Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Journal of Combinatorial Theory, Series A 115 (2008) 845–877

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

Alternating subgroups of Coxeter groups

Francesco Brenti [a], Victor Reiner [b,1], Yuval Roichman [c,2]

[a] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*
[b] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA*
[c] *Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel*

Received 7 February 2007

Available online 9 January 2008

Abstract

We study combinatorial properties of the alternating subgroup of a Coxeter group, using a presentation of it due to Bourbaki.

Keywords: Coxeter group; Alternating group; Presentation; Length; Poincaré series

Contents

E-mail addresses: brenti@mat.uniroma2.it (F. Brenti), reiner@math.umn.edu (V. Reiner), yuvalr@math.biu.ac.il (Y. Roichman).

1 Second author supported by NSF grant DMS-0245379.
2 Third author supported in part by the Israel Science Foundation grant No. 947/04.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.10.004

1. Introduction

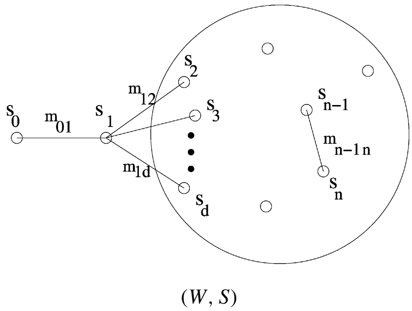
The problem of extending combinatorial identities on the symmetric group to other groups, including alternating groups, was posed by Foata and others in the early nineties. A solution for the alternating subgroup A_n of the symmetric group was given in [3,10]. The solution is based on a well-known observation, that viewing the symmetric group as a Coxeter group yields natural algebraic interpretations to classical permutation statistics. A simple, Coxeter-like presentation of A_n led to natural extended statistics and identities. The goal of this paper is to explore whether combinatorial properties of Coxeter groups may be extended, in general, to their alternating subgroups, using Coxeter-like presentations.

For any Coxeter system (W, S), its *alternating subgroup* W^+ is the kernel of the *sign character* that sends every $s \in S$ to -1. An exercise from Bourbaki gives a simple presentation for W^+, after one chooses a generator $s_0 \in S$. We will explore the combinatorial properties of this presentation, distinguishing in the four main sections of the paper different levels of generality (defined below) regarding the chosen generator s_0:

$$\begin{tikzcd}
 & \begin{array}{c} s_0 \text{ arbitrary} \\ (\textit{Section } 2) \end{array} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{array}{c} s_0 \text{ evenly-laced} \\ (\textit{Section } 3) \end{array} \arrow[dr, dash] & & \begin{array}{c} s_0 \text{ a leaf} \\ (\textit{Section } 4). \end{array} \arrow[dl, dash] \\
 & \begin{array}{c} s_0 \text{ an even leaf} \\ (\textit{Section } 5) \end{array} &
\end{tikzcd}$$

Section 2 reviews the presentation and explores some of its consequences in general for the *length function*, *parabolic subgroups*, a *Coxeter-like complex* for W^+, and the notion of *palindromes*, which play the role usually played by *reflections* in a Coxeter system. This section also defines *weak* and *strong* partial orders on W^+ and poses some basic questions about them.

Section 3 explores the special case where s_0 is *evenly-laced*, meaning that the order m_{0i} of $s_0 s_i$ is even (or infinity) for all i. It turns out that, surprisingly, this case is much better-behaved. Here the unique, length-additive factorization $W = W^J \cdot W_J$ for parabolic subgroups of W induces similar unique length-additive factorizations within W^+. One can compute generating functions for W^+ by length, or jointly by length and certain descent statistics. Here the palindromes which shorten an element determine that element uniquely, and satisfy a crucial



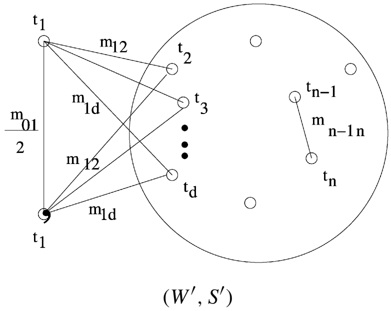


Fig. 1. Schematic of the relation between the diagrams for a Coxeter system (W, S) with even leaf node s_0, and the Coxeter system (W', S') derived from it, closely connected to the alternating group W^+. The unique neighbor of s_0 has been labeled s_1, so that m_{01} is *even*.

strong exchange property. This gives better characterizations of the weak and strong partial orders, and answers affirmatively all the questions about these orders from Section 2 in this case.

Section 4 examines how the general presentation simplifies to what we call a *nearly Coxeter* presentation when s_0 is a *leaf* in the Coxeter diagram, meaning that s_0 commutes with all but one of the other generators in $S - \{s_0\}$. Such leaf generators occur in many situations, e.g. when W is finite and for most affine Weyl groups.

Section 5 studies the further special case where s_0 is an evenly-laced leaf. The classification of finite and affine Coxeter systems shows that *all* evenly-laced nodes s_0 are even leaves when W is finite, and this is almost always the case for W affine. In particular, even leaves occur in the finite type $B_n = (C_n)$ and the affine types $\tilde{B}_n, \tilde{C}_n$. When s_0 is an even leaf, there is an amazingly close connection between the alternating group W^+ and a *different* index 2 subgroup W', namely the kernel of the homomorphism χ_0 sending s_0 to -1 and all other Coxeter generators to $+1$. It turns out that this subgroup W' is a (non-parabolic) reflection subgroup of W, carrying its own Coxeter presentation (W', S'), closely related to the Coxeter presentation of (W, S). This generalizes the inclusion of type D_n inside B_n, and although $W^+ \not\cong W'$, the connection allows one to reduce all the various combinatorial questions for the presentation (W^+, R) (length function, descent sets, partial orderings, reduced words) to their well-studied counterparts in the Coxeter system (W', S').[1]

Although most of our results are relatively straightforward once one knows what one wants to prove, among those that we consider the main results are Propositions 2.4.2, 2.5.5, and 3.4.3, Corollaries 3.3.3 and 5.2.4, and the quite surprising Theorem 3.5.1. We also pose several natural questions, namely Questions 2.5.7, 2.6.2, 2.6.3, 2.6.4, and 2.6.5.

2. The general case

2.1. Bourbaki's presentation

Let (W, S) be a Coxeter system with generators $S = \{s_0, s_1, \ldots, s_n\}$, that is, W has a presentation of the form

[1] While the combinatorics of W^+ and W' seems to be similar, the combinatorics of other subgroups of index 2 seems to be different; in particular, no nearly Coxeter presentation for these groups is known; see, e.g., [2].

$$W = \langle S = \{s_0, s_1, \dots, s_n\} \colon (s_i s_j)^{m_{ij}} = e \text{ for } 0 \leqslant i \leqslant j \leqslant n \rangle \tag{1}$$

where $m_{ij} = m_{ji} \in \{2, 3, \dots\} \cup \{\infty\}$ and $m_{ii} = 2$.

The *sign character* $\epsilon : W \to \{\pm 1\}$ is the homomorphism uniquely defined by $\epsilon(s) = -1$ for all $s \in S$. Its kernel $W^+ := \ker(\epsilon)$ is an index two subgroup called the *alternating subgroup* of W.

Once one has distinguished s_0 in S by its zero subscript, an exercise in Bourbaki [6, Chapter IV, Section 1, Exercise 9] suggests a simple presentation for W^+, which we recall here and prove along the lines suggested by Bourbaki.

Proposition 2.1.1. *Given a Coxeter system (W, S) with distinguished generator s_0, map the set $R = \{r_1, \dots, r_n\}_{i=1,2,\dots,n}$ into W^+ via $r_i \mapsto s_0 s_i$. Then this gives a set of generators for W^+ with the following presentation*:

$$W^+ \cong \langle R = \{r_1, \dots, r_n\} \colon r_i^{m_{0i}} = (r_i^{-1} r_j)^{m_{ij}} = e \textit{ for } 1 \leqslant i < j \leqslant n \rangle. \tag{2}$$

Proof. Consider the abstract group H^+ with the presentation by generators R given on the right-hand side of (2). One checks that the set map $\alpha : R \to H^+$ sending r_i to r_i^{-1} extends to an involutive group automorphism α on H^+: the relation $(r_i r_j^{-1})^{m_{ij}} = e$ follows from the relation $(r_i^{-1} r_j)^{m_{ij}} = e$ in H^+ by taking the inverse of both sides and then conjugating by r_j.

Thus the group $\mathbb{Z}/2\mathbb{Z} = \{1, \alpha\}$ acts on H^+, and one can form the semidirect product $H^+ \rtimes \mathbb{Z}/2\mathbb{Z}$ in which $(h_1 \alpha^i) \cdot (h_2 \alpha^j) = h_1 \alpha^i (h_2) \cdot \alpha^{i+j}$. This has either of the following two presentations:

$$\begin{aligned}
&H^+ \rtimes \mathbb{Z}/2\mathbb{Z} \\
&\quad \cong \langle r_1, \dots, r_n, \alpha \colon \alpha^2 = r_i^{m_{0i}} = (r_i^{-1} r_j)^{m_{ij}} = e \text{ for } 1 \leqslant i < j \leqslant n,\ \alpha r_i \alpha = r_i^{-1} \rangle \\
&\quad \cong \langle r_0, r_1, \dots, r_n, \alpha \colon r_0 = \alpha^2 = (r_i^{-1} r_j)^{m_{ij}} = e \text{ for } 0 \leqslant i < j \leqslant n,\ \alpha r_i \alpha = r_i^{-1} \rangle.
\end{aligned}$$

We claim that the following two maps are well-defined and inverse isomorphisms:

$$\begin{aligned}
&W \xrightarrow{\rho} H^+ \rtimes \mathbb{Z}/2\mathbb{Z}, \\
&s_i \mapsto \alpha r_i (= r_i^{-1} \alpha) \quad \text{for } i = 1, \dots, n, \\
&s_0 \mapsto \alpha r_0 (= \alpha), \\
&H^+ \rtimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sigma} W, \\
&r_i \mapsto s_0 s_i \qquad\qquad \text{for } i = 0, 1, \dots, n, \\
&\alpha \mapsto s_0.
\end{aligned}$$

To check that ρ is well-defined one must check that the (W, S) Coxeter relations $(s_i s_j)^{m_{ij}} = e$ for $0 \leqslant i \leqslant j \leqslant n$ map under ρ to relations in $H^+ \rtimes \mathbb{Z}/2\mathbb{Z}$. Bearing in mind that $r_0 = e$, this is checked as follows:

$$(s_i s_j)^{m_{ij}} = e \mapsto (\alpha r_i \alpha r_j)^{m_{ij}} = (r_i^{-1} \alpha \alpha r_j)^{m_{ij}} = (r_i^{-1} r_j)^{m_{ij}} = e.$$

To check that σ is well-defined one can check that the relations in the second presentation for $H^+ \rtimes \mathbb{Z}/2\mathbb{Z}$ map under σ to relations in W. These are checked as follows:

$$\begin{aligned}
&(r_i^{-1} r_j)^{m_{ij}} = e \mapsto (s_i s_0 s_0 s_j)^{m_{ij}} = (s_i s_j)^{m_{ij}} = e, \\
&\alpha^2 = e \mapsto s_0^2 = e, \\
&\alpha r_i \alpha = r_i^{-1} \mapsto s_0 (s_0 s_i) s_0 = (s_0 s_i)^{-1}.
\end{aligned}$$

Once one knows that ρ, σ are well-defined, it is easily checked that they are inverse isomorphisms by checking this on generators.

Since $\sigma(H^+) \subseteq W^+$, and both $\sigma(H^+)$, W^+ are subgroups of W of index 2, it must be that $\sigma(H^+) = W^+$. Hence σ restricts to the desired isomorphism between the abstractly presented group H^+ and W^+. $\square$

2.2. *Length with respect to $R \cup R^{-1}$*

The maps ρ, σ which appear in the proof of Proposition 2.1.1 lead to a nice interpretation for the length function of W^+ with respect to the symmetrized generating set $R \cup R^{-1}$.

Definition 2.2.1. Given a group G and subset $A \subset G$, let A^* denote the set of all words $\mathbf{a} = (a_1, \ldots, a_\ell)$ with letters a_i in A. Let $A^{-1} := \{a^{-1}\colon a \in A\}$.

Let $\ell_A(\cdot)$ denote the length function on G with respect to the set A, that is,

$$\ell_A(g) := \min\{\ell\colon g = a_1 a_2 \cdots a_\ell \text{ for some } a_i \in A\},$$

where by convention, we set $\ell_A(g) = \infty$ if there are no such expressions for g.

Given an A^*-word $\mathbf{a}$ that factors g in G, say that $\mathbf{a}$ is a *reduced word* for g if it achieves the minimum possible length $\ell_A(g)$.

Definition 2.2.2. Given a Coxeter system (W, S) with $S = \{s_0, s_1, \ldots, s_n\}$ as before, let $\nu(w)$ denote the minimum number of generators $s_j \neq s_0$ occurring in any expression $\mathbf{s} = (s_{i_1}, \ldots, s_{i_\ell}) \in S^*$ that factors w in W, i.e. $w = s_{i_1} \cdots s_{i_\ell}$.

Proposition 2.2.3. *For a Coxeter system (W, S) with $S = \{s_0, s_1, \ldots, s_n\}$ as before, and the presentation (W^+, R) in (2), one has*

$$\ell_{R \cup R^{-1}}(w) = \nu(w)$$

for all $w \in W^+$.

Proof. Assume $w \in W^+$. First we prove the inequality $\ell_{R \cup R^{-1}}(w) \geqslant \nu(w)$. Given an $(R \cup R^{-1})^*$-word $\mathbf{r}$ that factors w of the shortest possible length $\ell_{R \cup R^{-1}}(w)$, apply the map σ from before

$$r_i \mapsto s_0 s_i,$$
$$r_i^{-1} \mapsto s_i s_0,$$

to each letter and concatenate. This gives an S^*-word $\mathbf{s}$ that factors w, having $\ell_{R \cup R^{-1}}(w)$ occurrences of generators $s_j \neq s_0$. Hence the minimum possible such number $\nu(w)$ must be at most $\ell_{R \cup R^{-1}}(w)$.

