$ numerically.

3. Error estimates

We provide basic error estimates in this section and show that finite-dimensional subspaces used in Section 2 do provide the suitable approximation order. For the notational convenience, we restrict ourselves to the uniform partition case in the previous section. Under such setting, it is easier to describe the preconditioning method based on the multi-level nested subspaces. However, error estimates stated in this section hold true for the general nonuniform partition case. Furthermore, the preconditioning method developed in the previous section is readily generalized to the nonuniform partition case as long as the sequence of subspaces are nested.

For the general non-uniform partition defined by $0 = x_0 < x_1 < \cdots < x_M = 1$, let

$$\phi_j := \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise}. \end{cases}$$

We also let $h_j := x_j - x_{j-1}$, and $h := \max_{1 \leqslant j \leqslant M}\{h_j\}$, where the later quantity measures the mesh size. The finite-dimensional space is spanned by the nodal basis functions $\{\phi_j\}$,

$$V_h := \operatorname{span}\{\phi_0 + \phi_1, \phi_2, \ldots, \phi_{M-1}\}.$$

Then the Galerkin method is to find $u_h \in V_h$ such that

$$a(u_h, v) = \langle xf, v\rangle, \quad \forall v \in V_h.$$

We will follow several lemmas to obtain error estimates in this section. In the following, the solution u is assumed to be smooth ($u \in H^2$, where H^2 denotes the usual Sobolev space of functions with the weak derivatives up to order two in $L_2(0, 1)$) with the boundary conditions $u'(0) = u(1) = 0$. We let the same letter C which is independent of h denote the different constants in the different inequalities.

Lemma 5. *There exists a constant h_0 such that for all $h < h_0$,*

$$\|x^{1/2}(u' - u_I')\|_{L_2} \leqslant Ch\|x^{1/2}u''\|_{L_2},$$

where C is a constant depending on $\max_{i>1}\{x_i/x_{i-1}\}$ and that $u_I \in V_h$ is the interpolant of u defined by

$$u_I(x) := u(x_1)\big(\phi_0(x) + \phi_1(x)\big) + \sum_{i=2}^{M-1} u(x_j)\phi_j(x).$$

Proof. On the interval $I_i := (x_{i-1}, x_i)$, $i > 1$, similar to the proof of Lemma 2 of [9], we have

$$\|x^{1/2}(u' - u_I')\|_{L_2(I_i)} \leqslant Ch\|x^{1/2}u''\|_{L_2(I_i)}. \tag{26}$$

On the interval $I_1 = (0, x_1)$, let $e(x) := u'(x) - u'(x_1)$. Then $e(x_1) = 0$ and $e'(x) = u''(x)$. Following the idea in Lemma 1, we have

$$\begin{aligned}\int_0^{x_1} x\big|e(x)\big|^2 &= \int_0^{x_1} x\left|\int_x^{x_1} e'(t)\,dt\right|^2 dx \leqslant \int_0^{x_1} x\left|\int_x^{x_1}\left(\frac{t}{x}\right)^{1/2}\big|e'(t)\big|\right|^2 dt\\ &\leqslant \|t^{1/2}e'\|^2_{L_2(I_1)}h^2,\end{aligned}$$

and hence

$$\int_0^{x_1} x\big|u'(x) - u'(x_1)\big|^2\,dx \leqslant h^2\int_0^{x_1} x\big|u''(x)\big|^2\,dx. \tag{27}$$

Since

$$\int_0^{x_1} x\big|u'(x) - u'(x_1)\big|^2\,dx = \Gamma_1 + \Gamma_2 - 2\int_0^{x_1} xu'(x)u'(x_1)\,dx,$$

where $\Gamma_1 := \int_0^{x_1} x|u'(x)|^2\,dx$, and $\Gamma_2 := |u'(x_1)|^2\int_0^{x_1} x\,dx$, we have

$$\int_0^{x_1} x\big|u'(x) - u'(x_1)\big|^2\,dx \geqslant \Gamma_1 - \beta\Gamma_1 - \frac{1}{\beta}\Gamma_2 + \Gamma_2.$$

Let $\beta = 1/2$. Note that $u_I'(x) = 0$, $x \in I_1$. Then together with (27), we have

$$\|x^{1/2}(u' - u_I')\|_{L_2(I_1)} = \|x^{1/2}u'\|^2_{L_2(I_1)} = \Gamma_1 \leqslant 2h^2\|x^{1/2}u''\|^2_{L_2(I_1)} + 2\Gamma_2. \tag{28}$$

Combing (26) with (28) yields

$$\|x^{1/2}(u' - u_I')\|^2_{L_2(0,1)} \leqslant Ch^2\|x^{1/2}u''\|^2_{L_2(0,1)} + 2\Gamma_2.$$

