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Completion of a Prüfer domain[1]

B.G. Kang*, M.H. Park

Department of Mathematics, Pohang Institute of Science and Technology, Pohang 790-784, South Korea

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Abstract

Let V (resp. D) be a valuation domain (resp. SFT Prüfer domain), I a proper ideal, and $\hat{V}$ (resp. $\hat{D}$) be the I-adic completion of V (resp. D). We show that (1) $\hat{V}$ is a valuation domain, (2) Krull dimension of $\hat{V} = \dim V/I + 1$ if I is not idempotent, $\hat{V} \cong V/I$ if I is idempotent, (3) $\dim \hat{D} = \dim D/I + 1$, (4) $\hat{D}$ is an SFT Prüfer ring, and (5) $\hat{D}$ is a catenarian ring. © 1999 Elsevier Science B.V. All rights reserved.

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Throughout this paper, all rings are assumed to be commutative rings with identity. It is well known that for a Noetherian ring R and a proper ideal I of R, the Krull dimension of the I-adic completion $\hat{R}$ of R equals $\sup\{\mathrm{ht}\, M \mid M$ is a maximal ideal of R containing $I\}$ [7, Proposition 7.3, p. 35]. In this paper, we will study the completion of a valuation domain and a Prüfer domain and get a similar equation for the Krull dimension of the completion.

First we describe some properties of prime ideals of the power series ring $V[\![X]\!]$ of a valuation domain V. In [4], Arnold gave a collection of principal prime ideals of $V[\![X]\!]$, where V is a finite-dimensional valuation domain with the SFT-property, i.e., a finite-dimensional discrete valuation domain: Let Q be a prime ideal of $V[\![X]\!]$ and let $Q \cap V = P$. If $P[\![X]\!] \neq Q$ and $Q \neq P + (X)$ (i.e., $X \notin Q$), then Q is a principal ideal (and $Q \subset P_1 + (X)$, where P_1 is the prime ideal just above P). In a nondiscrete valuation domain or a non-SFT valuation domain, these conditions are not enough to guarantee Q to be a principal ideal (see the remark following Corollary 2). Under an additional hypothesis that Q contains a power series with unit content, we prove that Q is a

* Corresponding author.

E-mail address: bgkang@posmath.postech.ac.kr. (B.G. Kang)

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principal ideal. This result will enable us to extend a part of [4, Proposition 5] to the infinite-dimensional case. We will give a characterization of prime elements of $V[\![X]\!]$, V an SFT valuation domain. For $f \in V[\![X]\!]$, we denote by C_f the ideal of V generated by the coefficients of f. When C_f is a unit ideal, we usually write $C_f = 1$.

Lemma 1 (see Arnold [4, Proposition 5]). *Let V be a valuation domain with the maximal ideal M and Q a prime ideal of $V[\![X]\!]$ such that $X \notin Q$. If Q contains an element f such that $C_f = 1$, then Q is a principal ideal.*

Proof. Let $g = \sum_{i=0}^{\infty} a_i X^i \in Q$ with $C_g = 1$. Let $n(g)$ be the smallest integer such that a_n is a unit. Let $f \in Q$ be such that $n(f)$ is the smallest in the set $\{n(g) \mid g \in Q$ and $C_g = 1\}$. Let a_0 be the constant term of f. Note that $a_0 \neq 0$. For otherwise $f = Xh$, $h \in V[\![X]\!]$. Since $X \notin Q$, $h \in Q$. However, $n(h) = n(f) - 1$, which contradicts the minimality of $n(f)$. Thus $a_0 \neq 0$. We claim that the value $v(a_0)$ of the constant term a_0 of f is the minimum among the values of constant terms of elements in Q. Suppose not and let $g = b_0 + b_1 X + \cdots + b_n X^n + \cdots \in Q$ be an element such that $v(b_0) < v(a_0)$. For a $c \in M$, $a_0 = b_0 c$. Now $f - cg = X(\cdots + (u - cb_n)X^{n-1} + \cdots) \in Q$, where $n = n(f)$ and u is the unit coefficient of X^n in f. Since $X \notin Q$, $h = \cdots + (u - cb_n)X^{n-1} + \cdots \in Q$, contrary to the fact that n is minimal. Thus $v(a_0)$ is the minimum as claimed. We show that $Q = (f)$. Let $g \in Q$. For a $c_1 \in V$, $X \mid (g - c_1 f)$. Let $g - c_1 f = Xg_1$, $g_1 \in Q$. Likewise $g_1 - c_2 f = Xg_2$ for $c_2 \in V$ and $g_2 \in Q$. Then $g = c_1 f + Xg_1 = c_1 f + X(c_2 f + Xg_2) = c_1 f + Xc_2 f + X^2 g_2$. Continuing in this way, we get $g = \sum_{i=1}^{\infty} c_i X^{i-1} f = (\sum_{i=1}^{\infty} c_i X^{i-1}) f$. □

Corollary 2. *If V is a valuation domain with the maximal ideal M, Q a prime ideal of V such that $Q \cap V = (0)$ and $Q \nsubseteq M[\![X]\!]$, then Q is a principal ideal.*

Remark. In Lemma 1, the assumption that Q contains an element f with $C_f = 1$ is necessary: for a one-dimensional nondiscrete valuation domain V with maximal ideal M, it is well known that the ideal $M \cdot V[\![X]\!]$ is a nonprincipal prime ideal of $V[\![X]\!]$. In Corollary 2, the condition that $Q \nsubseteq M[\![X]\!]$ is essential. In [9], we constructed an infinite descending chain of prime ideals $\{P_n\}_{n=1}^{\infty}$ inside $M[\![X]\!]$ and such that $P_n \cap V = (0)$ for each $n \geqslant 1$. It is easy to see that these P_n are not principal ideals either by looking at the construction or by the observation that in a completely integrally closed domain, a nonzero principal prime ideal is necessarily a height 1 prime ideal.