Similarly we prove the opposite inequality $\ell_{R \cup R^{-1}}(w) \leqslant \nu(w)$. Given an S^*-word $\mathbf{s}$ that factors w with the minimum number $\nu(w)$ of occurrences of generators $s_j \neq s_0$, apply the map ρ from before

$$s_i \mapsto \alpha r_i \quad \text{for } i = 1, \ldots, n,$$
$$s_0 \mapsto \alpha,$$

to each letter and concatenate. This gives an $(R \cup \{\alpha\})^*$-word $\mathbf{r}$ that factors w, having $\nu(w)$ occurrences of generators r_i, and an *even* number of occurrences of α (because $w \in W^+$ implies

s has even length). Repeatedly using the relation $\alpha r_i \alpha^{-1} = r_i^{-1}$, one can bring all these evenly many occurrences of α in **r** to the right end of the word, where they will cancel out because $\alpha^2 = 1$. This leaves an $(R \cup R^{-1})^*$ word factoring w, having length $\nu(w)$. Hence $\ell_{R\cup R^{-1}}(w) \leqslant \nu(w)$. $\square$

For any $w \in W^+$, the proof of the inequality $\ell_{R\cup R^{-1}}(w) \leqslant \nu(w)$ describes in two steps a map (which we will also call ρ) from S^*-words **s** factoring w to $(R \cup R^{-1})^*$-words **r** factoring w. For future use, we point out that this map has the following simple explicit description:

- replace s_i with r_i or r_i^{-1}, respectively, depending upon whether the letter s_i occurs in an even or odd position of **s**, respectively, and
- remove all occurrences of s_0.

As an example,

$$\begin{array}{lccccccccc}
\text{position:} & (1, & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9), \\
S^*\text{-word:} & (s_0, & s_2, & s_0, & s_1, & s_2, & s_0, & s_0, & s_3, & s_1), \\
\big(R \cup \{\alpha\}\big)^*\text{-word:} & (\alpha, & \alpha r_2, & \alpha, & \alpha r_1, & \alpha r_2, & \alpha, & \alpha, & \alpha r_3, & \alpha r_1), \\
\big(R \cup R^{-1}\big)^*\text{-word:} & \big(& r_2, & & r_1, & r_2^{-1}, & & & r_3, & r_1^{-1}\big).
\end{array}$$

Proposition 2.2.4. *The map just described coincides with the map from S^*-words factoring w to $(R \cup R^{-1})^*$-words factoring w described in the proof of Proposition* 2.2.3.

Proof. Note that an occurrence of r_i in **r** which came from an occurrence of s_i in the kth position of **s** will start with k occurrences of α to its left in the $(R \cup \{\alpha\})^*$-word, and each of these α's "toggles" it between $r_i \leftrightarrow r_i^{-1}$ as that α moves past it to the right. $\square$

Example 2.2.5. Let (W, S) be the symmetric group $W = \mathfrak{S}_n$, with $S = \{s_0, s_1, \dots, s_{n-2}\}$ in which s_i is the adjacent transposition $(i+1, i+2)$, so $s_0 = (1, 2)$; this is the usual Coxeter system of type A_{n-1}. Then the length in $W^+ = \mathfrak{A}_n$ with respect to generating set $R \cup R^{-1} = R \cup \{r_1^{-1}\}$ was given combinatorially in [10, §1.3.2], which we recall here.

Given a permutation $w \in \mathfrak{S}_n$, let $\mathrm{lrmin}(w)$ denote its number of *left-to-right minima*, that is, the number of $j \in \{2, 3, \dots, n\}$ satisfying $w(i) > w(j)$ for $1 \leqslant i < j$. Let $\mathrm{inv}(w)$ denote its number of *inversions*, that is, the number of pairs (i, j) with $1 \leqslant i < j \leqslant n$ and $w(i) > w(j)$. It is well known [5, Proposition 1.5.2] that the Coxeter group length ℓ_S has the interpretation $\ell_S(w) = \mathrm{inv}(w)$.

Proposition 2.2.6. *For any $w \in \mathfrak{S}_n$, the maximum number of occurrences of s_0 in a reduced S^*-word for w is* $\mathrm{lrmin}(w)$. *Consequently,*

$$\ell_{R\cup R^{-1}}(w) = \mathrm{inv}(w) - \mathrm{lrmin}(w).$$

The proof is straightforward, and omitted.

In [10] it was shown that for (W, S) of type A_{n-1} with s_0 a leaf node as above, one has

$$\sum_{w \in W^+} q^{\ell_{R\cup R^{-1}}(w)} = (1+2q)\big(1+q+2q^2\big)\cdots\big(1+q+q^2+\cdots+q^{n-3}+2q^{n-2}\big), \tag{3}$$

and there are refinements of (3) that incorporate other statistics; see [10, Proposition 5.7(2), 5.11(2)]. The results of the current paper do not recover this, and are in a sense, complementary—they say more about the case where s_0 is evenly-laced.

2.3. *Parabolic subgroup structure for* (W^+, R)

The presentation (2) for W^+ with respect to the generating set R likens (W^+, R) to a Coxeter system, and suggests the following definition.

Definition 2.3.1. For any $J \subset R = \{r_1, \ldots, r_n\}$, the subgroup $W_J^+ = \langle J \rangle$ generated by J inside W^+ will be called a (*standard*) *parabolic subgroup*.

The structure of parabolic subgroups W_J for (W, S) is an important part of the theory. For (W^+, R) one finds that its parabolic subgroups are closely tied to the parabolic subgroups W_J containing s_0, via the following map.

Definition 2.3.2. Define $\tau : W \to W^+$ by

$$\tau(w) := \begin{cases} w & \text{if } w \in W^+, \\ ws_0 & \text{if } w \notin W^+. \end{cases}$$

In other words, $\tau(w)$ is the unique element in the coset $wW_{\{s_0\}} = \{w, ws_0\}$ that lies in W^+.

The following key property of τ is immediate from its definition.

Proposition 2.3.3. *The set map $\tau : W \to W^+$ is equivariant for the W^+-actions on W, W^+ by left-multiplication.*

In fact, τ induces a W^+-equivariant bijection $W/W_{\{s_0\}} \to W^+$, but we will soon see that more is true. Given any $J \subseteq S$ with $s_0 \in J$, let

$$\tau(J) := \{r_i \colon s_0 \neq s_i \in J\}.$$

Note that the map $J \mapsto \tau(J)$ is a bijection between the indexing sets for parabolic subgroups in W containing s_0 and for all parabolic subgroups of W^+.

Proposition 2.3.4. *For any $J \subseteq S$ with $s_0 \in J$, one has*

$$W_J \cap W^+ = W^+_{\tau(J)}.$$

Proof. The inclusion $W^+_{\tau(J)} \subseteq W_J \cap W^+$ should be clear. For the reverse inclusion, given $w \in W_J \cap W^+$, write a J^*-word $\mathbf{s}$ that factors w containing only $s_i \in J$. Applying the map ρ from Proposition 2.2.4 gives a $(\tau(J) \cup \tau(J)^{-1})^*$-word that factors w, showing that $w \in W^+_{\tau(J)}$. $\square$

Proposition 2.3.5. *For any $J \subseteq S$ with $s_0 \in J$, the* (*set*) *map τ induces a W^+-equivariant bijection*

$$W/W_J \xrightarrow{\tau} W^+/W^+_{\tau(J)}.$$

In particular, taking $J = \{s_0\}$, this is a W^+-equivariant bijection

$$W/W_{\{s_0\}} \xrightarrow{\tau} W^+.$$

Proof. One has a well-defined composite map of sets

$$W \xrightarrow{\tau} W^+ \to W^+/W^+_{\tau(J)}$$

sending w to $\tau(w)W^+_{\tau(J)}$. This composite surjects because $\tau : W \to W^+$ surjects.

It remains to show two things: the composite induces a *well-defined* map $W/W_J \xrightarrow{\tau} W^+/W^+_{\tau(J)}$, and that this induced map is *injective*. Both of these are shown simultaneously as follows: for any $u, v \in W$ one has

$$\begin{aligned} \tau(u)W^+_{\tau(J)} = \tau(v)W^+_{\tau(J)} \quad &\Leftrightarrow \quad \tau(v)^{-1}\tau(u) \in W^+_{\tau(J)} \\ &\Leftrightarrow \quad \tau\big(\tau(v)^{-1}u\big) \in W^+_{\tau(J)} \\ &\Leftrightarrow \quad \tau(v)^{-1}u \in W_J \\ &\Leftrightarrow \quad v^{-1}u \in W_J \\ &\Leftrightarrow \quad uW_J = vW_J \end{aligned}$$

where we have used throughout the fact that $s_0 \in J$, and where the second equivalence uses the W^+-equivariance of the set map $\tau : W \to W^+$ from Proposition 2.3.3. □

Note that Proposition 2.3.5 implies that for any $J \subseteq S$ with $s_0 \in J$, the set of minimum ℓ_S-length coset representatives W^J for W/W_J maps under τ to a set $\tau(W^J)$ of coset representatives for $W^+/W^+_{\tau(J)}$. It turns out that these coset representatives $\tau(W^J)$ are always of minimum $\ell_{R\cup R^{-1}}$-length. To prove this, we note a simple property of the function ν that was defined in Definition 2.2.2.

Proposition 2.3.6. *For any w in W one has*

$$\nu(s_0w) = \nu(w) = \nu(ws_0) = \ell_{R\cup R^{-1}}\big(\tau(w)\big).$$

Proof. Since $\nu(w^{-1}) = \nu(w)$, the first equality follows if one shows the middle equality. Also, since $\ell_{R\cup R^{-1}}(\tau(w)) = \nu(\tau(w))$ and since $\tau(w)$ is either w or ws_0, the last equality also follows from the middle equality.

To prove the middle equality, it suffices to show the inequality $\nu(ws_0) \leqslant \nu(w)$ for all $w \in W$; the reverse inequality follows since $w = ws_0 \cdot s_0$. But this inequality is clear: starting with an S^*-word $\mathbf{s}$ for w that has the minimum number $\nu(w)$ of occurrences of $s_j \neq s_0$, one can append an s_0 to the end to get an S^*-word that factors ws_0 having no more such occurrences. □

Corollary 2.3.7. *For any $J \subseteq S$ with $s_0 \in J$, the coset representatives $\tau(W^J)$ for $W^+/W^+_{\tau(J)}$ each achieve the minimum $\ell_{R\cup R^{-1}}$-length within their coset.*

Proof. Let $w \in W^J$, and $w' \in \tau(w)W^+_{\tau(J)}$. Given an S^*-word for w' that has the minimum number $\nu(w')$ of occurrences of $s_j \neq s_0$, one can extract from it an S^*-reduced subword for w'. Since $w' \in wW_J$ and $w \in W^J$, one has $w \leqslant w'$ in the *strong Bruhat* order on W, and hence one can extract from this a further S^*-subword factoring w [8, §5.10]. Consequently $\nu(w) \leqslant \nu(w')$. But then Proposition 2.3.6 says that

$$\ell_{R\cup R^{-1}}\big(\tau(w)\big) = \nu(w) \leqslant \nu(w') = \ell_{R\cup R^{-1}}(w')$$

as desired. □

Note that we have made no assertion here about an element of $\tau(W^J)$ being *unique* in achieving the minimum length $\ell_{R\cup R^{-1}}$ within its coset, nor have we asserted that the unique factorization $W^+ = \tau(W^J)\cdot W^+_{\tau(J)}$ has additivity of lengths $\ell_{R\cup R^{-1}}$. In fact, these properties fail in general (see Remark 3.4.2), but they will be shown in Section 3.3 to hold whenever s_0 is an evenly-laced node.

Remark 2.3.8. The proof of Corollary 2.3.7 contains a fact which we isolate here for future use.

Proposition 2.3.9. *Let (W, S) be an arbitrary Coxeter system. If $w < w'$ in the strong Bruhat order on W then $\nu(w) \leqslant \nu(w')$. In particular,*

(i) *for any $s \in S$, $w \in W$, if $\ell_S(ws) < \ell_S(w)$ then $\nu(ws) \leqslant \nu(w)$;*
(ii) *for $w, w' \in W^+$, if $w < w'$ in the strong Bruhat order on W then $\ell_{R\cup R^{-1}}(w) \leqslant \ell_{R\cup R^{-1}}(w')$.*

2.4. The Coxeter complex for (W^+, R)

Associated to every Coxeter system (W, S) is a simplicial complex $\Delta(W, S)$ known as its *Coxeter complex*, that has many guises (see [8, §1.15, 5.13] and [5, Exercise 3.16]):

(i) It is the *nerve* of the covering of the set W by the sets

$$\{wW_{S\setminus\{s\}}\}_{w\in W,\ s\in S}$$

which are all cosets of maximal (proper) parabolic subgroups.

(ii) It is the unique simplicial complex whose face poset has elements indexed by the collection

$$\{wW_J\}_{w\in W,\ J\subseteq S}$$

of all cosets of all parabolic subgroups, with ordering by *reverse inclusion*.

(iii) It describes the decomposition by reflecting hyperplanes into cells (actually spherical simplices) of the unit sphere intersected with the Tits cone in the *contragradient representation* V^* of W.

The Coxeter complex $\Delta(W, S)$ enjoys many nice combinatorial, topological, and representation-theoretic properties (see [4], [5, Exercise 3.16]), such as:

(i) It is a pure $(|S| - 1)$-dimensional simplicial complex, and is *balanced* in the sense that if one colors the typical vertex $\{wW_{S\setminus\{s\}}\}_{w\in W, s\in S}$ by the element $s \in S$, then every maximal face of $\Delta(W, S)$ contains exactly one vertex of each color $s \in S$.
(ii) It is a *shellable pseudomanifold*, homeomorphic to either an $(|S| - 1)$-dimensional sphere or open ball, depending upon whether W is finite or infinite.
(iii) When W is finite, the homology $H_*(\Delta(W, S), \mathbb{Z})$, which is concentrated in the top dimension $|S| - 1$, carries the *sign* character of W.
(iv) For each $J \subseteq S$, the *type-selected* subcomplex $\Delta(W, S)_J$, induced on the subset of vertices with colors in J, inherits the properties of being pure $(|J| - 1)$-dimensional, balanced, and shellable. Consequently, although $\Delta(W, S)_J$ is no longer homeomorphic to a sphere, it is homotopy equivalent to a wedge of $(|J| - 1)$-dimensional spheres. Furthermore, the W-action on its top homology has an explicit decomposition into Kazhdan–Lusztig cell representations.

The results of Section 2.3 allow us to define a Coxeter-like complex for (W^+, R) in the sense of [1], and the map τ allows one to immediately carry over many of the properties of $\Delta(W, S)$.

Definition 2.4.1. Given a Coxeter system (W, S) with $S = \{s_0, s_1, \ldots, s_n\}$, and the ensuing presentation (2) for W^+ via the generators $R = \{r_1, \ldots, r_n\}$, define the *Coxeter complex* to be the simplicial complex $\Delta(W^+, R)$ which is the *nerve* of the covering of the set W^+ by the maximal (proper) parabolic subgroups

$$\left\{wW^+_{R\setminus\{r\}}\right\}_{w\in W^+,\ r\in R}.$$

Recall this means $\Delta(W^+, R)$ has vertex set in bijection with these maximal parabolic subgroups, and a subset of its vertices spans a simplex exactly when their corresponding maximal parabolic subgroups have non-empty intersection.

Proposition 2.3.5 and the usual properties of the Coxeter complex $\Delta(W, S)$ immediately imply the following.