The proof will be completed by estimating Γ_2:

$$2\Gamma_2 = \big|u'(x_1)\big|^2(x_1)^2 \leqslant h^2\big|u'(x_1)\big|^2 = h^2\left|\int_0^{x_1} u''(t)\,dt\right|^2 \leqslant \frac{h^3}{2}\|u''\|^2_{L_2(I_1)}.$$

Consequently, there exists a constant h_0 such that

$$h\|u''\|^2_{L_2(0,x_1)} \leqslant \left\|x^{1/2}u''\right\|^2_{L_2(0,1)}, \quad \forall h < h_0,$$

and thus the proof is completed. □

Theorem 2. *There exists a constant h_0 such that for any $h < h_0$,*

$$\left\|x^{1/2}\left(u' - u_h'\right)\right\|_{L_2(0,1)} \leqslant Ch\left\|x^{1/2}u''\right\|_{L_2(0,1)}, \tag{29}$$

and

$$\left\|x^{1/2}(u - u_h)\right\|_{L_2(0,1)} \leqslant Ch^2\left\|x^{1/2}u''\right\|_{L_2(0,1)} \tag{30}$$

hold.

Proof. Inequality (29) is a standard error estimate. We may obtain inequality (30) by a duality argument. Let w solve $a(v,w) = \langle x(u-u_h), v\rangle$, $\forall v \in \dot{H}^1_0$. Then we have

$$a(u-u_h, w) = \left\langle x(u-u_h), u-u_h\right\rangle = \left\|x^{1/2}(u-u_h)\right\|^2_{L_2},$$

and

$$\begin{aligned} a(u-u_h, w) = a(u-u_h, w-w_I) &\leqslant C\left\|x^{1/2}\left(u'-u_h'\right)\right\|_{L_2}\left\|x^{1/2}\left(w'-w_I'\right)\right\|_{L_2} \\ &\leqslant Ch^2\left\|x^{1/2}u''\right\|_{L_2}\left\|x^{1/2}w''\right\|_{L_2}. \end{aligned}$$

This shows that once the regularity of w is verified, i.e.,

$$\left\|x^{1/2}w''\right\|_{L_2} \leqslant C\left\|x^{1/2}(u-u_h)\right\|_{L_2},$$

(30) holds true. w satisfies the following equation (from $-(xw')' = x(u-u_h-qw)$):

$$w(x) = \int_x^1 \frac{1}{t}\int_0^t \left(u(s) - u_h(s) - q(s)w(s)\right)s\,ds\,dt.$$

Differentiating both sides of the above expression twice, we have

$$w''(x) = \frac{1}{x^2}\int_0^x \left(u(s) - u_h(s) - q(s)w(s)\right)s\,ds - \left(u(x) - u_h(x)\right) + q(x)w(x),$$

and thus

$$\begin{aligned} \left\|x^{1/2}w''\right\|_{L_2} \leqslant{} & \left\|\frac{1}{x}\int_0^x \left(\frac{s}{x}\right)^{1/2} s^{1/2}(u-u_h-qw)\,ds\right\|_{L_2} \\ & + \left\|x^{1/2}(u-u_h)\right\|_{L_2} + C\left\|x^{1/2}w\right\|_{L_2}. \end{aligned}$$

By the Hardy's inequality

$$\left\|\frac{1}{x}\int_0^x f(t)\,dt\right\|_{L_2} \leqslant 2\|f\|_{L_2},$$

we get

$$\|x^{1/2}w''\|_{L_2} \leqslant \left\| \frac{1}{x} \int_0^x \big(s^{1/2}(u-u_h) - s^{1/2}qw\big)\,ds \right\|_{L_2} + \|x^{1/2}(u-u_h)\|_{L_2} + C\|x^{1/2}w\|_{L_2}$$
$$\leqslant C\big[\|x^{1/2}(u-u_h)\|_{L_2} + \|x^{1/2}w\|_{L_2}\big].$$

Finally, we prove the inequality $\|x^{1/2}w\|_{L_2} \leqslant C\|x^{1/2}(u-u_h)\|_{L_2}$ as follows:

$$\|x^{1/2}w\|_{L_2}\|x^{1/2}(u-u_h)\|_{L_2} \geqslant \langle x(u-u_h), w\rangle = a(w,w) \geqslant C\|x^{1/2}w'\|_{L_2}^2$$
$$\geqslant C\|x^{1/2}w\|_{L_2}^2.$$

This completes the proof. □

Finally, for the case $q(x)=0$, we provide error estimates for $\|u'-u_h'\|_{L_2}$ and $\|u-u_h\|_{L_2}$.