A ring R is called an SFT-ring (strong finite type ring) if for each ideal J of R, there exists a finitely generated ideal $I \subseteq J$ and a natural number n such that $j^n \in I$ for each $j \in J$. This class of rings is extensively studied in [1, 3]. An SFT Prüfer domain (resp. SFT valuation domain) is a Prüfer domain (resp. valuation domain) that is also an SFT-ring. Recall that a valuation domain is said to be discrete if every primary ideal is a power of its radical. For a finite-dimensional valuation domain V, V is discrete if and only if V is an SFT-ring (see Lemma 2.7 and Proposition 3.1 of [1]). Although an SFT valuation domain is always a discrete valuation domain, the converse does not hold: see the example in [9]. In the proof of [4, Proposition 5], it is implicitly assumed that V is finite-dimensional. Using Lemma 1, we will fill this gap. In fact the prime ideals of $V[\![X]\!]$ satisfying the hypotheses of [4, Proposition 5] always satisfy those hypotheses of Lemma 1 as the next result shows. We also give a characterization of the prime elements of $V[\![X]\!]$, which is an answer to a question posed in [4].

Corollary 3 (see Arnold [4, Proposition 5]). *Let V be an SFT valuation domain* (*not necessarily finite-dimensional*) *and let Q be a prime ideal of $V[\![X]\!]$.*

(1) *If $Q \cap V = P$, $P[\![X]\!] \subsetneq Q$, and $Q \neq P + (X)$, then Q is a principal ideal. In fact, Q is generated by an element f such that $C_f = 1$.*

(2) *An element f of $V[\![X]\!]$ is a prime if and only if either f is irreducible and $C_f = 1$ or f is an associate of p, where (p) is the maximal ideal of V.*

Proof. (1) Let M be the maximal ideal of V. In view of [1, Corollary 3.6], it is obvious that $Q \nsubseteq M[\![X]\!]$. Moreover $X \notin Q$. Now the conclusion follows from Lemma 1.

(2) Since $M^2 \neq M$, M is a principal ideal, say $M = (p)$. Let (f) be a principal ideal of $V[\![X]\!]$ which is not an associate of p. Then $(f) \nsubseteq M[\![X]\!]$, for otherwise $(f) = P[\![X]\!]$ for a prime ideal P of V [1, Corollary 3.6]. In this case, P is a principal ideal and hence necessarily the maximal ideal M, a contradiction. So $C_f = 1$. For the reverse implication, let $f \in V[\![X]\!]$ be an irreducible element such that $C_f = 1$ and f is not an associate of X. It is not difficult to see that $X \notin \sqrt{(f)}$. So there exists a prime ideal Q minimal over (f), which does not contain X. Lemma 1 implies that Q is a principal ideal, say $Q = (g)$. Thus $(f) = (g)$ since f is irreducible, whence f is a prime element. □

Let D be a domain, (a) a principal ideal of D, and $\hat{D}$ the (a)-adic completion of D. Let $\theta : D \to \hat{D}$ be the canonical ring homomorphism and $\varphi : D[\![X]\!] \to \hat{D}$ be the natural ring epimorphism such that $\varphi(\sum_{i=0}^{\infty} a_i X^i) = \sum_{i=0}^{\infty} \theta(a_i) a^i$.

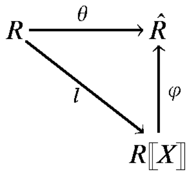
Lemma 4 (Greco and Salmon [7, Proposition 3.4]). $\ker \varphi = (X - a)$ *and thus* $\hat{D} \cong D[\![X]\!]/(X - a)$.

Proof. Let $g = a_0 + a_1 X + \cdots + a_k X^k + \cdots \in \ker \varphi$. In $\hat{D}$, $a_0 + a_1 a + a_2 a^2 + \cdots = 0$. If we regard $\hat{D}$ as $\varprojlim D/(a^n)$, then $a_0 \in (a)$, $a_0 + a_1 a \in (a^2)$, $a_0 + a_1 a + a_2 a^2 \in (a^3)$, …. Let $a_0 = ab_0$, $b_0 \in D$. From $a_0 + a_1 a = ab_0 + a_1 a \in (a^2)$, we get $a_1 = ab_1 - b_0$ for $b_1 \in D$. From $a_0 + a_1 a + a_2 a^2 \in (a^3)$, we get $a^2 b_1 + a_2 a^2 \in (a^3)$. So $a_2 = ab_2 - b_1$ for $b_2 \in D$. Proceeding this way, we get $a_{n+1} = ab_{n+1} - b_n$ for $b_0, b_1, \ldots, b_n, \ldots$ in D. From these, we deduce that $a_0 + a_1 X + \cdots = (X - a)(-b_0 - b_1 X - b_2 X^2 - \cdots - b_n X^n - \cdots)$. □

Let $\theta : D \to \hat{D}$ be the canonical ring homomorphism. Recall that $\ker \theta = \bigcap_{n=1}^{\infty} (a^n)$ [5].