Proposition 2.4.2. *The map $\tau : W \to W^+$ induces a W^+-equivariant simplicial isomorphism*

$$\Delta(W, S)_{S\setminus\{s_0\}} \cong \Delta\big(W^+, R\big)$$

where $\Delta(W, S)_{S\setminus\{s_0\}}$ denotes the type-selected subcomplex obtained by deleting all vertices of color s_0 from $\Delta(W, S)$.

Consequently $\Delta(W^+, R)$ is a pure $(n - 1)$-dimensional shellable simplicial complex, which is balanced with color set R.

Similarly for any $J \subseteq R$, its type-selected subcomplex $\Delta(W^+, R)_J$ is W^+-equivariantly isomorphic to the type-selected subcomplex $\Delta(W, S)_{\tau^{-1}(J)}$.

We remark that the isomorphism in this proposition shows that $\Delta(W^+, R)$ has an alternate description similar to description (ii) for $\Delta(W, S)$ above: $\Delta(W^+, R)$ is the unique simplicial complex whose face poset has elements indexed by the collection $\{wW^+_J\}_{w\in W^+,\ J\subseteq R}$ of all cosets of parabolic subgroups, ordered by reverse inclusion.

One might view the fact that $\Delta(W^+, R)$ is isomorphic to a (non-spherical) subcomplex of $\Delta(W, S)$ as providing an analogue of description (iii) for $\Delta(W, S)$. In fact, Proposition 2.4.2 has strong consequences for the topology of this non-spherical subcomplex. Let $\mathbb{Z}[W/W_{S-\{s_0\}}]$ denote the permutation action of W^+ on cosets of the maximal parabolic $W_{S-\{s_0\}}$. In other words,

$$\mathbb{Z}[W/W_{S-\{s_0\}}] = \mathrm{Res}^W_{W^+}\mathrm{Ind}^W_{W_{S-\{s_0\}}}\mathbf{1}.$$

If W is finite, denote by $\mathbb{Z}v$ the unique copy of the trivial representation contained inside $\mathbb{Z}[W/W_{S-\{s_0\}}]$, spanned by the sum v of all cosets $wW_{S-\{s_0\}}$.

Corollary 2.4.3. *The reduced homology $\tilde{H}_*(\Delta(W^+, R), \mathbb{Z})$ is concentrated in dimension $n - 1$, and carries the W^+-representation which is the restriction from W of the representation on the top homology of $\Delta(W, S)_{S\setminus\{s_0\}}$. More concretely,*

$$H_*\big(\Delta\big(W^+, R\big), \mathbb{Z}\big) \cong \begin{cases} \mathbb{Z}[W/W_{S-\{s_0\}}] & \text{when } W \text{ is infinite,} \\ \mathbb{Z}[W/W_{S-\{s_0\}}]/\mathbb{Z}v & \text{when } W \text{ is finite.} \end{cases}$$

Proof. The first assertions follow from Proposition 2.4.2 and the fact that a pure shellable d-dimensional complex has reduced homology concentrated in dimension d.

The more concrete description of the W^+-action is derived as follows. One can always apply *Alexander duality* to the embedding of $\Delta(W,S)_{S-\{s_0\}}$ inside a certain $(|S|-1)$-dimensional sphere $\mathbb{S}^{|S|-1}$; this sphere $\mathbb{S}^{|S|-1}$ is either $\Delta(W,S)$ or its one-point compactification, depending upon whether W is finite or infinite. In both cases, W acts on the top homology $\tilde{H}_{|S|-1}(\mathbb{S}^{|S|-1},\mathbb{Z})=\mathbb{Z}$ of this sphere by the sign character ϵ, giving the following isomorphism of W-representations (cf. [12, Theorem 2.4]):

$$\tilde{H}_{|S\setminus J|-1}\big(\Delta(W,S)_{S\setminus J},\mathbb{Z}\big)\cong\epsilon\otimes\big(\tilde{H}_{|J|-1}\big(\Delta(W,S)_J,\mathbb{Z}\big)\big)^*$$

for any $J\subseteq S$; here U^* denotes the contragradient of a representation U, and when W is infinite, the space $\Delta(W,S)_J$ appearing on the right should be replaced by its disjoint union $\Delta(W,S)_J\cup\{*\}$ with the compactification point $*$ of the sphere.

Taking $J=\{s_0\}$, one obtains a W-representation isomorphism between the homology $\tilde{H}_{|S|-2}(\Delta(W,S)_{S-\{s_0\}},\mathbb{Z})$ and the twist by ϵ of either $\mathbb{Z}[W/W_{S-\{s_0\}}]$ or $\mathbb{Z}[W/W_{S-\{s_0\}}]/\mathbb{Z}v$, depending upon whether W is infinite or finite. Restricting this isomorphism to W^+, the twist by ϵ becomes trivial, and one gets the statement of the corollary. $\square$

Example 2.4.4. Let (W,S) be of type A_3, so that $W=\mathfrak{S}_3$, having Coxeter diagram which is a path with three nodes. If one labels the generators S as

$$S=\{s_0,s_1,s_2\}=\big\{(1,2),(2,3),(3,4)\big\},$$

so that s_0 is a leaf node in the Coxeter diagram, then Fig. 2(a) shows the Coxeter complex $\Delta(W^+,R)$ with facets labeled by W^+. Fig. 2(b) shows the isomorphic type-selected subcomplex $\Delta(W,S)_{S-\{s_0\}}$ with facets labeled by $W^{\{s_0\}}$.

Fig. 2(c) shows the resulting Coxeter complex $\Delta(W^+,R)$ with facets labeled by W^+ after one relabels

$$S=\{s_0,s_1,s_2\}=\big\{(2,3),(1,2),(3,4)\big\},$$

so that now s_0 is the central node, not a leaf, and s_1,s_2 commute.

2.5. *Palindromes versus reflections*

For a Coxeter system (W,S), the set of reflections

$$\mathrm{T}:=\bigcup_{\substack{w\in W\\ s\in S}} wsw^{-1}$$

plays an important role in the theory. A similar role for (W^+,R) is played by the set of *palindromes*, particularly when s_0 is evenly-laced. Palindromes will also give the correct way to define the analogues of the strong Bruhat order defined in Section 2.6 below.

Definition 2.5.1. Given a pair (G,A) where G is a group generated by a set A, say that an element g in G is an *(odd) palindrome* if there is an $(A\cup A^{-1})^*$-word $\mathbf{a}=(a_1,\dots,a_\ell)$ factoring g with ℓ odd such that $a_{\ell+1-i}=a_i$ for all i. Denote the set of (odd) palindromes in G by $\mathrm{P}(G)$.

The set of palindromes for (G,A) is always closed under taking inverses. For a Coxeter system (W,S), since S consists entirely of involutions, the set of palindromes is the same as the set T of reflections.

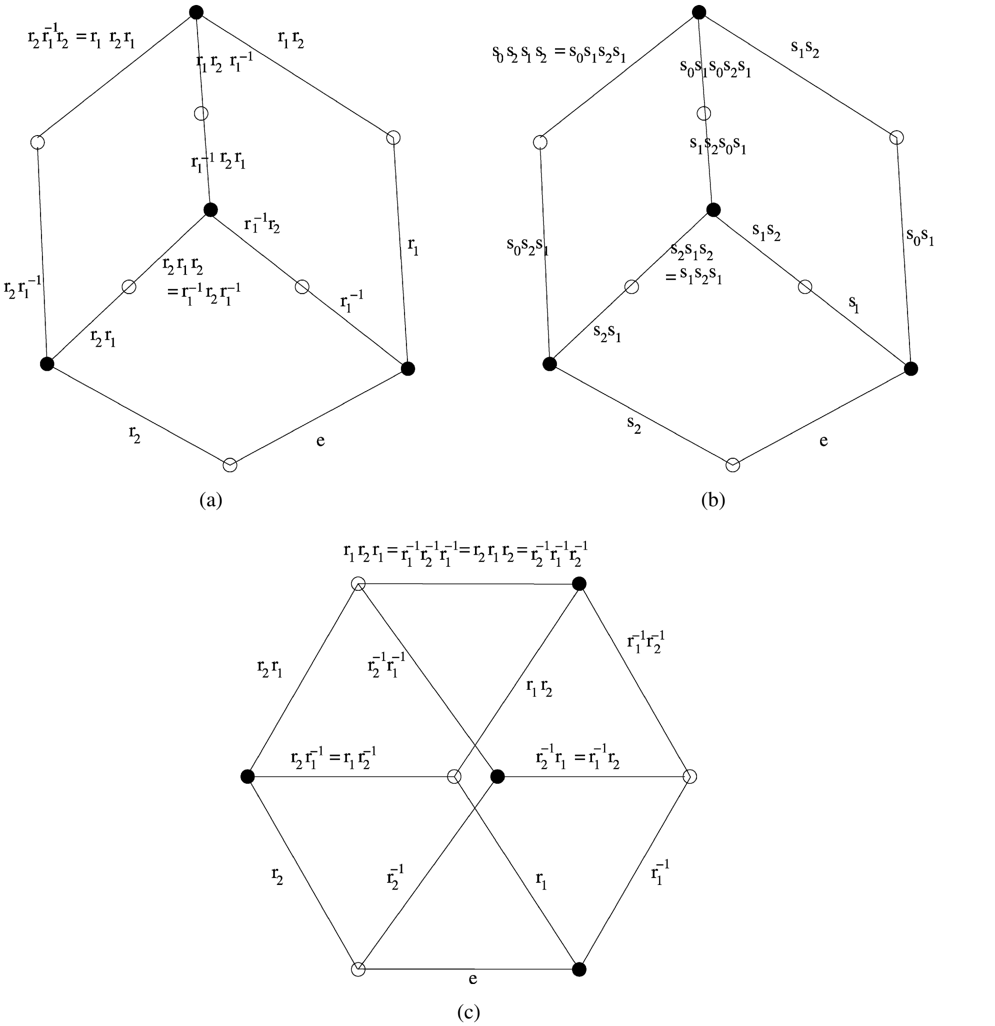


Fig. 2. Coxeter complexes for (W^+, R) with (W, S) of type A_3, and two different choices for the distinguished node s_0. Part (a) shows $\Delta(W^+, R)$ when s_0 is a leaf node, that is, the Coxeter diagram is labeled $s_0 - s_1 - s_2$, while part (b) shows the isomorphic complex $\Delta(W, S)_{S-\{s_0\}}$. Part (c) shows $\Delta(W^+, R)$ when s_0 is the non-leaf node, that is, the Coxeter diagram is labeled $s_1 - s_0 - s_2$.

When s_0 is not evenly-laced in (W, S), the palindromes $\mathtt{P}(W^+)$ can behave unexpectedly, e.g. the identity element e is a palindrome: if m_{01} is odd, one has the odd palindromic expression $e = r_1^{m_{01}}$. See also Example 2.5.6 below.

Nevertheless, one does have in general a very close relation between palindromes for (W^+, R) and palindromes (= reflections) for (W, S). Let

$$\hat{\mathtt{T}} := \bigcup_{\substack{w \in W \\ s \in S \setminus \{s_0\}}} wsw^{-1}$$

denote the set of reflections in W that are conjugate to at least one $s \neq s_0$.

Proposition 2.5.2. *The inclusion* $\hat{\mathtt{T}} \subset \mathtt{T}$ *is proper if and only if* s_0 *is evenly-laced.*

Proof. When s_0 is not evenly laced, say m_{01} is odd, then s_0 is conjugate to s_1 and hence $\hat{\mathtt{T}} = \mathtt{T}$.

When s_0 *is* evenly-laced, the character $\chi_0 : W \to \{\pm 1\}$ taking value -1 on s_0 and $+1$ on $s_1, \dots, s_n$ shows that s_0 is not conjugate to any of $s_1, \dots, s_n$, and hence the inclusion $\hat{\mathtt{T}} \subsetneq \mathtt{T}$ is proper. □

Proposition 2.5.3. *For any Coxeter system* (W, S)*, one has*

$$\mathtt{P}(W^+)s_0 = \hat{\mathtt{T}} = s_0\mathtt{P}(W^+).$$

In other words, an element $w \in W^+$ *is a palindrome with respect to* R *if and only if* ws_0 *(or equivalently* $s_0 w$*) is a reflection lying in the subset* $\hat{\mathtt{T}}$*, and vice versa.*

Proof. Since $\mathtt{P}(W^+) = \mathtt{P}(W^+)^{-1}$, it suffices to show the first equality.

Assume $w \in W^+$ is a palindrome, say $w = r^{(1)} \cdots r^{(k-1)} r^{(k)} r^{(k-1)} \cdots r^{(1)}$ with each $r^{(i)} \in R \cup R^{-1}$. Then

$$\begin{aligned} ws_0 &= r^{(1)} \cdots r^{(k-1)} r^{(k)} r^{(k-1)} \cdots r^{(1)} s_0 = r^{(1)} \cdots r^{(k-1)} r^{(k)} s_0 \left(r^{(k-1)}\right)^{-1} \cdots \left(r^{(1)}\right)^{-1} \\ &= u r^{(k)} s_0 u^{-1} \end{aligned}$$

for $u := r^{(1)} \cdots r^{(k-1)}$, and where we have used the fact that $rs_0 = s_0 r^{-1}$ for any $r \in R \cup R^{-1}$. Since $r^{(k)} s_0$ is either $s_0 s_i s_0$ or $s_i s_0 s_0 = s_i$ for some $i = 1, 2, \dots, n$, one concludes that ws_0 lies in $\hat{\mathtt{T}}$.

Conversely, given $ws_i w^{-1}$ in $\hat{\mathtt{T}}$, write any S^*-word $\mathbf{s}$ for w. Its reverse $\mathbf{s}^{rev}$ is a word for w^{-1}, and $(\mathbf{s}, s_i, \mathbf{s}^{rev}, s_0)$ is a word for $ws_i w^{-1} s_0$. Applying the map from Proposition 2.2.4 to this word yields an $(R \cup R^{-1})^*$ word $\mathbf{r}$ for $ws_i w^{-1} s_0$, which will be palindromic because there is an odd distance in the word $(\mathbf{s}, s_i, \mathbf{s}^{rev}, s_0)$ between any two corresponding occurrences of s_j for $j = 1, 2, \dots, n$. □

Definition 2.5.4. Given $w \in W$, recall that its set of *left-shortening reflections* is

$$\mathtt{T}_L(w) := \{t \in \mathtt{T}\colon \ell_S(tw) < \ell_S(w)\}.$$

Given $w \in W^+$, define its set of *left-shortening palindromes* by

$$\mathtt{P}_L(w) := \{p \in \mathtt{P}(W^+)\colon \ell_{R \cup R^{-1}}(pw) < \ell_{R \cup R^{-1}}(w)\}.$$

In a Coxeter system (W, S), it is well known ([5, Chapter 1], [8, §5.8]) that for any w in W, the set $\mathtt{T}_L(w)$ enjoys these properties:

(a) $\ell_S(w) = |\mathtt{T}_L(w)|$.
(b) (*Strong exchange property*) For any $t \in \mathtt{T}$, and any reduced S^*-word $\mathbf{s} = (s^{(1)}, \dots, s^{(\ell)})$ for w, the following are equivalent:
  (i) $t \in \mathtt{T}_L(w)$, that is, $\ell_S(tw) < \ell_S(w)$.
  (ii) $t = t_k := s^{(1)} \cdots s^{(k-1)} s^{(k)} s^{(k-1)} \cdots s^{1)}$ for some k.
  (iii) $tw = s^{(1)} \cdots s^{(k-1)} s^{(k+1)} \cdots s^{(\ell)}$ for some k.
  In other words, $\mathtt{T}_L(w) = \{t_1, \dots, t_\ell\}$.
(c) The set $\mathtt{T}_L(w)$ determines w uniquely.