Theorem 3. *If $q(x)=0$, then*

$$\|u'-u_h'\|_{L_2} \leqslant Ch\|u''\|_{L_2}.$$

Proof. u_h satisfies the following equation:

$$\langle x(u'-u_h'), v'\rangle = 0, \quad \forall v \in V_h. \tag{31}$$

Let $V_h' := \{v'\colon v \in V_h\}$. It is easily seen that $V_h' = \{w = \sum_{j=2}^M c_j\chi_{I_j}\colon \{c_j\} \in \mathbb{R}^{M-1}\}$. In other words, V_h' is the linear span of the piecewise constants on each interval I_j, $j>1$ with 0 on the interval I_1. Let $u_h' = \sum_{j=2}^M c_j\chi_{I_j}$. Then (31) is equivalent to

$$\int_{x_{i-1}}^{x_i} \big(u'(x) - c_i\big)x\,dx = 0, \quad i > 1.$$

Therefore,

$$c_i = \int_{x_{i-1}}^{x_i} u'(t)t\,dt \Big/ \int_{x_{i-1}}^{x_i} t\,dt.$$

Setting $\Gamma := \int_{x_{i-1}}^{x_i} t\,dt$, we obtain

$$\|u'-u_h'\|_{L_2(I_i)}^2 = \int_{x_{i-1}}^{x_i} \big(u'(x) - u_h'(x)\big)^2\,dx$$
$$= \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_i} \big(u'(x)-u'(t)\big)t\,dt \right)^2 dx/\Gamma^2, \quad i>1. \tag{32}$$

Estimating the last term in the above equation gives

$$\int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_i} \big(u'(x)-u'(t)\big)t\,dt \right)^2 dx \leqslant \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_i} \big|u'(x)-u'(t)\big|^2\,dt \int_{x_{i-1}}^{x_i} t^2\,dt \right) dx. \tag{33}$$

Now we have

$$\int_{x_{i-1}}^{x_i} \big(u'(x)-u'(t)\big)^2\,dt = \int_{x_{i-1}}^{x_i} \left|\int_x^t u''(s)\,ds\right|^2 dt$$
$$\leqslant \int_{x_{i-1}}^{x_i} \left|\int_x^t \big|u''(s)\big|^2\,ds\right|\left|\int_x^t 1^2\,ds\right| dt \leqslant \|u''\|^2_{L_2(I_i)} h_i^2. \tag{34}$$

Hence, after plugging (34) into (33), we estimate the term $\|u'-u_h'\|^2_{L_2(I_i)}$ in (32) by

$$\int_{x_{i-1}}^{x_i} \big|u'(x)-u_h'(x)\big|^2\,dx \leqslant h_i^3\|u''\|^2_{L_2(I_i)} \frac{\int_{x_{i-1}}^{x_i} t^2\,dt}{(\int_{x_{i-1}}^{x_i} t\,dt)^2} \leqslant h_i^3\|u''\|^2_{L_2(I_i)} \frac{x_i^2\int_{x_{i-1}}^{x_i} dt}{(x_{i-1}\int_{x_{i-1}}^{x_i} dt)^2}$$
$$\leqslant \left(\frac{x_i}{x_{i-1}}\right)^2 h^2\|u''\|^2_{L_2(I_i)}.$$

On the interval I_1,

$$\int_0^{x_1}\big|u'(x)\big|^2\,dx = \int_0^{x_1}\left|\int_0^x u''(s)\,ds\right|^2 dx$$
$$\leqslant \int_0^{x_1}\left|\int_0^x \big|u''(s)\big|^2\,ds \int_0^x 1^2\,ds\right| dx \leqslant \|u''\|^2_{L_2(0,x_1)} h^2.$$

Thus the proof is completed. □

The following is another theorem on the error estimate of $\|u-u_h\|_{L_2(0,1)}$:

Theorem 4. *If $q(x)=0$, there exists a constant h_0 such that for any $h<h_0$,*

$$\|u-u_h\|_{L_2(0,1)} \leqslant Ch\big\|u'-u_h'\big\|_{L_2(0,1)}.$$

4. Examples

Now we shall provide the numerical results of some examples to confirm the theory developed earlier. For the first example, let $q(x)=0$ and $f(x)=-\frac{\pi}{2}x^{-1}(\sin\frac{\pi}{2}x + x\frac{\pi}{2}\cos\frac{\pi}{2}x)$. Then the solution is $u=-\cos\frac{\pi}{2}x$. To define the coarsest grid, we split the domain (0, 1) into $2^1=2$ pieces, and then divide each piece into 2 pieces of equal length. We keep on splitting until there are 2^n pieces. Therefore, we have n levels of nested subspaces. Concerning the singular boundary value problem (5), once the finite-dimensional subspace V_n is fixed, the stiffness matrix A_n is also fixed, as well as the preconditioner PH. Therefore, we will demonstrate the performance of the preconditioner first by comparing the condition numbers of A_n with those of $(PH)A_n(PH)^{\mathrm{T}}$ with different n. In Table 1, we display the maximum eigenvalues, minimum eigenvalues and the condition numbers of the two matrices A_n and $(PH)A_n(PH)^{\mathrm{T}}$ for the different n. Computing results in Table 1 illustrate that condition numbers of the preconditioned stiffness matrices are uniformly bounded by 2. It strongly support the corollary in Section 2. We use the Galerkin method to solve the problem with mesh size $1/2^n$ and let u, and u_h denote the solution and the Galerkin solution of the singular problem, respectively. $|(u-u_h)|_{H^1}$, $\|(u-u_h)\|_{L_2}$ with different n are listed in Table 2. As predicted by Theorems 3, 4, the Galerkin method with the piecewise linear nodal basis preserves $O(h)$, $O(h^2)$ convergence rates for $|(u-u_h)|_{H^1}$, $\|(u-u_h)\|_{L_2}$, respectively.

Table 1
Condition numbers of the matrix A_n and $(PH)A_n(PH)^{\mathrm{T}}$

n	5	6	7	8	9	10
$\lambda_{\max,A_n}$	109.8	232.9	482.7	986.9	2.001×10^3	4.037×10^3
$\lambda_{\min,A_n}$	0.0477	0.0232	0.0115	0.0057	0.0028	0.0014
$\kappa(A_n)$	2.302×10^3	1.002×10^4	4.213×10^4	1.735×10^5	7.061×10^5	2.854×10^6
$\lambda_{\max,PHA_n(PH)^{\mathrm{T}}}$	1.9688	1.984	1.992	1.996	1.998	1.999
$\lambda_{\min,PHA_n(PH)^{\mathrm{T}}}$	1.031	1.016	1.008	1.004	1.003	1.001
$\kappa(PHA_n(PH)^{\mathrm{T}})$	1.909	1.954	1.977	1.988	1.994	1.997

Table 2
Estimates of $|u-u_h|_{H^1}$, $\|u-u_h\|_{L_2}$

n	$\|u-u_h\|_{H^1}$	$\\|u-u_h\\|_{L_2}$
5	0.0307	1.73×10^{-4}
6	0.0134	4.53×10^{-5}
7	0.0062	1.20×10^{-5}
8	0.0029	3.16×10^{-6}
9	0.0014	8.17×10^{-7}
10	7.05×10^{-4}	2.12×10^{-7}

Table 3
Estimates of $|u-u_h|_{H^1}$, $\|u-u_h\|_{L_2}$

n	$\|u-u_h\|_{H^1}$	$\\|u-u_h\\|_{L_2}$
5	0.0306	2.07×10^{-4}
6	0.0141	5.49×10^{-5}
7	0.0067	1.44×10^{-5}
8	0.0033	3.70×10^{-6}
9	0.0016	9.45×10^{-7}
10	8.03×10^{-4}	2.39×10^{-7}

Table 4
Condition numbers of the matrix A_n and $(PH)A_n(PH)^{\mathrm{T}}$

n	5	6	7	8	9	10
$\lambda_{\max,A_n}$	109.8	232.9	482.7	986.9	2.001×10^3	4.037×10^3
$\lambda_{\min,A_n}$	0.0538	0.0261	0.0129	0.0064	0.0032	0.0016
$\kappa(A_n)$	2.043×10^3	8.907×10^3	3.746×10^4	1.543×10^5	6.283×10^5	2.540×10^6
$\lambda_{\max,PHA_n(PH)^{\mathrm{T}}}$	2.016	2.021	2.023	2.024	2.024	2.024
$\lambda_{\min,PHA_n(PH)^{\mathrm{T}}}$	1.036	1.018	1.009	1.004	1.002	1.001
$\kappa(PHA_n(PH)^{\mathrm{T}})$	1.946	1.986	2.005	2.015	2.019	2.022

Similar to the example shown in [9], we let $q(x) = 1 - x^2$, $f(x) = (1 - x^2)^2 + 4$ in our second example. Subspace level is up to $n = 10$. In this situation, $u(x) = 1 - x^2$, and $|(u - u_h)|_{H^1}$, $\|(u - u_h)\|_{L_2}$ are computed in Table 3 for different n. Condition numbers of the preconditioned system are shown in Table 4. Similar computing results to example one are obtained. These numerical results confirm the performance of our preconditioning method.

Acknowledgments

The author is very grateful to Professor Rong-Qing Jia for his helpful conversations and suggestions. The author also thanks the reviewers for their careful reviews and constructive comments, which lead to the quality improvement of the paper.

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