Lemma 5. *Let R be a ring and a an element of R. Then $(X-a)R[\![X]\!]\cap R\subseteq\bigcap_{n=1}^{\infty}(a^n)$. If R is a domain, then the converse holds*: $(X-a)R[\![X]\!]\cap R=\bigcap_{n=1}^{\infty}(a^n)$.

Proof. Since the diagram



commutes, $(X-a)R[\![X]\!]\cap R\subseteq\ker\theta=\bigcap_{n=1}^{\infty}(a^n)$. Suppose R is a domain. By Lemma 4, $\bigcap_{n=1}^{\infty}(a^n)\subseteq\ker\varphi=(X-a)R[\![X]\!]$. $\square$

Theorem 6. *Let V be a valuation domain and a a nonunit element of V. Then $X-a$ is a prime element of $V[\![X]\!]$.*

Proof. We may assume that $a\neq 0$. Let $P=\bigcap_{n=1}^{\infty}(a^n)$, which is a prime ideal of V [6, Theorem 17.1]. P is contained in $(X-a)$ by Lemma 5. It is easy to see that $P[\![X]\!]\subseteq(X-a)$. By Lemma 5, $\sqrt{(X-a)}\cap V=P$ where $\sqrt{(X-a)}$ is the nil radical of $(X-a)$. Passing to V/P, we may assume that $\bigcap_{n=1}^{\infty}(a^n)=(0)$, so that $\sqrt{(X-a)}\cap V=(0)$. Pick a prime ideal Q minimal over $(X-a)$ such that $Q\cap V=(0)$. (Note that if $Q\cap V\neq(0)$ for every Q, then $Q\cap V\supseteq\sqrt{(a)}$, which is the height 1 prime ideal of V. This leads to $\sqrt{(X-a)}\cap V\supseteq\sqrt{(a)}$). Clearly, Q satisfies the hypotheses of Corollary 2. So Q is a principal ideal, say $Q=(f)$. We claim that $(f)=(X-a)$. Since $X-a\in Q$, $X-a=fg$ for a $g\in V[\![X]\!]$. Suppose g is not a unit. Then the coefficient of X in fg would not be a unit. So g is a unit and hence $X-a$ is a prime. $\square$

Theorem 7. *Let V be a valuation domain, I a proper ideal of V, and $\hat{V}$ the I-adic completion of V. Then $\hat{V}$ is a valuation domain and the value group $v(\hat{V})$ of $\hat{V}$ is isomorphic to $v(V/\bigcap_{n=1}^{\infty}I^n)$.*

Proof. If $I=I^2$, then I is a prime ideal by [6, Theorem 17.1]. So $\hat{V}\cong V/I$ is a valuation domain. Now let us assume that $I\neq I^2$. Choose a such that $a\in I\backslash I^2$. Since $I^2\subseteq(a)\subseteq I$, the (a)-adic completion of V is isomorphic to the I-adic completion of V by the bounded difference. Thus, we may assume that I is a principal ideal, say $I=(a)$. By Lemma 4 and Theorem 6, $\hat{V}$ is a domain. We identify $\hat{V}$ with $V[\![X]\!]/(X-a)$. To prove $\hat{V}$ is a valuation domain, let $\overline{f(X)}\in\hat{V}$. Let n be such that $\overline{f(X)}=a^n\overline{g(X)}$ and $a\nmid\overline{g(X)}$ where $\overline{g(X)}\in\hat{V}$. Such an n exists since $\bigcap_{i=1}^{\infty}(a^i\hat{V})=(0)$, which follows from the fact that $\hat{V}$ is complete w.r.t. the $a\hat{V}$ topology and so $\hat{\hat{V}}\cong\hat{V}$ canonically. (By [5, Lemma 10.1, Proposition 10.5], $\bigcap_{i=1}^{\infty}\widehat{(a^iV)}=(0)$. Since $\bigcap_{i=1}^{\infty}(a^i\hat{V})\subseteq\bigcap_{i=1}^{\infty}\widehat{(a^iV)}$, $\bigcap_{i=1}^{\infty}(a^i\hat{V})=(0)$.) Let $\overline{g(X)}=\overline{b_0+b_1X+\cdots+b_nX^n+\cdots}=\bar{b}_0+\bar{X}\overline{(b_1+b_2X+\cdots)}=\bar{b}_0+\bar{a}\overline{(b_1+b_2X+\cdots)}=\bar{b}_0+\bar{a}\bar{h}$, where $h=b_1+b_2X+\cdots$. Since $a\nmid\overline{g(X)}$ in $\hat{V}$, $a\nmid b_0$ in