Analogously, given a reduced $(R \cup R^{-1})^*$-word $\mathbf{r} = (r^{(1)}, \ldots, r^{(\nu(w))})$ that factors w in W^+, one can define for $k = 1, 2, \ldots, \nu(w)$ the palindromes

$$p_k := \left(r^{(1)}\right)^{-1}\left(r^{(2)}\right)^{-1} \cdots \left(r^{(k)}\right)^{-1} \cdots \left(r^{(2)}\right)^{-1}\left(r^{(1)}\right)^{-1}.$$

One can relate this to $\mathtt{P}_L(w)$ and to $T_L(w)$ in general; define for $w \in W$ the set

$$\hat{\mathtt{T}}_L(w) := \mathtt{T}_L(w) \cap \hat{\mathtt{T}}.$$

Proposition 2.5.5. *For any choice of distinguished generator s_0, and for any $w \in W^+$, with the above notation one has inclusions*

$$\{p_1, \ldots, p_{\nu(w)}\} \subseteq \mathtt{P}_L(w) \subseteq \hat{\mathtt{T}}_L(s_0 w)s_0. \tag{4}$$

When s_0 is evenly-laced, both inclusions are equalities:

$$\{p_1, \ldots, p_{\nu(w)}\} = \mathtt{P}_L(w) = \hat{\mathtt{T}}_L(s_0 w)s_0.$$

Proof. The first inclusion in (4) is straightforward, as one calculates

$$p_k w = \left(r^{(1)}\right)^{-1}\left(r^{(2)}\right)^{-1} \cdots \left(r^{(k-1)}\right)^{-1} r^{(k+1)} r^{(k+2)} \cdots r^{(\nu(w))}$$

and hence $\ell_{R\cup R^{-1}}(p_k w) < \nu(w) = \ell_{R\cup R^{-1}}(w)$.

For the second inclusion in (4), given a palindrome $p \in \mathtt{P}(W^+)$, we know from Proposition 2.5.3 that $t := ps_0$ is a reflection in $\hat{\mathtt{T}}$, and conversely any reflection t in $\hat{\mathtt{T}}$ will have $p := ts_0$ a palindrome in $\mathtt{P}(W^+)$. Thus it remains to show that

$$\ell_{R\cup R^{-1}}(pw) < \ell_{R\cup R^{-1}}(w) \quad \text{implies} \quad \ell_S(ts_0 w) < \ell_S(s_0 w).$$

Using $\ell_{R\cup R^{-1}} = \nu$, along with the fact that $\nu(s_0 w) = \nu(w)$ by Proposition 2.3.6, and setting $w' := s_0 w$, one can rewrite this desired implication as

$$\nu(tw') < \nu(w') \quad \text{implies} \quad \ell_S(tw') < \ell_S(w'). \tag{5}$$

We show the contrapositive: if $\ell_S(tw') \geqslant \ell_S(w')$ then tw' is greater than w' in the Bruhat order on W, and hence $\nu(tw') \geqslant \nu(w')$ by Proposition 2.3.9.

For the assertions of equality, assuming s_0 is evenly-laced, it suffices to show that the two sets $\{p_1, \ldots, p_{\nu(w)}\}$ and $\hat{\mathtt{T}}_L(s_0 w)s_0$ both have the same cardinality, namely $\nu(w)$.

For the first set, it suffices to show that $p_i \neq p_j$ for $1 \leqslant i < j \leqslant \nu(w)$. Supposing $p_i = p_j$ for the sake of contradiction, one has

$$w = p_i^{-1} p_j w = r^{(1)} \cdots r^{(i-1)}\left(r^{(i+1)}\right)^{-1} \cdots \left(r^{(j-1)}\right)^{-1} r^{(j+1)} \cdots r^{(\nu(w))} \tag{6}$$

which gives the contradiction that $\ell_{R\cup R^{-1}}(w) < \nu(w)$.

For the second set, let $\ell := \ell_S(s_0 w)$ and choose a reduced S^*-word $\mathbf{s} = (s^{(1)}, \ldots, s^{(\ell)})$ that factors $s_0 w$. Defining

$$t_k = t^{(1)} t^{(2)} \cdots t^{(k)} \cdots t^{(2)} t^{(1)}$$

for $1 \leqslant k \leqslant \ell$, one has [5, Corollary 1.4.4], [8, §5.8] that the t_k are all distinct, and $\mathtt{T}_L(s_0 w) := \{t_k\}_{1\leqslant k\leqslant \nu(w)}$. Since $\nu(w) = \nu(s_0 w)$, there will be exactly $\nu(w)$ indices $\{i_1, \ldots, i_{\nu(w)}\}$ for which $s^{(i_j)} \neq s_0$. As $t_k \in \hat{\mathtt{T}}$ if and only if $s^{(k)} \neq s_0$ (due to s_0 being evenly-laced), this means

$$\left|\hat{\mathtt{T}}_L(s_0 w)s_0\right| = \left|\hat{\mathtt{T}}_L(s_0 w)\right| = \left|\mathtt{T}_L(s_0 w) \cap \hat{\mathtt{T}}\right| = \left|\{t_{i_1}, \ldots, t_{i_{\nu(w)}}\}\right| = \nu(w). \qquad \square$$

Example 2.5.6. When (W, S) is the dihedral Coxeter system $I_2(m)$ in which $S = \{s_0, s_1\}$ with $m := m_{01}$, then (W^+, R) is simply the cyclic group of order m. If one chooses m to be odd, then every element $w \in W^+$ is a palindrome, i.e. $\mathtt{P}(W^+) = W^+$, and one has

$$\mathtt{P}(W^+)s_0 = W^+s_0 = \hat{\mathtt{T}} = \mathtt{T}.$$

Furthermore, if one picks m odd and sufficiently large, it illustrates the potential bad behavior of palindromes when s_0 is not evenly-laced. For example, in this situation, $w = r_1^{-1}r_1^{-1}$ will have both inclusions strict in (4):

$$\begin{array}{ccccc} \{p_1, \ldots, p_{\nu(w)}\} & \subsetneq & \mathtt{P}_L(w) & \subsetneq & \hat{\mathtt{T}}_L(s_0 w)s_0 \\ \| & & \| & & \| \\ \{r_1, r_1 r_1\} & & \{r_1, r_1 r_1, r_1 r_1 r_1\} & & \{e, r_1, r_1 r_1, r_1 r_1 r_1, r_1 r_1 r_1 r_1\}. \end{array}$$

This dihedral example also shows why replacing the set $\mathtt{P}(W^+)$ of palindromes for (W^+, R) with the set of *conjugates of* $R \cup R^{-1}$

$$\bigcup_{\substack{w \in W^+ \\ r \in R \cup R^{-1}}} wrw^{-1} \tag{7}$$

would be the wrong thing to do: in this example, W^+ is cyclic and hence abelian, so that this set of conjugates in (7) is no larger than $R \cup R^{-1} = \{r_1, r_1^{-1}\}$ itself!

Example 2.5.6 shows that the analogues for palindromes in (W^+, R) of the properties $\ell_S(w) = |\mathtt{T}_L(w)|$ and the strong exchange property for reflections in (W, S) can fail when s_0 is not evenly-laced. They do hold under the evenly-laced assumption—see Theorem 3.5.1 below, which furthermore asserts that the set $\mathtt{P}_L(w)$ determines $w \in W^+$ uniquely when s_0 is evenly-laced. This raises the following question.

Question 2.5.7. *When s_0 is chosen arbitrarily, does $\mathtt{P}_L(w)$ determine $w \in W^+$ uniquely?*

2.6. *Weak and strong orders*

For a Coxeter system (W, S) there are two related partial orders (the weak and strong Bruhat orders) on W which form graded posets with rank function ℓ_S. Here we define analogues for (W^+, R).

Definition 2.6.1. Define the (*left*) *strong order* $\leqslant_{LS}$ on W^+ as the reflexive and transitive closure of the relation $w \xrightarrow{p} pw$ if $p \in \mathtt{P}(W^+)$ and $\ell_{R\cup R^{-1}}(w) < \ell_{R\cup R^{-1}}(pw)$. Similarly define the (*right*) *strong order* $\leqslant_{RS}$.

Define the (*left*) *weak order* $\leqslant_{LW}$ on W^+ as the reflexive and transitive closure of the relation $w <_{LW} rw$ if $r \in R \cup R^{-1}$ and $\ell_{R\cup R^{-1}}(w) + 1 = \ell_{R\cup R^{-1}}(rw)$. Similarly define the (*right*) *weak order* $\leqslant_{RW}$.

Several things should be fairly clear from these definitions:

(i) Because these are reflexive transitive binary relations on W^+ that are weaker than the partial ordering by the length function $\ell_{R\cup R^{-1}}$, they are actually partial orders on the set W^+. In other words, taking the transitive closure creates no directed cycles.

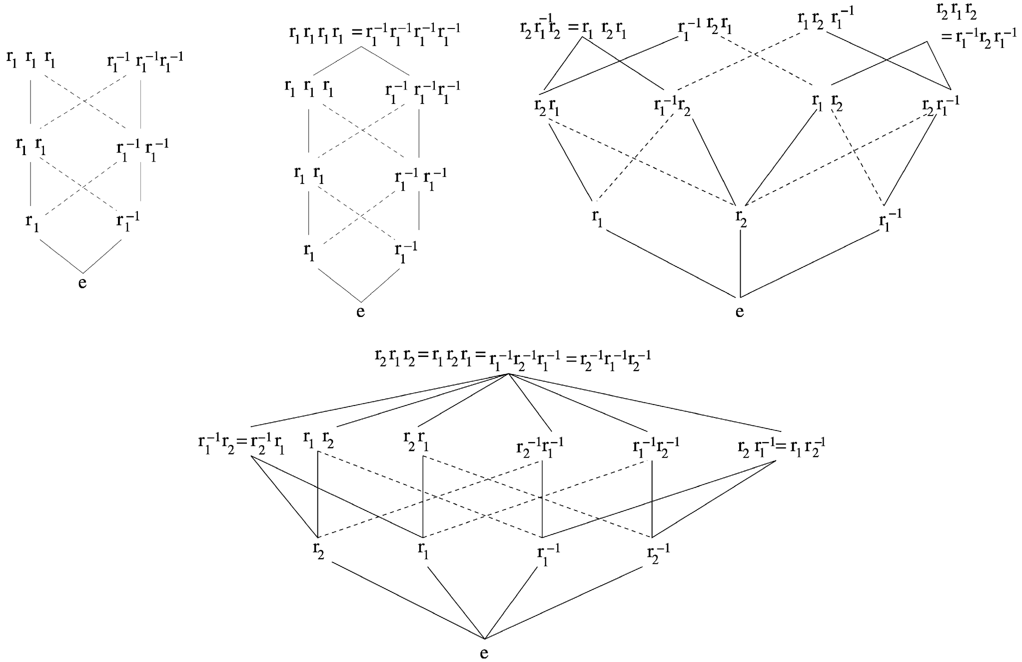


Fig. 3. Examples of the left weak (solid edges) and left strong orders (solid and dotted edges) on W^+ for $(W, S) = I_2(7)$, $I_2(8)$, and A_3 with s_0 labeling a leaf node versus a non-leaf node.

(ii) Because the map $w \mapsto w^{-1}$ preserves the set of palindromes $\mathtt{P}(W^+)$ and the length function $\ell_{R \cup R^{-1}}$, it induces an isomorphism between the left and right versions of the two orders.
(iii) The identity $e \in W^+$ is the unique minimum element in all of these orders.
(iv) The (left, right, respectively) strong order is stronger than the (left, right, respectively) weak order, meaning that if $x \leqslant y$ in the weak version of the order then $x \leqslant y$ in the strong version (or the strong version of the order will have, in general, *more* edges in its Hasse diagram).
(v) For every $u, v \in W^+$, $v \leqslant_{RW} u$ implies $P_L(v) \subseteq P_L(u)$.

Question 2.6.2. *Does the inclusion $P_L(v) \subseteq P_L(u)$ imply $v \leqslant_{RW} u$?*

Fig. 3 shows the left weak and left strong orders on W^+ for the two dihedral Coxeter systems $I_2(7)$, $I_2(8)$, as well as for type A_3 with the two different choices for the node labeled s_0, as in Fig. 2.

The usual weak and strong orders on a Coxeter group W have several good properties (see [5, Chapters 2,3]): they are all *graded* by the length function ℓ_S, the left and right weak orders are both *meet semilattices*, and the strong order is *shellable*. A glance at Fig. 3 then raises several obvious questions about the analogous orders on W^+.

Question 2.6.3. *Are all of these orders graded by the function $\ell_{R \cup R^{-1}}$, that is, do all maximal chains have the same length?*

Question 2.6.4. *Do the weak orders form a meet semilattice in general?*

Question 2.6.5. *Is the strong order shellable?*

We will see in Section 3.6 that the answers to all of these questions are affirmative when s_0 is evenly-laced. Furthermore, in Section 5 it will be shown that when s_0 is an evenly-laced leaf node, the strong and weak orders coincide with the usual Coxeter group strong and weak orders for the related Coxeter system (W', S') defined there.

Remark 2.6.6. Some things are clearly *not* true of the various orders, even in the best possible situation where s_0 is an even leaf.

Although the left weak/strong orders are isomorphic to the right weak/strong orders, they are not the *same* orders. For example, when (W, S) is of type B_3 with s_0 the even leaf one can check that r_1 is below r_2r_1 in both the left weak and left strong orders, but this fails in both the right weak and right strong orders.

None of the four orders (left/right weak/strong) on W^+ coincides with the restriction from W to W^+ of the analogous left/right weak/strong order on W. For example, suppose that (W, S) has W finite with an odd number $|T|$ of reflections, and s_0 is evenly-laced (this occurs in type B_n for n odd). Then there will be a maximum element, namely $\tau(w_0) = w_0s_0 = s_0w_0$, for all four orders on W^+, where here w_0 is the longest element in W; see Proposition 3.6.6 below. But $\tau(w_0)$ will *not* be a maximum element when one restricts any of the left/right weak or strong orders from W to W^+: the elements w_0s_j for $j \geqslant 1$ will also lie in W^+, and have the same length

$$\ell_S(w_0s_j) = |T| - 1 = \ell_S\bigl(\tau(w_0)\bigr)$$

and hence will be incomparable to $\tau(w_0)$.