V. So $b_0 \mid a$ in V and $a = b_0 c$ for a nonunit c of V. Now $\overline{g(X)} = \bar{b}_0(1+\bar{c}\bar{h})$ and $1+\bar{c}\bar{h}$ is a unit since $\bar{c}$ is a nonunit and $\hat{V}$ is a local ring. Thus $\overline{f(X)} = a^n \overline{g(X)} = a^n b_0 u = c'u$ where $a^n b_0 = cc'$ and u is a unit of $\hat{V}$. This completes the proof of the first assertion. Recall that $\hat{V} \cong (\widehat{V/\bigcap I^n})$. We assume that $\bigcap_{n=1}^{\infty} I^n = \{0\}$, and under this hypothesis, we show that $v(\hat{V}) \cong v(V)$. Then the general case easily follows: $v(\hat{V}) \cong v(\widehat{V/\bigcap I^n}) \cong v(V/\bigcap I^n)$. Let K (resp. F) be the quotient field of V (resp. $\hat{V}$), K^* (resp. F^*) the nonzero elements of K (resp. F), and $\mathscr{U}$ (resp. $\mathscr{V}$) the group of the units of V (resp. $\hat{V}$). Since the natural ring homomorphism $\theta : V \to \hat{V}$ is an injection, K can be embedded into F. Clearly $\mathscr{U} \subseteq \mathscr{V}$. Next we show that $\mathscr{V} \cap V = \mathscr{U}$. Let $\alpha \in \mathscr{V} \cap V$. Then $\bar{\alpha}$ is a unit of $\hat{V} = V[\![X]\!]/(X - a)$. So $\bar{\alpha} \notin \overline{M + (X)}$, which is the unique maximal ideal of $\hat{V}$, where M is the maximal ideal of V. Hence $\alpha \notin M$. So α is a unit of V, i.e., $\alpha \in \mathscr{U}$. From this, we obtain another embedding $\phi : K^*/\mathscr{U} \to F^*/\mathscr{V}$. It remains to show that ϕ is onto. Let $y = \overline{(a/b)} \in F^*/\mathscr{V}$; $a, b \in \hat{V}$. As is shown previously, $a = cu$, $b = dv$ for $c, d \in V$ and $u, v \in \mathscr{V}$. Now $\overline{a/b} = \overline{(c/d)uv^{-1}} = \phi(\overline{c/d})$. So ϕ is onto. □

Theorem 8. *Let V be a finite-dimensional valuation domain, I a nonidempotent ideal of V, and $\hat{V}$ the I-adic completion of V. Then*:

(1) $\dim \hat{V} = \dim(V/\bigcap_{n=1}^{\infty} I^n) = \dim V/I + 1$.

(2) $\operatorname{Spec}(\hat{V}) = \{P\hat{V} \mid P \in \operatorname{Spec}(V) \text{ and } P \supseteq \bigcap_{n=1}^{\infty} I^n\}$.

(3) *For $P_1, P_2 \in \operatorname{Spec}(V)$ with $P_1, P_2 \supseteq \bigcap_{n=1}^{\infty} I^n$, $P_1\hat{V} \supseteq P_2\hat{V} \Leftrightarrow P_1 \supseteq P_2$ and $P_1\hat{V} = P_2\hat{V} \Leftrightarrow P_1 = P_2$.*

Proof. (1) follows from Theorem 7. In proving (2), we give another proof of (1). Choose $a \in I \setminus I^2$. As in the proof of Theorem 7, we may assume that $I = (a)$. Since $\hat{V} \cong (\widehat{V/\bigcap_{n=1}^{\infty}(a^n)})$ and $V/\bigcap_{n=1}^{\infty}(a^n)$ is a valuation domain, we may also assume that $\bigcap_{n=1}^{\infty}(a^n) = \{0\}$. So a is contained in the minimal prime ideal P of V. Let Q be a prime ideal of $V[\![X]\!]$ properly containing $(X - a)$. By Corollary 2 and Theorem 6, $Q \cap V \neq \{0\}$. So $P \subseteq Q \cap V$, which implies $P + (X) \subseteq Q$. So $\dim \hat{V} \leq \dim(V[\![X]\!]/(P + (X))) + 1 = \dim(V/P) + 1 = \dim(V/I) + 1$. Let $0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be the chain of all the prime ideals of V. The chain $0 \subsetneq \overline{P_1 + (X)} \subsetneq \overline{P_2 + (X)} \subsetneq \cdots \subsetneq \overline{P_n + (X)}$ is a chain of prime ideals of $V[\![X]\!]/(X - a)$. So $\dim \hat{V} \geq n = \dim V/I + 1$. Hence $\dim \hat{V} = \dim V/I + 1 = n$ and so $0 \subsetneq \overline{P_1 + (X)} \subsetneq \overline{P_2 + (X)} \subsetneq \cdots \subsetneq \overline{P_n + (X)}$ is the chain of all prime ideals of $\hat{V}$. Note that $\overline{P_i + (X)} = P_i\hat{V}$.

(3) It is routine to check this. □

Now we consider the global case.

Lemma 9. *Let D be a finite-dimensional SFT Prüfer domain. Let $(a_1, \ldots, a_n) \subsetneq D$ be a proper ideal. Then any prime ideal Q of $D[\![X_1, \ldots, X_n]\!]$ containing $(X_1 - a_1, \ldots, X_n - a_n)$ has height $\geq n$.*