Similarly, none of the four orders on W^+ coincides, via the bijection $\tau : W^+ \to W^{\{s_0\}}$, to the restriction from W to $W^{\{s_0\}}$ of the analogous order on W. This can be seen already for (W, S) of type $I_2(4) = B_2$, where all four orders on W^+ are isomorphic to a rank two Boolean lattice, while the various strong/weak orders restricted from W to $W^{\{s_0\}}$ turn out either to be total orders or non-lattices.

3. The case of an evenly-laced node

When the distinguished generator s_0 in $S = \{s_0, s_1, \ldots, s_n\}$ for the Coxeter system (W, S) has the extra property that m_{0i} is even for $i = 1, 2, \ldots, n$, we say that s_0 is an *evenly-laced node* of the Coxeter diagram. This has many good consequences for the presentation (W^+, R) explored in the next few subsections. Note that there do exist evenly-laced nodes for the irreducible finite Coxeter systems of types $B_n(= C_n)$, $I_2(m)$ with m even, and for the affine Coxeter systems of types $\tilde{A}_1(= I_2(\infty))$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{G}_2$.

3.1. Length revisited

Definition 3.1.1. Part of Tits' solution to the word problem for the Coxeter system (W, S) asserts [5, §3.3] that one can connect any two reduced S^*-words for w in W by a sequence of *braid moves* of the form

$$\underbrace{s_is_js_is_j\cdots}_{m_{ij}\text{ letters}} = \underbrace{s_js_is_js_i\cdots}_{m_{ij}\text{ letters}}. \tag{8}$$

When s_0 is evenly-laced, there will always be the same number of occurrences of s_0 on either side of (8), and hence the number of occurrences of s_0 in *any* reduced word is the same; denote this quantity $\ell_0(w)$.

The Coxeter presentation for (W, S) also allows one to define, when s_0 is evenly-laced, a homomorphism

$$\begin{aligned} &\chi_0 : W \to \{\pm 1\},\\ &\chi_0(s_0) = -1, \\ &\chi_0(s_j) = +1 \quad \text{for } j = 1, 2, \ldots, n. \end{aligned} \tag{9}$$

Note that $\chi_0(w) = (-1)^{\ell_0(w)}$.

Recalling that $\nu(w)$ was defined to be the minimum number of $s_j \neq s_0$ occurring in an S^*-word that factors w, one immediately concludes the following reinterpretation for the length function of (W^+, R).

Proposition 3.1.2. *Assume s_0 is evenly-laced. Then for every $w \in W$ one has*

$$\nu(w) = \ell_S(w) - \ell_0(w).$$

Consequently, for any $w \in W^+$, the length function $\ell_{R \cup R^{-1}}(w) (= \nu(w))$ can be computed from any reduced S^*-word for w.

3.2. Length generating function

When s_0 is evenly-laced, the simpler interpretation for the length function $\ell_{R \cup R^{-1}}$ allows one to compute its generating function for (W^+, R), by relating it to known variations on the usual Coxeter group length generating function for (W, S).

The usual diagram-recursion methods [8, §5.12] for writing down the Poincaré series

$$W(S; q) := \sum_{w \in W} q^{\ell_S(w)}$$

as a rational function in q turn out to generalize straightforwardly [9,11], allowing one to write down the finer Poincaré series

$$W(S; q_0, q) := \sum_{w \in W} q_0^{\ell_0(w)} q^{\nu(w)}.$$

This power series in q_0, q will actually end up being a rational function of q_0, q for any Coxeter system (W, S) with s_0 evenly-laced. The key point is that in the unique factorization

$$W = W^J \cdot W_J,$$

both statistics $\ell_0(w)$, $\nu(w)$ behave additively (see [8, §5.12] or [9,11]), yielding the factorization

$$W(S; q_0, q) = W^J(S; q_0, q) \cdot W_J(S; q_0, q).$$

Here we are using the notation for any subset $A \subset W$ that

$$A(S; q_0, q) := \sum_{w \in A} q_0^{\ell_0(w)} q^{\nu(w)}.$$

Definition 3.2.1. Define the $\ell_{R\cup R^{-1}}$ length generating function on W^+:

$$W^+(R\cup R^{-1};q):=\sum_{w\in W^+} q^{\ell_{R\cup R^{-1}}(w)}.$$

Corollary 3.2.2. *When s_0 is evenly-laced,*

$$W^+(R\cup R^{-1};q)=\bigl[W^{\{s_0\}}(S;q_0,q)\bigr]_{q_0=1}=\left[\frac{W(S;q_0,q)}{1+q_0}\right]_{q_0=1}.$$

Proof. Since the map $\tau: W^{\{s_0\}}\to W^+$ is a bijection by Proposition 2.3.5, and since $\ell_{R\cup R^{-1}}(\tau(w))=\nu(w)$ by Proposition 2.2.3, one has

$$\begin{aligned} W^+(R\cup R^{-1};q)&=\bigl[W^{\{s_0\}}(S;q_0,q)\bigr]_{q_0=1}=\left[\frac{W(S;q_0,q)}{W_{\{s_0\}}(S;q_0,q)}\right]_{q_0=1}\\ &=\left[\frac{W(S;q_0,q)}{1+q_0}\right]_{q_0=1}. \qquad \square \end{aligned}$$

Example 3.2.3. Let (W,S) be the Coxeter system of type $B_n(=C_n)$, so that W is the group of *signed permutations* acting on $\mathbb{R}^n$. Index $S=\{s_0,s_1,\dots,s_{n-1}\}$ so that s_0 is the special generator that negates the first coordinate, and s_i swaps the ith, $(i+1)$st coordinates when $i\geqslant 1$. The Coxeter presentation has

$$\begin{aligned} &m_{01}=4,\\ &m_{i,i+1}=3 \quad \text{for } i=1,2,\dots,n-1,\\ &m_{ij}=2 \quad \text{for } |i-j|\geqslant 2. \end{aligned}$$

It is well known (see [7,9,11]) and not hard to check that

$$W(S;q_0,q)=(-q_0;q)_n[n]!_q$$

where

$$\begin{aligned} &(x;q)_n:=(1-x)(1-xq)\bigl(1-xq^2\bigr)\cdots\bigl(1-xq^{n-1}\bigr),\\ &[n]!_q:=\frac{(q;q)_n}{(1-q)^n}=[n]_q[n-1]_q\cdots[2]_q[1]_q,\\ &[n]_q:=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-1}. \end{aligned}$$

Consequently, Corollary 3.2.2 implies

$$\begin{aligned} W^+(R\cup R^{-1};q)&=\left[\frac{(-q_0;q)_n[n]!_q}{1+q_0}\right]_{q_0=1}=(-q;q)_{n-1}[n]!_q=[n]_q\prod_{j=1}^{n-1}\bigl(1+q^j\bigr)[j]_q\\ &=[n]_q\prod_{j=1}^{n-1}[2j]_q. \end{aligned}$$

The same formula will be derived differently in Example 5.2.6.

3.3. Parabolic coset representatives revisited

Recall that for any subset $J \subseteq S$ with $s_0 \in J$, the map τ sends the distinguished minimum ℓ_S-length coset representatives W^J for W/W_J to a collection $\tau(W^J)$ of coset representatives for $W^+/W_{\tau(J)}$, each of which achieves the minimum $\ell_{R\cup R^{-1}}$-length in its coset. Thus for every $w \in W^+$ one has a unique factorization

$$w = \tau(x)y \tag{10}$$

with $x \in W^J$ and $y \in W^+_{\tau(J)}$ unique. One can make a stronger assertion when s_0 is evenly-laced.

Proposition 3.3.1. *Assume s_0 is evenly-laced. Then in the unique factorization* (10) *one has additivity of lengths*:

$$\ell_{R\cup R^{-1}}(w) = \ell_{R\cup R^{-1}}\big(\tau(x)\big) + \ell_{R\cup R^{-1}}(y).$$

Proof. Since elements $w \in W^+$ have $\ell_{R\cup R^{-1}}(w) = \nu(w)$, one must show that in the factorization (10), one has

$$\nu(w) = \nu\big(\tau(x)\big) + \nu(y). \tag{11}$$

Because s_0 is evenly-laced, Definition 3.1.1 and Proposition 3.1.2 imply that in the usual length-additive parabolic factorization for $w \in W$ as $w = w^J w_J$ with $w^J \in W^J$, $w_J \in W_J$, one has additivity of ν:

$$\nu(w) = \nu\big(w^J\big) + \nu(w_J). \tag{12}$$

Note that (10) implies that the usual parabolic factorization $w = w^J \cdot w_J$ in W must either take the form $w = x \cdot y$ (if $\tau(x) = x$) or the form $w = x \cdot s_0 y$ (if $\tau(x) = xs_0$). In either case, the desired additivity (11) follows from (12), using Proposition 2.3.6. $\square$

This immediately implies the following.

Corollary 3.3.2. *When s_0 is evenly-laced, the coset representatives $\tau(W^J)$ for $W^+/W^+_{\tau(J)}$ can be distinguished intrinsically in any of the following ways*:

(i) $\tau(W^J)$ *are the unique representatives within each coset $wW^+_{\tau(J)}$ achieving the minimum $\ell_{R\cup R^{-1}}$-length.*

(ii) $$\tau\big(W^J\big) := \big\{x \in W^+\colon\ \ell_{R\cup R^{-1}}(xy) > \ell_{R\cup R^{-1}}(x) \text{ for all } y \in W^+_{\tau(J)}\big\}.$$

(iii) $$\tau\big(W^J\big) := \big\{x \in W^+\colon\ \ell_{R\cup R^{-1}}(xr) > \ell_{R\cup R^{-1}}(x) \text{ for all } r \in \tau(J) \cup \tau(J)^{-1}\big\}.$$

One also has the following immediate corollary, giving a factorization for the $\ell_{R\cup R^{-1}}$ generating function. Define the notation for any subset $A \subset W^+$ that

$$A\big(R \cup R^{-1}; q\big) := \sum_{w\in A} q^{\ell_{R\cup R^{-1}}(w)}.$$

Corollary 3.3.3. *For every subset $J \subseteq R$*

$$W^+\big(R \cup R^{-1}; q\big) = W^{+J}\big(R \cup R^{-1}; q\big) \cdot W^+_J\big(R \cup R^{-1}; q\big).$$

Note that the factorization in Corollary 3.3.3 fails in general when s_0 is not evenly-laced. For example, in the case of type A_{n-1} where $W = \mathfrak{S}_n$ and s_0 is a leaf node of the Coxeter diagram, $W^+(R \cup R^{-1}; q)$ was given explicitly earlier in factored form as (3), but is not divisible by $W^+_{\{r_i\}}(R \cup R^{-1}; q) = 1 + q$ for any of the generators r_i with $i > 1$. See also Example 3.4.2 below.

3.4. *Descent statistics*

For a Coxeter system (W, S), aside from the length statistic $\ell_S(w)$ for $w \in W$, one often considers the *descent set* and *descent number* of w defined by

$$\begin{aligned} \mathrm{Des}_S(w) &:= \bigl\{s \in S\colon\ \ell_S(ws) < \ell_S(w)\bigr\} \subseteq S, \\ \mathrm{des}_S(w) &:= \bigl|\mathrm{Des}_S(w)\bigr|. \end{aligned}$$

Generating functions counting W jointly by ℓ_S and $\mathrm{Des}_S(w)$ are discussed in [11].

When (W, S) is arbitrary, for the alternating group W^+ and its generating set R there are several reasonable versions of the descent set one might consider.

Definition 3.4.1. Given $w \in W^+$, define its *descent set* $\mathrm{Des}_{R\cup R^{-1}}(w)$, *symmetrized descent set* $\widehat{\mathrm{Des}}_R(w)$, *weak descent set* (or *non-ascent set*) $\mathrm{Nasc}_{R\cup R^{-1}}(w)$ and its *symmetrized weak descent set* $\widehat{\mathrm{Nasc}}_R(w)$ as follows:

$$\begin{aligned} \mathrm{Des}_{R\cup R^{-1}}(w) &:= \bigl\{r \in R \cup R^{-1}\colon\ \ell_{R\cup R^{-1}}(wr) < \ell_{R\cup R^{-1}}(w)\bigr\} \subseteq R \cup R^{-1}, \\ \widehat{\mathrm{Des}}_R(w) &:= \bigl\{r \in R\colon \text{ either } r \text{ or } r^{-1} \in \mathrm{Des}_{R\cup R^{-1}}(w)\bigr\} \subseteq R, \\ \mathrm{Nasc}_{R\cup R^{-1}}(w) &:= \bigl\{r \in R \cup R^{-1}\colon\ \ell_{R\cup R^{-1}}(wr) \leqslant \ell_{R\cup R^{-1}}(w)\bigr\} \subseteq R \cup R^{-1}, \\ \widehat{\mathrm{Nasc}}_R(w) &:= \bigl\{r \in R\colon \text{ either } r \text{ or } r^{-1} \in \mathrm{Nasc}_{R\cup R^{-1}}(w)\bigr\} \subseteq R. \end{aligned}$$

Part of the justification for considering weak descents comes from the type A_{n-1} example where $W = \mathfrak{S}_n$: in [10, Theorem 1.10(2)], it was shown that the resulting major index (i.e., the sum of the indices of the weak descents) is equidistributed with the length $\ell_{R\cup R^{-1}}$.

Note that one did not have to worry about *weak* descents for (W, S) because the existence of the sign character shows that one always has $\ell_S(ws) \neq \ell_S(w)$ for any $s \in S$. This can fail for (W^+, R) and the length function $\ell_{R\cup R^{-1}}$ in general.

Example 3.4.2. Continuing Example 2.5.6, let (W, S) be the dihedral Coxeter system $I_2(m)$ with $m = 2k + 1$. Then the two elements r_1^k, r_1^{-k} both achieve the maximum $\ell_{R\cup R^{-1}}$-length value of k, but differ by multiplication on the right by elements of $R \cup R^{-1}$:

$$\begin{aligned} r_1^k \cdot r_1 &= r_1^{-k}, \\ r_1^{-k} \cdot r_1^{-1} &= r_1^k. \end{aligned}$$

Note that this also illustrates the failure of both Proposition 3.3.1 and Corollary 3.3.2 without the assumption that s_0 is evenly-laced: they fail on the coset $r_1^k W^+_{\tau(J)} = r_1^{-k} W^+_{\tau(J)}$, where $J = \{s_0, s_1\}$ and $\tau(J) = \{r_1\}$.