Proof. Let $Q_0 = Q \cap D[\![X_1, \ldots, X_{n-1}]\!]$. In [3], it is shown that $\dim D[\![X_1, \ldots, X_n]\!] = (\dim D)n + 1 < \infty$. According to [1], the power series ring over a non-SFT-ring is infinite-Krull-dimensional. From this, we deduce that $D[\![X_1,\ldots,X_{n-1}]\!]$ is an SFT-ring. Since $Q_0 \cdot D[\![X_n]\!] \subseteq Q$ and $D[\![X_1,\ldots,X_{n-1}]\!]$ is an SFT-ring, we have $Q_0[\![X_n]\!] = \sqrt{Q_0 \cdot D[\![X_1,\cdots,X_{n-1}]\!][\![X_n]\!]} \subseteq Q$ [2, Theorem 1]. Since $1 \notin Q$, $X_n - a_n \notin Q_0[\![X_n]\!]$. So $Q_0[\![X_n]\!] \subsetneq Q$. By induction on n, we get the inequality $\operatorname{ht} Q_0 \geq n-1$, so that $\operatorname{ht} Q_0[\![X_n]\!] \geq n-1$ (note that $(X_1 - a_1,\ldots,X_{n-1}-a_{n-1}) \subseteq Q_0$). So $\operatorname{ht} Q \geq n$. For the case $n=1$, note that any prime ideal containing $X-a$ is nonzero, and thus $\operatorname{ht} Q \geq 1$. □

Let D be a commutative ring, I an ideal of D, $\hat{D}$ the I-adic completion of D, and $S = \{i_\alpha \mid \alpha \in A\}$ a generating set of I. Let $\theta : D \to \hat{D}$ be the canonical mapping and $\varphi : D[\![\{X_\alpha \mid \alpha \in A\}]\!]_2 \to \hat{D}$ the canonical epimorphism defined by $\varphi(X_\alpha) = \theta(i_\alpha)$ for each $\alpha \in A$ and $\varphi(d) = \theta(d)$ for $d \in D$. For the definition of the second type power series ring $D[\![\{X_\alpha \mid \alpha \in A\}]\!]_2$, readers are referred to [6, p. 6].

Lemma 10. *If Q is a prime ideal of $D[\![\{X_\alpha \mid \alpha \in A\}]\!]_2$ containing $\ker\varphi$, then the radical $\sqrt{I}$ of I and $Q \cap D$ are not coprime. If $f \in \ker\varphi$, then the constant term of f is in I.*

Proof. Suppose $\sqrt{I} + Q \cap D = D$. For an $x \in \sqrt{I}$ and a $d \in Q \cap D$, $x + d = 1$. Choose l so that $x^l \in I$. For an $e \in Q \cap D$, we have $x^l + e = 1$. Put $y = x^l$. Since $y \in I$, $y = d_1 i_1 + \cdots + d_k i_k$ for $d_1,\ldots,d_k \in D$ and $i_1,\ldots,i_k \in S$. Put $h = -d_1(X_1 - i_1) - \cdots - d_k(X_k - i_k) + e$. Then $h \in Q$ since $(X_1 - i_1,\ldots,X_k - i_k) \subseteq \ker\varphi \subseteq Q$ and $e \in Q \cap D$. Now $h = (d_1 i_1 + \cdots + d_k i_k) + e - d_1 X_1 - d_2 X_2 - \cdots - d_k X_k = 1 - d_1 X_1 - \cdots - d_k X_k$ is a unit in $D[\![\{X_\alpha \mid \alpha \in A\}]\!]_2$, which contradicts that $Q \neq (1)$. So $\sqrt{I}$ and $Q \cap D$ are not coprime. Let $f = a_0 + f_1 + f_2 + \cdots \in \ker\varphi$, where f_i is homogeneous of degree i in $D[\{X_\alpha \mid \alpha \in A\}]$. If we realize $\hat{D}$ as the inverse limit $\varprojlim D/I^n$, then $0 = \varphi(f) = (\bar{a}_0, \overline{a_0 + f_1(X_\alpha = i_\alpha \mid \alpha \in A)}, \ldots)$. Since $\bar{a}_0 = 0$ in D/I, $a_0 \in I$. □

Let D be a finite-dimensional SFT Prüfer domain, $I = (a_1,\ldots,a_n)$ a finitely generated proper ideal of D, $\hat{D}$ the I-adic completion of D, and φ the canonical ring epimorphism from $D[\![X_1,\ldots,X_n]\!]$ to $\hat{D}$. For a nonzero prime ideal P of D, we denote by $\mathscr{B}(P)$ the prime ideal of D just below P.

Lemma 11. *Let D be a finite-dimensional SFT Prüfer domain. Suppose the radical $\sqrt{I}$ of I is a prime ideal P. Then* (1) *the prime ideal $P + (X_1,\ldots,X_n)$ is not minimal over* $\ker\varphi$ *and* (2) *if Q is a minimal prime ideal of* $\ker\varphi$, *then $Q \cap D = \mathscr{B}(P)$.*