When s_0 is an evenly-laced node, restricting the character χ_0 to W^+ one has

$$\chi_0(w) = (-1)^{\nu(w)} = (-1)^{\ell_{R\cup R^{-1}}(w)}.$$

This shows that for any $r \in R$ one has $\ell_{R \cup R^{-1}}(wr) \neq \ell_{R \cup R^{-1}}(w)$, and hence, in this case, weak descents are the same as descents:

$$\mathrm{Nasc}_{R \cup R^{-1}}(w) = \mathrm{Des}_{R \cup R^{-1}}(w) = \bigl\{r \in R \cup R^{-1}\colon \ell_{R \cup R^{-1}}(wr) < \ell_{R \cup R^{-1}}(w)\bigr\},$$
$$\widehat{\mathrm{Nasc}}_R(w) = \widehat{\mathrm{Des}}_R(w) = \bigl\{r \in R\colon \text{either } r \text{ or } r^{-1} \in \mathrm{Des}_{R \cup R^{-1}}(w)\bigr\}.$$

Note also that the set $\mathrm{Nasc}_{R \cup R^{-1}}(w)$ completely determines the set $\widehat{\mathrm{Nasc}}_R(w)$, and hence is finer information about w. It would be nice to have generating functions counting W^+ jointly by $\ell_{R \cup R^{-1}}$ and either $\mathrm{Nasc}_{R \cup R^{-1}}$ or $\widehat{\mathrm{Nasc}}_R$. These seem hard to produce in general. However, when s_0 is evenly-laced, we next show how to produce such a generating function for the pair $(\ell_{R \cup R^{-1}}, \widehat{\mathrm{Nasc}}_R)$. In Section 5, we will do the same for the finer information $(\ell_{R \cup R^{-1}}, \mathrm{Nasc}_{R \cup R^{-1}})$ under the stronger hypothesis that s_0 is an evenly-laced *leaf*.

It turns out that non-ascents in (W^+, R) relate to descents in (W, S) of the minimum length parabolic coset representatives $W^{\{s_0\}}$ for $W/W_{\{s_0\}}$. This is mediated by the inverse τ^{-1} to the bijection $\tau : W^{\{s_0\}} \to W^+$ that comes from taking $J = \{s_0\}$ in Proposition 2.3.5.

Our starting point is a relation for general (W, S) between $\widehat{\mathrm{Nasc}}_R$ on W^+ and Des_S on $W^{\{s_0\}}$. For the purpose of comparing subsets of $R = \{r_1, \ldots, r_n\}$ and $S \setminus \{s_0\} = \{s_1, \ldots, s_n\}$, identify both of these sets of generators with their subscripts $[n] := \{1, 2, \ldots, n\}$.

Proposition 3.4.3. *After the above identification of subscripts, for any Coxeter system (W, S) and $s_0 \in S$ and $w \in W^+$, one has a (possibly proper) inclusion*

$$\widehat{\mathrm{Nasc}}_R(w) \supseteq \mathrm{Des}_S\bigl(\tau^{-1}(w)\bigr). \tag{13}$$

When s_0 is evenly-laced, this inclusion becomes an equality:

$$\bigl(\mathrm{Des}_R(w) =\bigr)\widehat{\mathrm{Nasc}}_R(w) = \mathrm{Des}_S\bigl(\tau^{-1}(w)\bigr). \tag{14}$$

Proof. To show the inclusion, given $w \in W^+$, assume $s_j \in \mathrm{Des}_S(\tau^{-1}(w))$, and one must show that $r_j \in \widehat{\mathrm{Nasc}}_R(w)$ (note that $j \neq 0$ since $\tau^{-1}(w) \in W^{\{s_0\}}$). Since $\ell_S(\tau^{-1}(w)s_j) < \ell_S(\tau^{-1}(w))$, by Proposition 2.3.9(i) one has

$$\nu\bigl(\tau^{-1}(w)s_j\bigr) \leqslant \nu\bigl(\tau^{-1}(w)\bigr).$$

If $\tau^{-1}(w) = w$ then this gives

$$\ell_{R \cup R^{-1}}\bigl(wr_j^{-1}\bigr) = \nu(ws_js_0) = \nu(ws_j) \leqslant \nu(w) = \ell_{R \cup R^{-1}}(w)$$

using Proposition 2.3.6. If $\tau^{-1}(w) = ws_0$ then

$$\ell_{R \cup R^{-1}}(wr_j) = \nu(ws_0s_j) \leqslant \nu(ws_0) = \nu(w) = \ell_{R \cup R^{-1}}(w)$$

again using Proposition 2.3.6. Either way, one has $r_j \in \widehat{\mathrm{Nasc}}(w)$.

Now assume s_0 is evenly-laced, and $r_j \in \widehat{\mathrm{Nasc}}_R(w)(= \widehat{\mathrm{Des}}_R(w))$. One must show that $s_j \in \mathrm{Des}_S(\tau^{-1}(w))$. Consider these cases:

Case 1. $r_j \in \mathrm{Des}_{R \cup R^{-1}}(w)$. Then

$$\nu(ws_0s_j) = \ell_{R \cup R^{-1}}(wr_j) < \ell_{R \cup R^{-1}}(w) = \nu(w) = \nu(ws_0),$$

which forces $\ell_S(ws_0s_j) < \ell_S(ws_0)$ by Proposition 2.3.9(i). Thus $s_j \in \mathrm{Des}_S(ws_0)$.

If $\tau^{-1}(w) = ws_0$, then we are done. If $\tau^{-1}(w) = w$, and one assumes for the sake of contradiction that $s_j \notin \mathrm{Des}_S(w)$, then one has

$$w \in W^{\{s_0\}} \cap W^{\{s_j\}} = W^{\{s_0, s_j\}}.$$

This gives the contradiction

$$\ell_S(ws_0s_j) = \ell_S(w) + 2 \not< \ell_S(w) + 1 = \ell_S(ws_0).$$

Case 2. $r_j^{-1} \in \mathrm{Des}_{R\cup R^{-1}}(w)$. Then

$$\nu(ws_j) = \nu(ws_js_0) = \ell_{R\cup R^{-1}}\big(wr_j^{-1}\big) < \ell_{R\cup R^{-1}}(w) = \nu(w),$$

which forces $\ell_S(ws_j) < \ell_S(w)$ by Proposition 2.3.9(i). Thus $s_j \in \mathrm{Des}_S(w)$.

If $\tau^{-1}(w) = w$, then we are done. If $\tau^{-1}(w) = ws_0$, and one assumes for the sake of contradiction that $s_j \notin \mathrm{Des}_S(ws_0)$, then one has

$$ws_0 \in W^{\{s_0\}} \cap W^{\{s_j\}} = W^{\{s_0, s_j\}}.$$

This gives the contradiction

$$\ell_S(ws_j) = \ell_S(ws_0 \cdot s_0s_j) = \ell_S(ws_0) + 2 = \ell_S(w) + 1 \not< \ell_S(w). \qquad \square$$

Remark 3.4.4. To see that the inclusion in (13) can be proper, consider the Coxeter system (W, S) of type A_3 with s_0 chosen to be a leaf node, as in Fig. 2(a). Here if one takes $w = r^{-1}r_2r_1$ then $\tau^{-1}(w) = s_1s_2s_0s_1$, with $\widehat{\mathrm{Nasc}}_R(w) = \{r_1, r_2\}$ but $\mathrm{Des}_S(\tau^{-1}(w)) = \{s_1\}$.

We should also point out that this problem cannot be fixed by using $\widehat{\mathrm{Des}}_R$ instead of $\widehat{\mathrm{Nasc}}_R(w)$. Not only would this not give equality in (13), but one would no longer in general have an inclusion: For example, for $w = r_1r_2r_1^{-1} \in A_3$ with s_0 chosen to be a leaf node as above,

$$\widehat{\mathrm{Des}}_R(w) = \{r_1\},$$
$$\mathrm{Des}_S\big(\tau^{-1}(w)\big) = \mathrm{Des}_S(s_0s_1s_0s_2s_1) = \{s_1, s_2\}.$$

Proposition 3.4.3 immediately implies the following.

Corollary 3.4.5. *When s_0 is evenly-laced (so $\widehat{\mathrm{Nasc}}_R = \widehat{\mathrm{Des}}_R$), one has*

$$\sum_{w\in W^+} \mathbf{t}^{\widehat{\mathrm{Des}}_R(w)} q^{\ell_{R\cup R^{-1}}(w)} = \sum_{w\in W^{\{s_0\}}} \mathbf{t}^{\mathrm{Des}_S(w)} q^{\nu(w)}$$
$$= \left[\sum_{w\in W} \mathbf{t}^{\mathrm{Des}_S(w)} q_0^{\ell_0(w)} q^{\nu(w)}\right]_{q_0=1, t_0=0}$$

where the elements in $\widehat{\mathrm{Des}}_R(w)$ and $\mathrm{Des}_S(w)$ are identified with their subscripts as before, and $\mathbf{t}^A := \prod_{j\in A} t_j$.

This last generating function for W is easily computed using the techniques from [11].

Example 3.4.6. Consider the Coxeter system (W, S) of type B_n, labeled as in Example 3.2.3. Then [11, §II, Theorem 3] shows that

$$\sum_{w\in W} \mathbf{t}^{\mathrm{Des}_S(w)} q_0^{\ell_0(w)} q^{\nu(w)} = (-q_0; q)_n [n]!_q \det[a_{ij}]_{i,j=-1,0,1,2,\ldots,n-1}$$

where

$$a_{ij}=\begin{cases}0 & \text{for } j<i-1,\\ t_i-1 & \text{for } j=i-1,\\ \frac{t_i}{(-q_0;q)_{j+1}[j+1]!_q} & \text{for } j\geqslant i=-1,\\ \frac{t_i}{[j-i+1]!_q} & \text{for } j\geqslant i\geqslant 0\end{cases}$$

with the convention $t_{-1}=1$. As an example, for $n=3$, one has

$$\sum_{w\in W}\mathbf{t}^{\mathrm{Des}_S(w)}q_0^{\ell_0(w)}q^{\nu(w)}$$
$$=(-q_0;q)_3[3]!_q\det\begin{bmatrix}1 & \frac{1}{(-q_0;q)_1[1]!_q} & \frac{1}{(-q_0;q)_2[2]!_q} & \frac{1}{(-q_0;q)_3[3]!_q}\\ t_0-1 & \frac{t_0}{[1]!_q} & \frac{t_0}{[2]!_q} & \frac{t_0}{[3]!_q}\\ 0 & t_1-1 & \frac{t_1}{[1]!_q} & \frac{t_1}{[2]!_q}\\ 0 & 0 & t_2-1 & \frac{t_2}{[1]!_q}\end{bmatrix}$$

thus Corollary 3.4.5 gives

$$\sum_{w\in W^+}\mathbf{t}^{\widehat{\mathrm{Nasc}}_{R\cup R^{-1}}(w)}q^{\ell_{R\cup R^{-1}}(w)}$$
$$=2(-q;q)_2[3]!_q\det\begin{bmatrix}1 & \frac{1}{2(-q;q)_0[1]!_q} & \frac{1}{2(-q;q)_1[2]!_q} & \frac{1}{2(-q;q)_2[3]!_q}\\ -1 & 0 & 0 & 0\\ 0 & t_1-1 & \frac{t_1}{[1]!_q} & \frac{t_1}{[2]!_q}\\ 0 & 0 & t_2-1 & \frac{t_2}{[1]!_q}\end{bmatrix}$$
$$=1+q(2t_1+t_2)+q^2(3t_1+2t_2)++q^3(3t_1+t_2+2t_1t_2)$$
$$+q^4(2t_1+t_2+2t_1t_2)+q^5(t_1+2t_1t_2)+q^6t_1t_2.$$

3.5. Palindromes revisited

When s_0 is evenly-laced, the set of palindromes for (W^+,R) behaves much more like the set of reflections in a Coxeter system (W,S), and plays a more closely analogous role.

Theorem 3.5.1. *Assume (W,S) has s_0 evenly-laced. Then for any $w\in W^+$, one has the following.*

(a) $\ell_{R\cup R^{-1}}=|\mathrm{P}_L(w)|$.
(b) (*Strong exchange property*) *For any reduced $(R\cup R^{-1})^*$-word*

$$\mathbf{r}=\left(r^{(1)},\ldots,r^{(\nu(w))}\right)$$

factoring w, one has $\mathrm{P}_L(w)=\{p_k\}_{1\leqslant k\leqslant\nu(w)}$ where

$$p_k:=\left(r^{(1)}\right)^{-1}\left(r^{(2)}\right)^{-1}\cdots\left(r^{(k)}\right)^{-1}\cdots\left(r^{(2)}\right)^{-1}\left(r^{(1)}\right)^{-1}.$$

In other words, for a palindrome p and reduced $(R\cup R^{-1})$-word $\mathbf{r}=(r^{(1)},\ldots,r^{(\nu(w))})$, one has

$$\ell_{R\cup R^{-1}}(pw) < \ell_{R\cup R^{-1}}(w) \quad \textit{if and only if}$$
$$p = p_k \textit{ for some } k = 1, 2, \ldots, \nu(w) \quad \textit{if and only if}$$
$$pw = \bigl(r^{(1)}\bigr)^{-1}\bigl(r^{(2)}\bigr)^{-1} \cdots \bigl(r^{(k-1)}\bigr)^{-1} r^{(k+1)} \cdots r^{(\nu(w))}$$
$$\textit{for some } k = 1, 2, \ldots, \nu(w).$$

(c) *The set* $\mathtt{P}_L(w)$ *determines* w *uniquely.*

Proof. Assertions (a) and (b) are immediate from the assertion of equality in Proposition 2.5.5.

For (c), one must show that for any $w, w' \in W^+$, if $\mathtt{P}_L(w) = \mathtt{P}_L(w')$ then $w = w'$. Via Proposition 2.5.5, it is equivalent to show the following for any w, w' in W:

$$\hat{\mathtt{T}}_L(w) = \hat{\mathtt{T}}_L(w') \quad \text{implies} \quad w' \in wW_{\{s_0\}}\bigl(= \{w, ws_0\}\bigr).$$

We will prove this assertion by induction on $\nu(w)$.

In the base case, if $\nu(w) = 0$, then $w \in W_{\{s_0\}}$, which forces $\hat{\mathtt{T}}_L(w') = \hat{\mathtt{T}}_L(w) = \emptyset$, and hence also $w' \in W_{\{s_0\}}$.

In the inductive step, we make use of the following property [5, Exercise 1.12] of $\mathtt{T}_L(w)$:

$$\begin{aligned}
&\mathtt{T}_L(sw) = \{s\} \vartriangle s\mathtt{T}_L(w)s \quad \text{for any } s \in S, \text{ and hence}\\
&\hat{\mathtt{T}}_L(s_0w) = s_0\hat{\mathtt{T}}_L(w)s_0,\\
&\hat{\mathtt{T}}_L(s_iw) = \{s_i\} \vartriangle s_i\hat{\mathtt{T}}_L(w)s_i \quad \text{for } i = 1, 2, \ldots, n,
\end{aligned} \tag{15}$$

where $A \vartriangle B := (A \setminus B) \sqcup (B \setminus A)$ denotes the symmetric difference of the sets A, B. We treat two cases for w.