Proof. (1) $P \nsubseteq \sqrt{\ker\varphi}$. For otherwise some power of P is contained in $\ker\varphi \cap D$, which is $\bigcap_{m=1}^{\infty} I^m = \mathscr{B}(P)$. This would lead to the contradiction $P \subseteq \mathscr{B}(P)$. Thus $P \nsubseteq \sqrt{\ker\varphi}$ and so there exists a prime ideal Q minimal over $\ker\varphi$ such that $P \nsubseteq Q$. By Lemma 10, $Q \cap D$ and P are not coprime. Since D is a Prüfer domain, this implies that $Q \cap D$ and P are comparable. So either $Q \cap D \subseteq P$ or $P \subseteq Q \cap D$. However $P \nsubseteq Q$. So $Q \cap D \subsetneq P$. Since $\mathscr{B}(P) \subseteq Q \cap D \subset P$, $Q \cap D = \mathscr{B}(P)$. Let Q_1 be a prime ideal just above Q so that height $(Q_1/Q)=1$. By Lemma 9, height $\bar{Q}\geq n$ in $(D/\mathscr{B}(P))[\![X_1,\dots,X_n]\!]$, and so height $Q_1\geq n+1$, from which it follows that $\bar{P}_1=\bar{Q}_1\cap\bar{D}\neq\{0\}$ [3, Lemma 3.5]. By [4, Lemma 1], $Q_1\subseteq P_1+(X_1,\dots,X_n)$ and $\mathrm{ht}\,(P_1/\mathscr{B}(P))=1$. Since $I\subseteq P_1$, $P=\sqrt{I}\subseteq P_1$ and so $P=P_1$. Since $\ker\varphi\subseteq Q\subsetneq Q_1\subseteq P_1+(X_1,\dots,X_n)=P+(X_1,\dots,X_n)$, $P+(X_1,\dots,X_n)$ is not minimal over $\ker\varphi$. (2) Let Q be a minimal prime ideal of $\ker\varphi$. By (1), $P\not\subseteq Q$. Then the proof of (1) validates the claim. □

Theorem 12. *Let D be a finite-dimensional SFT Prüfer domain, I a proper ideal of D, and $\hat{D}$ the I-adic completion of D. Then* $\dim\hat{D}=\dim D/I+1$.

Proof. Let $P_1,\dots,P_m$ be the minimal primes of I [1, Corollary 2.6]. $\sqrt{I}=P_1\cap\cdots\cap P_m$. For any $l\geq 0$, $(\sqrt{I})^l=P_1^l\cap\cdots\cap P_m^l=P_1^l\cdots P_m^l$ since $P_1^l,\dots,P_m^l$ are pairwise coprime. By the Chinese remainder theorem, $D/(\sqrt{I})^l\cong D/P_1^l\oplus\cdots\oplus D/P_m^l$. So $\hat{D}\cong\hat{D}_1\oplus\cdots\oplus\hat{D}_m$, where $\hat{D}_i$ is the P_i-adic completion of D. Since $\dim\hat{D}=\max_i(\dim\hat{D}_i)$, we will assume that $\sqrt{I}=P$ is a prime ideal and $I=(a_1,\dots,a_n)$. Let $\varphi:D[\![X_1,\dots,X_n]\!]\to\hat{D}$ be the canonical ring epimorphism. Since $\hat{D}\cong\widehat{(D/\bigcap_{m=1}^{\infty}I^m)}$, we may assume that $\bigcap_{m=1}^{\infty}I^m=\{0\}$. Since $\mathscr{B}(P)=\bigcap_{m=1}^{\infty}P^m=\bigcap_{m=1}^{\infty}I^m=\{0\}$, $\mathrm{ht}\,P=1$. Let $l=\dim\hat{D}$ and $\ker\varphi\subseteq Q_0\subset\cdots\subset Q_l$ be a chain of prime ideals which gives the dimension l. By Lemma 9, $\mathrm{ht}\,Q_0\geq n$. By [3, Lemma 3.5], $Q_1\cap D\neq\{0\}$. By Lemma 10, P and $Q_1\cap D$ are comparable. Since $Q_1\cap D\neq\{0\}$ and $\mathrm{ht}\,P=1$, $P\subseteq Q_1\cap D$. So $(X_1,\dots,X_n)\subseteq Q_1$ and hence $P+(X_1,\dots,X_n)\subseteq Q_1$. From this, we deduce that $l-1\leq\dim D/P$. So $l\leq\dim D/P+1=\dim D/I+1$. For the reverse inequality, let $m=\dim D/I$ and let $I\subseteq P_0\subset\cdots\subset P_m$ be a chain of prime ideals of D which gives the dimension of D/I. By Lemma 10, $\ker\varphi\subseteq I+(X_1,\dots,X_n)\subseteq P_0+(X_1,\dots,X_n)$. Now $\ker\varphi\subset P_0+(X_1,\dots,X_n)\subset\cdots\subset P_m+(X_1,\dots,X_n)$. By Lemma 11, $P_0+(X_1,\dots,X_n)$ is not minimal over $\ker\varphi$. So $\dim\hat{D}\geq m+1=\dim D/I+1$. Therefore, $\dim\hat{D}=\dim D/I+1$. □

Lemma 13. *Let R be a ring, I a finitely generated ideal of R such that $\bigcap_{n=1}^{\infty}I^n=(0)$, and $\hat{R}$ the I-adic completion of R. Then*

(1) $I^n\hat{R}=\widehat{(I^n)}$ *and so* $R/I^n\cong\hat{R}/I^n\hat{R}$ *canonically*,

(2) *if M is a maximal ideal of R such that $I\subseteq M$, then $M\hat{R}$ is a maximal ideal of $\hat{R}$ and $M\hat{R}=\hat{M}$, and*

(3) $\mathrm{Max}(\hat{R})=\{\hat{M}\mid M\in\mathrm{Max}(R)$ *and* $M\supseteq I\}$.

Proof. (1) By [5, Proposition 10.13], $I^n\hat{R}=\widehat{(I^n)}$. By [5, Proposition 10.4], $R/I^n\cong\hat{R}/\widehat{(I^n)}$. So $R/I^n\cong\hat{R}/I^n\hat{R}$. (2) Let $I=(a_1,\dots,a_l)$. We identify $\hat{R}$ with $R[\![X_1,\dots,X_l]\!]/\ker\varphi$ through the canonical ring epimorphism $\varphi:R[\![X_1,\dots,X_l]\!]\to\hat{R}$. Obviously $\ker\varphi\supseteq(X_1-a_1,\dots,X_l-a_l)$. So $M+(X_1-a_1,\dots,X_l-a_l)=M+(X_1,\dots,X_l)$. Since $M\hat{R}\supseteq\overline{M+(X_1-a_1,\dots,X_l-a_l)}$, $M\hat{R}\supseteq\overline{M+(X_1,\dots,X_l)}$. Now $\overline{M+(X_1,\dots,X_l)}\subseteq M\hat{R}\subseteq\hat{M}\subsetneq\hat{R}$. Since $\overline{M+(X_1,\dots,X_l)}$ is a maximal ideal of $R[\![X_1,\dots,X_l]\!]/\ker\varphi$, we conclude that $\overline{M+(X_1,\dots,X_l)}=M\hat{R}=\hat{M}$. (3) Statement (2) implies $\mathrm{Max}(\hat{R})\supseteq\{\hat{M}\mid M\in\mathrm{Max}(R)$ and $M\supseteq I\}$. Every maximal ideal of $R[\![X_1,\dots,X_l]\!]/\ker\varphi$ is of the form

$\overline{M+(X_1,\ldots,X_l)}$, where M is a maximal ideal of R and $M+(X_1,\ldots,X_l)\supseteq\ker\varphi$. Since $X_1-a_1,\ldots,X_l-a_l\in\ker\varphi$, $(a_1,\ldots,a_l)\subseteq M$. Also note that $\overline{M+(X_1,\ldots,X_l)}=M(R[\![X_1,\ldots,X_l]\!]/\ker\varphi)=M\hat{R}$. $\square$

We present a partial converse of Theorem 7.

Lemma 14. *Let R be a ring and I a finitely generated ideal of R such that $\bigcap_{n=1}^{\infty} I^n=(0)$ and I is contained in the Jacobson radical $J(R)$ of R. If the I-adic completion $\hat{R}$ of R is a valuation domain, then R is also a valuation domain.*

Proof. Since $\bigcap_{n=1}^{\infty} I^n=(0)$, R can be embedded into $\hat{R}$ through the canonical homomorphism $\theta: R\to\hat{R}$. So R is an integral domain. Let K be the quotient field of R. Since $\hat{R}$ is a valuation domain, $\hat{R}\cap K$ is also a valuation domain. We claim that $\hat{R}\cap K=R$. It suffices to show that $a\hat{R}\cap R=aR$ for all $a\in R$. Let $0\neq a\in R$. Since $\bigcap_{n=1}^{\infty} I^n\hat{R}=(0)$ and $\hat{R}$ is a valuation domain, there exists an $n\geq 1$ such that $I^n\hat{R}\subseteq a\hat{R}$. So $I^n\subseteq a\hat{R}\cap R$. From this and the fact that $a\hat{R}\cap R\subseteq\bigcap_{k=1}^{\infty}(aR+I^k)$, it follows that $aR+I^n=aR+I^{n+1}=\cdots$. Let $\bar{R}=R/aR$, $\bar{I}=(I+aR)/aR$. Then $\bar{I}$ is a finitely generated ideal of $\bar{R}$ and $\bar{I}\subseteq J(\bar{R})$. Recall that $aR+I^n=aR+I^{n+1}$. From this, we get another observation that $(\bar{I})^n=\overline{(I^n)}=\overline{(I^{n+1})}=(\bar{I})^{n+1}$. By Nakayama's Lemma, $(\bar{I})^n=0$, i.e., $I^n+aR\subseteq aR$. Thus, $a\hat{R}\cap R\subseteq I^n+aR=aR$, and hence $a\hat{R}\cap R=aR$. $\square$

Theorem 15. *Let D be an SFT Prüfer domain, I a proper ideal of D, and $\hat{D}$ the I-adic completion of D. Then*:
(1) *$\hat{D}$ is an SFT Prüfer ring.*
(2) *$\hat{D}$ is an SFT Prüfer domain if and only if radical $\sqrt{I}$ of I is a prime ideal.*