Case 1. $\mathtt{T}_L(w) \cap S \neq \{s_0\}$, say $s_i \in \mathtt{T}_L(w)$ for some $i = 1, 2, \ldots, n$. Then

$$\hat{\mathtt{T}}_L(s_iw) = \{s_i\} \vartriangle s_i\hat{\mathtt{T}}_L(w)s_i = \{s_i\} \vartriangle s_i\hat{\mathtt{T}}_L(w')s_i = \hat{\mathtt{T}}_L(s_iw').$$

As $s_i \in \mathtt{T}_L(w)$ implies $\nu(s_iw) < \nu(w)$, so one can apply induction to conclude that $s_iw' \in s_iwW_{\{s_0\}}$, which implies $w' \in wW_{\{s_0\}}$ as desired.

Case 2. $\mathtt{T}_L(w) \cap S = \{s_0\}$. In this case

$$\hat{\mathtt{T}}_L(s_0w) = s_0\hat{\mathtt{T}}_L(w)s_0 = s_0\hat{\mathtt{T}}_L(w')s_0 = \hat{\mathtt{T}}_L(s_0w'),$$

and $\nu(s_0w) = \nu(w)$, but $\mathtt{T}_L(s_0w) \cap S \neq \{s_0\}$, so that Case 1 applies. □

Note that we have already seen in Example 2.5.6 that, without the assumption that s_0 is evenly-laced, the assertions of Theorem 3.5.1 can fail.

3.6. Orders revisited

When s_0 is evenly-laced, the strong exchange property for palindromes (Theorem 3.5.1(b)) has consequences for the weak and strong orders on W^+, analogous to what happens for the weak and strong orders on W. The next four propositions follow, along the lines of the usual proofs from [5, Chapters 2, 3].

Proposition 3.6.1. *When s_0 is evenly-laced, $u, w \in W^+$ satisfy $u \leqslant_{RW} w$ if and only if $\mathtt{P}_L(u) \subseteq \mathtt{P}_L(w)$.*

A similar statement holds for the left weak order $\leqslant_{LW}$, replacing left-shortening palindromes $\mathtt{P}_L(-)$ with right-shortening palindromes $\mathtt{P}_R(-)$.

Proposition 3.6.2. *When s_0 is evenly-laced, the left, right weak orders on W^+ are meet-semilattices.*

Proposition 3.6.3. *When s_0 is evenly-laced, $u, w \in W^+$ satisfy $u \leqslant_{LS} w$ if and only if for some (equivalently, every) reduced $(R \cup R^{-1})^*$-word $\mathbf{r} = (r^{(1)}, \ldots, r^{(\ell)})$ factoring w, there exists a reduced $(R \cup R^{-1})^*$-word factoring u which is a "subword" in the following sense:*

> *it can be obtained by deleting some of the $r^{(i)}$ from $\mathbf{r}$ and replacing any $r^{(i)}$ remaining that have an odd number of letters deleted to their right with their inverse $(r^{(i)})^{-1}$.*

A similar statement holds for the right strong order $\leqslant_{RS}$, replacing "right" with "left."

Recall that a poset is *thin* if every interval $[x, y]$ of rank 2 has exactly 4 elements, namely $\{x \leqslant u, v \leqslant y\}$.

Proposition 3.6.4. *When s_0 is evenly-laced, the left, right strong orders on W^+ are thin and shellable, and hence have every open interval homeomorphic to a sphere.*

Remark 3.6.5. Note that when s_0 is not evenly-laced, the strong order need not be thin, as illustrated by the existence of several upper intervals of rank 2 having 5 elements in Fig. 3(b).

When (W, S) is finite, the examples of $I_2(7)$, A_3 from Fig. 3 show that one need not have a unique maximum element in any of these orders if s_0 is not evenly-laced. However, if s_0 is evenly-laced, there is an obvious candidate for such a top element, namely $\tau(w_0)$, where w_0 is the longest element of W.

Proposition 3.6.6. *When (W, S) has s_0 evenly-laced and W finite, one has $w_0 s_0 = s_0 w_0$. Furthermore, the element $\tau(w_0) \in W^+ \cap w_0 W_{s_0}$ is the unique maximum element in all four (left or right, weak or strong) orders on W^+.*

Proof. For the first assertion note that, by [5, Proposition 2.3.2]

$$\ell_S(w_0 s_0 w_0) = \ell_S(w_0) - \big(\ell_S(w_0) - \ell_S(s_0)\big) = \ell_S(s_0) = 1$$

which shows $w_0 s_0 w_0$ lies in S. But since $w_0^{-1} = w_0$, it is also conjugate to s_0, so in the case where s_0 is evenly-laced, one must have $w_0 s_0 w_0 = s_0$, i.e., $w_0 s_0 = s_0 w_0$.

To see that $\tau(w_0)$ is the maximum in all four orders, one can easily check using Proposition 2.5.5 that $\mathtt{P}_L(\tau(w_0)) = \mathtt{P}(W^+)$. Hence $\tau(w_0)$ is the maximum for the right weak order by Proposition 3.6.1. Since $\tau(w_0)$ is either w_0 or $w_0 s_0 = s_0 w_0$, in either case one has $\tau(w_0)^{-1} = \tau(w_0)$, and hence it is also the maximum for the left weak order. It is then also the maximum for the left and right strong orders because they are stronger than the corresponding weak orders. $\square$

4. The case of a leaf node

The presentation (2) for W^+ becomes very close to a Coxeter presentation when s_0 is a *leaf* node, that is, s_0 commutes with $s_2, \ldots, s_n$, i.e., one has $m_{0i} = 2$ for $i = 2, \ldots, n$ (although m_{01} may be greater than 2). Note that every (irreducible) finite and affine Coxeter system (W, S), with the exception of the family $\tilde{A}_n$, has Coxeter diagram shaped like a tree, and hence will have some leaf node s_0.

4.1. Nearly Coxeter presentations

Proposition 4.1.1. *Let (W, S) be a Coxeter system with $S = \{s_0, s_1, \ldots, s_n\}$ and s_0 a leaf node. Then W^+ is generated by the set*

$$R := \{r_i = s_0 s_i \mid s_i \in S \setminus s_0\}$$

with the following presentation:

$$W^+ \cong \langle R = \{r_1, \ldots, r_n\}\colon r_1^{m_{01}} = r_i^2 = (r_i r_j)^{m_{ij}} = e \text{ for } 1 \leqslant i < j \leqslant n \rangle, \tag{16}$$

where m_{ij} is the order of $s_i s_j$ and s_1 is the neighbor of the leaf s_0.

Proof. Starting with the presentation in (2), note that given any $1 \leqslant i < j \leqslant n$, the relation $(r_i^{-1} r_j)^{m_{ij}} = e$ is equivalent to $(r_j^{-1} r_i)^{m_{ij}} = e$ by taking the inverse of both sides. However, since $j \geqslant 2$, one has $r_j^2 = e$ and so $r_j^{-1} = r_j$. Thus this relation is equivalent to $(r_j r_i)^{m_{ij}} = e$, which is also equivalent to $(r_i r_j)^{m_{ij}} = e$ via conjugation by r_j. □

Definition 4.1.2. Call a presentation for an abstract group having the form in (16) a *nearly Coxeter* presentation, meaning that all but one of the generators r_i is an involution and all other relations are of the form $(r_i r_j)^{m_{ij}}$ for some $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$.

Corollary 4.1.3. *Every abstract group A with a nearly Coxeter presentation is isomorphic to the alternating subgroup W^+ of some Coxeter system (W, S).*

In particular, if A is finite and has a nearly Coxeter presentation, then it is isomorphic to a product

$$A \cong W_0^+ \times W_1 \times \cdots \times W_r \tag{17}$$

in which each of the (W_i, S_i) are finite irreducible Coxeter systems (*and hence classified*).

Proof. If A is an abstract group with a nearly Coxeter presentation, as in (16), one can write down a corresponding Coxeter system (W, S) as in (1). Theorem 2 then shows that $A \cong W^+$.

Furthermore, if A is finite, then since $A \cong W^+$ and $[W : W^+] = 2$, one concludes that W is also finite. Consequently

$$W \cong W_0 \times W_1 \times \cdots \times W_r$$

for some finite irreducible Coxeter systems (W_i, S_i). Without loss of generality, one can index so that s_0, s_1 belong to (W_0, S_0). The isomorphism (17) then follows from examining the presentation. □

5. The case of an even leaf node

When the distinguished node s_0 is both a leaf and evenly-laced, that is, m_{01} is even and $m_{0j}=2$ for $j=2,3,\dots,n$, we shall say that s_0 is an *even leaf*. In this situation, the alternating subgroup W^+ has an amazingly close connection to a *different* index 2 non-parabolic reflection subgroup. Note that in every finite and affine Coxeter system containing an evenly-laced node s_0, namely types $B_n(=C_n)$, $I_2(m)$ with m even, $\tilde{A}_1(=I_2(\infty))$, $\tilde{B}_n, \tilde{C}_n, \tilde{G}_2$, this evenly-laced node is actually an even leaf,[2] to which the results below apply.

5.1. The Coxeter system (W', S')

Assume (W,S) is a Coxeter system with $S=\{s_0,s_1,\dots,s_n\}$ having s_0 as an even leaf. Since s_0 is evenly laced, recall that one has the linear character $\chi_0: W\to\{\pm 1\}$ from (9), taking value -1 on s_0 and $+1$ on all other $s_j\in S$. Let $W' := \ker\chi_0$, a subgroup of W of index 2.

We wish to show that W' is a reflection subgroup of W, and has a natural Coxeter presentation (W',S') extremely close to (W,S). Let $S' := \{t_1,t_2,\dots,t_n\}\cup\{t_1'\}$ be a set, and consider the set map

$$\begin{aligned}
&S' \xrightarrow{f} W',\\
&t_j \overset{f}{\mapsto} s_j \quad \text{for } j=1,2\dots,n,\\
&t_1' \overset{f}{\mapsto} s_0s_1s_0.
\end{aligned}$$

Proposition 5.1.1. *The set map f above extends to an isomorphism*

$$\begin{aligned}
W' \cong \big\langle S' = \{t_1,\dots,t_n\}\cup\{t_1'\}:\ &(t_i)^2 = (t_1')^2 = (t_it_j)^{m_{ij}} = e \text{ for } 1\leqslant i\leqslant j\leqslant n,\\
&(t_1't_j)^{m_{1j}} = e,\ (t_1't_1)^{\frac{m_{01}}{2}} = e\big\rangle,
\end{aligned} \tag{18}$$

which makes (W',S') a Coxeter system.

A schematic picture of the relation between the Coxeter diagrams of (W,S) and (W',S') was shown in Fig. 1. Note that the embedding $W'\subset W$ as a reflection (but not parabolic) subgroup generalizes the finite/affine Weyl group inclusions

$$\begin{aligned}
&W(D_n)\subseteq W(B_n)\big(=W(C_n)\big),\\
&W(\tilde{D}_n)\subseteq W(\tilde{B}_n), \quad \text{and}\\
&W(\tilde{B}_n)\subseteq W(\tilde{C}_n)
\end{aligned}$$

in which one always has $m_{01}=4$ so that t_1, t_1' commute, and are a pair of *oriflamme/fork* nodes at the end of the Coxeter diagram for (W',S').

Proof of Proposition 5.1.1. We employ a similar trick to Bourbaki's from Proposition 2.1.1. Consider the abstract group G with the Coxter presentation given on the right-hand side of (18).

[2] With a single affine exception: the affine type $\tilde{C}_2$ has the middle node in its diagram evenly-laced, but not an even leaf!

Since t_1, t_1' play identical roles in this presentation, the set map $\beta : S' \to G$ which fixes $t_2, \ldots, t_n$ and swaps t_1', t_1 extends to an involutive group automorphism $\beta : G \to G$.

Thus the group $\mathbb{Z}/2\mathbb{Z} = \{1, \beta\}$ acts on G, and one can form the semidirect product $G \rtimes \mathbb{Z}/2\mathbb{Z}$ in which $(g_1\beta^i) \cdot (g_2\beta^j) = g_1\beta^i(g_2) \cdot \beta^{i+j}$. This has the following presentation:

$$\begin{aligned} G \rtimes \mathbb{Z}/2\mathbb{Z} \cong \bigl\langle t_1, \ldots, t_n, \beta:\ & \beta^2 = \bigl(t_1'\bigr)^2 = (t_i t_j)^{m_{ij}} = e \text{ for } 1 \leqslant i \leqslant j \leqslant n, \\ & \bigl(t_1' t_j\bigr)^{m_{1j}} = e,\ \bigl(t_1' t_1\bigr)^{\frac{m_{01}}{2}} = e,\ \beta t_j = t_j \beta \text{ for } 2 \leqslant j \leqslant n,\ \beta t_1 = t_1'\beta \bigr\rangle. \end{aligned}$$

We claim that the following maps g, f are well-defined and inverse isomorphisms:

$$\begin{aligned} & W \xrightarrow{g} G \rtimes \mathbb{Z}/2\mathbb{Z}, \\ & s_i \mapsto t_i \qquad \text{for } i = 1, \ldots, n, \\ & s_0 \mapsto \beta, \\ & G \rtimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} W, \\ & t_i \mapsto s_i \qquad \text{for } i = 1, \ldots, n, \\ & t_1' \mapsto s_0 s_1 s_0, \\ & \beta \mapsto s_0. \end{aligned}$$

Here are the relations in (W, S) going to relations in $G \rtimes \mathbb{Z}/2\mathbb{Z}$ needed to check that f is well-defined:

$$\begin{aligned} & s_0^2 = e \mapsto \beta^2 = e, \\ & (s_i s_j)^{m_{ij}} = e \mapsto (t_i t_j)^{m_{ij}} = e \quad \text{for } 1 \leqslant i \leqslant j \leqslant n, \\ & (s_0 s_j)^2 = e \mapsto (\beta t_j)^2 = \beta t_j \beta t_j = \beta\beta t_j t_j = e \quad \text{for } 2 \leqslant j \leqslant n, \\ & (s_0 s_1)^{m_{01}} = e \mapsto (\beta t_1)^{m_{01}} = \underbrace{\beta t_1 \cdot \beta t_1 \cdots}_{m_{01} \text{ times}} = \underbrace{\beta t_1 t_1' \beta \cdot \beta t_1 t_1' \beta \cdots}_{\frac{m_{01}}{2} \text{ times}} = \beta \bigl(t_1 t_1'\bigr)^{\frac{m_{01}}{2}} \beta = e. \end{aligned}$$

Here are the relations in $G \rtimes \mathbb{Z}/2\mathbb{Z}$ going to relations in (W, S) needed to check that g is well-defined:

$$\begin{aligned} & \beta^2 = e \mapsto s_0^2 = e, \\ & (t_i t_j)^{m_{ij}} = e \mapsto (s_i s_j)^{m_{ij}} = e \quad \text{for } 1 \leqslant i \leqslant j \leqslant n, \\ & \bigl(t_1'\bigr)^2 = e \mapsto (s_0 s_1 s_0)^2 = e, \\ & \bigl(t_1' t_j\bigr)^{m_{1j}} = e \mapsto (s_0 s_1 s_0 s_j)^{m_{1j}} = (s_0 s_1 s_j s_0)^{m_{1j}} = s_0 (s_1 s_j)^{m_{1j}} s_0 = e, \\ & \bigl(t_1 t_1'\bigr)^{\frac{m_{01}}{2}} = e \mapsto (s_1 s_0 s_1 s_0)^{\frac{m_{01}}{2}} = (s_1 s_0)^{m_{01}} = e, \\ & \beta t_j = t_j \beta \mapsto s_0 s_j = s_j s_0 \quad \text{for } 2 \leqslant j \leqslant n, \\ & \beta t_1 = t_1' \beta \mapsto s_0 s_1 = s_0 s_1 s_0 s_0. \end{aligned}$$

Once one knows that f, g are well-defined, it is easily checked that they are inverse isomorphisms by checking this on generators.

Since $f(G) \subseteq W'$, and both $W', f(G)$ are subgroups of W of index 2, it must be that $f(G) = W'$. Hence f restricts to the desired isomorphism presenting W' as the Coxeter group G. $\square$

5.2. Relating W^+ to W'

We next discuss the tight relation between (W^+, R) and (W', S'), which is mediated by the following map.

Proposition 5.2.1. *When s_0 is an even leaf in (W, S), the following formulae*

$$W^+ \xrightarrow{\theta} W',$$
$$w \mapsto w \cdot s_0^{\ell_{R\cup R^{-1}}(w)} = \begin{cases} w & \text{if } w \in W', \\ ws_0 & \text{if } w \notin W' \end{cases}$$

define the same set map $\theta : W^+ \to W'$. In other words, $\theta(w)$ is the unique element in the coset $wW_{\{s_0\}} = \{w, ws_0\}$ that lies in W'.

Furthermore, θ is a bijection, and equivariant for the action of the subgroup $W^+ \cap W'$ by left-multiplication on W^+ and W'.

Proof. Note that

$$\chi_0(r_i) = \chi_0(s_0 s_i) = -1 = \chi_0(s_i s_0) = \chi_0\bigl(r_i^{-1}\bigr) \quad \text{for all } i$$

and hence $\chi_0(w) = (-1)^{\ell_{R\cup R^{-1}}(w)}$. This shows the equivalence of the two formulae for $\theta(w)$.

The $W^+ \cap W'$-equivariance of θ follows from either formula. Bijectivity of θ follows, for example, since one can check that the map $\tau : W \to W^+$ from Definition 2.3.2 when restricted to W' satisfies $\tau|_{W'} = \theta^{-1}$. □

Note that the bijection $\theta : W^+ \to W'$ is *not* a group isomorphism, and that W^+, W' are generally not isomorphic as groups. For example, when (W, S) is a dihedral Coxeter system $I_2(m)$ with m even, W^+ is always cyclic of order m, while W' is not cyclic for $m \geqslant 4$.

Nevertheless, the map θ is about as close as one can get to an isomorphism of the presentations (W^+, R) and (W', S'), in that θ lifts to the following map on words in the generating sets.

Definition 5.2.2. When s_0 is an even leaf in (W, S), define a set map $\Theta : (R \cup R^{-1})^* \to (S')^*$ by mapping a word $\mathbf{r} = (r^{(1)}, \dots, r^{(\ell)})$ one letter at a time according to the following rules:

$$r_j \mapsto t_j \quad \text{for } j = 2, 3, \dots, n,$$
$$r_1 \mapsto \begin{cases} t_1 & \text{if } r_1 = r^{(k)} \text{ with } k \text{ even}, \\ t_1' & \text{if } r_1 = r^{(k)} \text{ with } k \text{ odd}, \end{cases}$$
$$r_1^{-1} \mapsto \begin{cases} t_1' & \text{if } r_1 = r^{(k)} \text{ with } k \text{ even}, \\ t_1 & \text{if } r_1 = r^{(k)} \text{ with } k \text{ odd}. \end{cases}$$

The maps θ, Θ are related as follows.

Proposition 5.2.3. *Let s_0 be an even leaf in (W, S). Then for any $(R \cup R^{-1})^*$-word $\mathbf{r}$ of length ℓ that factors $w \in W^+$, its image $\Theta(\mathbf{r})$ is an S^*-word of the same length that factors $\theta(w)$.*

Proof. Given $\mathbf{r} = (r^{(1)}, \dots, r^{(\ell-1)}, r^{(\ell)})$ of length ℓ factoring $w \in W^+$, denote by w' the element in W' factored by $\Theta(\mathbf{r})$. One must show that $w^{-1}w' = s_0^{\ell}$, where $\ell = \ell_{R\cup R^{-1}}(w)$.

Proceed by induction on ℓ, where the base case $\ell = 0$ is trivial. In the inductive step, let u denote the element in W^+ factored by $(r^{(1)}, \dots, r^{(\ell-1)})$, and u' the element in W' factored by $\Theta(r^{(1)}, \dots, r^{(\ell-1)})$. By induction, $u^{-1}u' = s_0^{\ell-1}$. Then

$$w^{-1}w' = \left(r^{(\ell)}\right)^{-1} \cdot u^{-1}u' \cdot \Theta\left(r^{(\ell)}\right) = \left(r^{(\ell)}\right)^{-1} \cdot s_0^{\ell-1} \cdot \Theta\left(r^{(\ell)}\right),$$

which one must show coincides with s_0^ℓ. Consider the following cases:

Case 1. $r^{(\ell)} = r_j$ for some $j = 2, 3, \dots, n$. Then $r^{(\ell)} = s_0 s_j$ and $\Theta(r^{(\ell)}) = t_j = s_j$, so one gets

$$(s_0 s_j)^{-1} \cdot s_0^{\ell-1} \cdot s_j = s_0^\ell$$

because s_0, s_j commute.

Case 2a. $r^{(\ell)} = r_1$ and ℓ is even. Then $r^{(\ell)} = s_0 s_1$ and $\Theta(r^{(\ell)}) = t_1 = s_1$, so one gets

$$(s_0 s_1)^{-1} \cdot s_0^{\ell-1} \cdot s_1 = s_1 s_0 \cdot s_0^{\ell-1} \cdot s_1 = s_1 s_0^\ell s_1 = e = s_0^\ell.$$

Case 2b. $r^{(\ell)} = r_1$ and ℓ is odd. Then $r^{(\ell)} = s_0 s_1$ and $\Theta(r^{(\ell)}) = t_1' = s_0 s_1 s_0$, so one gets

$$(s_0 s_1)^{-1} \cdot s_0^{\ell-1} \cdot s_0 s_1 s_0 = s_1 s_0 \cdot s_0^{\ell-1} \cdot s_0 s_1 s_0 = s_0 = s_0^\ell.$$

Case 3a. $r^{(\ell)} = r_1^{-1}$ and ℓ is even. Then $r^{(\ell)} = s_1 s_0$ and $\Theta(r^{(\ell)}) = t_1' = s_0 s_1 s_0$, so one gets

$$(s_1 s_0)^{-1} \cdot s_0^{\ell-1} \cdot s_0 s_1 s_0 = s_0 s_1 \cdot s_0^{\ell-1} \cdot s_0 s_1 s_0 = e = s_0^\ell.$$

Case 3b. $r^{(\ell)} = r_1^{-1}$ and ℓ is odd. Then $r^{(\ell)} = s_1 s_0$ and $\Theta(r^{(\ell)}) = t_1 = s_1$, so one gets

$$(s_1 s_0)^{-1} \cdot s_0^{\ell-1} \cdot s_1 = s_0 s_1 \cdot s_0^{\ell-1} \cdot s_1 = s_0 = s_0^\ell. \qquad \square$$

Corollary 5.2.4. *Let s_0 be an even leaf in (W, S). Then for any $w \in W^+$, the bijections θ, Θ have the following properties*:

(i) $\ell_{R \cup R^{-1}}(w) = \ell_{S'}(\theta(w))$.
(ii) *Θ bijects the set of reduced $(R \cup R^{-1})^*$-words for w with the reduced $(S')^*$-words for $\theta(w)$.*
(iii) *Given any $w \in W^+$, the bijection $R \cup R^{-1} \to S'$ defined by*

$$r_j \mapsto t_j \quad \text{for } j = 2, 3, \dots, n,$$
$$r_1, r_1^{-1} \mapsto \begin{cases} t_1', t_1 & \text{if } \ell_{R \cup R^{-1}}(w) \text{ is even}, \\ t_1, t_1' & \text{if } \ell_{R \cup R^{-1}}(w) \text{ is odd}, \end{cases}$$

bijects $\mathrm{Des}_{R \cup R^{-1}}(w)$ *with* $\mathrm{Des}_{S'}(\theta(w))$.
(iv) *The map θ is a poset isomorphism $(W^+, \leqslant_{RW}) \to (W', \leqslant_{RW})$.*
(v) *The map θ is a poset isomorphism $(W^+, \leqslant_{RS}) \to (W', \leqslant_S)$.*

Proof. Assertions (i), (ii), (iii) and (iv) are straightforward from Proposition 5.2.3, while assertion (v) follows from it via Proposition 3.6.3. $\square$

Corollary 5.2.5. *When s_0 is an even leaf in (W, S), one has*

$$\sum_{w \in W^+} \mathbf{t}^{\mathrm{Des}_{R \cup R^{-1}}(w)} q^{\ell_{R \cup R^{-1}}(w)} = \boldsymbol{\Theta}\left[\sum_{w \in W'} \mathbf{t}^{\mathrm{Des}_{S'}(w)} q^{\ell_{S'}(w)} \right], \tag{19}$$

where $\boldsymbol{\Theta}$ is the obvious operator on monomials $\mathbf{t}^A q^\ell$ corresponding to the mapping from Corollary 5.2.4(iii). *In particular, letting* $\mathrm{des}_{R\cup R^{-1}}(w) := |\mathrm{Des}_{R\cup R^{-1}}(w)|$,

$$\sum_{w\in W^+} t^{\mathrm{des}_{R\cup R^{-1}}(w)} q^{\ell_{R\cup R^{-1}}(w)} = \sum_{w\in W'} t^{\mathrm{des}_{S'}(w)} q^{\ell_{S'}(w)}, \tag{20}$$

and

$$W^+\bigl(R\cup R^{-1}; q\bigr) = W'(S'; q). \tag{21}$$

Example 5.2.6. Let (W, S) be the Coxeter system of type B_n, labeled as in Example 3.2.3. Here (W', S') is the Coxeter system of type D_n, whose exponents are known to be $n-1, 1, 3, 5, \ldots, 2n-5, 2n-3$. Hence one can rederive the length generating function for (W^+, R) using Corollary 5.2.5 and a well-known result in the theory of Coxeter groups (see, e.g., [5, Theorem 7.1.5] or [8, Theorem 3.15]) as follows:

$$W^+\bigl(R\cup R^{-1}; q\bigr) = W'(S'; q) = [n]_q \prod_{j=1}^{n-1} [2j]_q.$$

Furthermore, [11, Theorem 7] gives generating functions incorporating the distributions of descents and length simultaneously for all groups D_n, and hence Eq. (20) allows one to derive the generating functions of $\mathrm{des}_{R\cup R^{-1}}$ and $\ell_{R\cup R^{-1}}$ simultaneously for W^+ of all of the groups B_n when s_0 is chosen to be the even leaf. When $n=3$ this gives, for example

$$\sum_{w\in W^+} t^{\mathrm{des}_{R\cup R^{-1}}(w)} q^{\ell_{R\cup R^{-1}}(w)} = 1 + 3qt + q^2\bigl(4t+t^2\bigr) + q^3\bigl(3t+3t^2\bigr) + q^4\bigl(t+4t^2\bigr) + 3q^5t^2 + q^6t^3.$$

Example 5.2.7. For (W, S) of affine type $\tilde{C}_n$, one has (W', S') equal to the affine Coxeter system of type $\tilde{B}_n$. Using Corollary 5.2.5, the known exponents $1, 3, 5, \ldots, 2n-1$ for the finite type B_n, and Bott's formula (see, e.g., [5, Theorem 7.1.10] or [8, §8.9]) for the length generating function of an affine Weyl group, one has that

$$W^+\bigl(R\cup R^{-1}; q\bigr) = W'(S'; q) = \prod_{j=1}^{n} \frac{[2j]_q}{1-q^{2j-1}}.$$

Similarly, for (W, S) of affine type $\tilde{B}_n$, one has that (W', S') is the affine Coxeter system of type $\tilde{D}_n$, and one derives

$$W^+\bigl(R\cup R^{-1}; q\bigr) = W'(S'; q) = \frac{[n]_q}{1-q^{n-1}} \prod_{j=1}^{n-1} \frac{[2j]_q}{1-q^{2j-1}}.$$

A refinement may be obtained using [11, Theorems 7 and 8], which give generating functions incorporating the distributions of descents and length simultaneously for all groups $\tilde{B}_n$, $\tilde{D}_n$. Hence, Eq. (20) allows one to derive the generating functions of $\mathrm{des}_{R\cup R^{-1}}$ and $\ell_{R\cup R^{-1}}$ simultaneously for W^+ of all of the groups $\tilde{C}_n$, $\tilde{B}_n$, when s_0 is chosen to be an even leaf.

Acknowledgments

The authors thank John Stembridge for pointing them to the Bourbaki exercise from Section 2.1 at an early stage of this investigation.

References

[1] E. Babson, V. Reiner, Coxeter-like complexes, Discrete Math. Theor. Comput. Sci. 6 (2004) 223–251 (electronic).
[2] D. Bernstein, A maj-inv bijection for $C_2 \wr A_n$, preprint, arXiv: math.CO/0509239, 2005.
[3] D. Bernstein, A. Regev, A Foata bijection for the alternating group and for q-analogues, Sém. Lothar. Combin. 53 (2004/06), Art. B53b, 16 pp. (electronic).
[4] A. Björner, Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings, Adv. Math. 52 (1984) 173–212.
[5] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Grad. Texts in Math., vol. 231, Springer, New York, 2005.
[6] N. Bourbaki, Lie Groups and Lie Algebras, Springer, 2002, English translation by Andrew Pressley.
[7] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, European J. Combin. 15 (1994) 417–441.
[8] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge University Press, Cambridge, 1990.
[9] I.G. Macdonald, The Poincaré series of a Coxeter group, Math. Ann. 199 (1972) 161–174.
[10] A. Regev, Y. Roichman, Permutation statistics on the alternating group, Adv. in Appl. Math. 33 (2004) 676–709.
[11] V. Reiner, The distribution of descents and length in a Coxeter group, Electron. J. Combin. 2 (1995) R25 (electronic).
[12] R.P. Stanley, Some aspects of groups acting on finite posets, J. Combin. Theory Ser. A 32 (1982) 132–161.