Proof. Let $\{P_1,\ldots,P_r\}$ be the set of minimal prime ideals of I. As is shown in the proof of Theorem 12, $\hat{D}\cong\hat{D}_1\oplus\cdots\oplus\hat{D}_r$, where $\hat{D}_i$ is the P_i-adic completion of D for $i=1,\ldots,r$. Now, let P be a prime ideal of D and $\hat{D}$ the P-adic completion of D. We will show that $\hat{D}$ is a Prüfer domain. Put $Q=\bigcap_{n=1}^{\infty}P^n$. Then Q is a prime ideal of D [6, Theorem 23.3(b) and (d)] and D/Q is an SFT Prüfer domain [3]. Since $\hat{D}\cong(\widehat{D/Q})$, we may assume that $Q=(0)$, i.e., $\bigcap_{n=1}^{\infty}P^n=(0)$. Since D is an SFT-ring, there exists a finitely generated ideal J of D contained in P and $l\geq 1$ such that $P^l\subseteq J$. Then, by the bounded difference, the P-adic completion of D is isomorphic to the J-adic completion of D. Now we replace P by J. Let $M\in\operatorname{Max}(D)$ be such that $M\supseteq J$. The natural mapping $\hat{D}\xrightarrow{\iota}(\widehat{D_M})$ is an injection: let $x=(\bar{x}_1,\bar{x}_2,\cdots)\in\varprojlim D/J^n$ be such that $\iota(x)=(\bar{x}_1,\bar{x}_2,\cdots)=0$ in $\varprojlim D_M/J^nD_M$. For an arbitrary integer $k\geq 1$, $x_{lk}\in J_M^{lk}\cap D\subseteq P_M^{lk}\cap D=P^{lk}\subseteq J^k$. Since $\bar{x}_k=\bar{x}_{lk}$ in D/J^k, $\bar{x}_k=0$ in D/J^k for all $k\geq 1$, which implies that $x=0$ and so ι is an injection. Since D_M is a valuation domain, $(\widehat{D_M})$ is also a valuation domain by Theorem 7. So the subring $\hat{D}$ is an integral domain. Note that $(\widehat{\hat{D}_{M\hat{D}}})=\varprojlim(\hat{D}_{M\hat{D}}/J^n\hat{D}_{M\hat{D}})\cong\varprojlim(\hat{D}/J^n\hat{D})_{M\hat{D}/J^n\hat{D}}\cong\varprojlim(\hat{D}/(\widehat{J^n}))_{\hat{M}/(\widehat{J^n})}\cong\varprojlim(D/J^n)_{M/J^n}\cong\varprojlim D_M/J_M^n=(\widehat{D_M})$, where [6, Proposition 5.8] is used for the first isomorphism and the fourth, the second isomorphism follows from Lemma 13, and the third isomorphism follows from Lemma 13 and [5, Corollary 10.4]. Thus $\widehat{(\hat{D}_{M\hat{D}})} \overset{\phi}{\cong} \widehat{(D_M)}$ is a valuation domain. We observed that $\hat{D} \overset{\iota}{\to} \widehat{(D_M)}$ is an injection. We claim that $\hat{D} \backslash M\hat{D} \subseteq$ the set of units of $\widehat{(D_M)}$, so that ι induces an embedding $\hat{D}_{M\hat{D}} \overset{\iota}{\to} \widehat{(D_M)}$: Since $M\widehat{D_M} \cap \hat{D} \supseteq M\hat{D}$, $M\hat{D}$ is a maximal ideal of $\hat{D}$ (Lemma 13), and $1 \notin M\widehat{D_M}$, we have $M\widehat{D_M} \cap \hat{D} = M\hat{D}$. So $\hat{D} \backslash M\hat{D} \subseteq \widehat{D_M} \backslash M\widehat{D_M}$. By Theorem 7, $M\widehat{D_M}$ is a maximal ideal of the valuation domain $\widehat{(D_M)}$, i.e., the unique maximal ideal of $\widehat{(D_M)}$. Thus $\hat{D} \backslash M\hat{D}$ is a set of units of $\widehat{(D_M)}$. Let ϕ be the natural isomorphism $\widehat{(D_M)} \overset{\phi}{\underset{\sim}{\to}} \widehat{(\hat{D}_{M\hat{D}})}$ obtained earlier. Consider the composition $\hat{D}_{M\hat{D}} \overset{\iota}{\hookrightarrow} \widehat{(D_M)} \overset{\phi}{\underset{\sim}{\to}} \widehat{(\hat{D}_{M\hat{D}})}$, which is identical with the natural ring homomorphism $\theta : \hat{D}_{M\hat{D}} \to \widehat{(\hat{D}_{M\hat{D}})}$, i.e., $\theta = \phi \circ \iota$. Since ι an injection and ϕ is an isomorphism, $\theta = \phi \circ \iota$ is an injection and hence $\bigcap_{n=1}^{\infty} (J^n \hat{D}_{M\hat{D}}) = \ker \theta = (0)$. By Lemma 14, $\hat{D}_{M\hat{D}}$ is also a valuation domain. In view of Lemma 13, $\mathrm{Max}(\hat{D}) = \{M\hat{D} \mid M \in \mathrm{Max}(D) \text{ and } M \supseteq J\}$. Hence $\hat{D}$ is a Prüfer domain. This completes the 'if' half of (2). In the general case, $\hat{D} \cong \hat{D}_1 \oplus \cdots \oplus \hat{D}_r$, where each $\hat{D}_i$ is a Prüfer domain. So $\hat{D}$ is a Prüfer domain if and only if $r = 1$, i.e., $\sqrt{I} = P$ is a prime ideal. Thus (2) is done. We are ready to prove (1). Put $Q_i = \hat{D}_1 \oplus \cdots \oplus \hat{D}_{i-1} \oplus \{0\} \oplus \hat{D}_{i+1} \oplus \cdots \oplus \hat{D}_r$. Then the set of minimal prime ideals of $\hat{D}$ is $\{Q_1, \ldots, Q_r\}$ and $Q_1 \cap \cdots \cap Q_r = \{0\}$. So $\hat{D}$ is a reduced ring. Note that $\hat{D}/Q_i \cong \hat{D}_i$. Let K_i be the quotient field of $\hat{D}_i$. By [8, Lemma 8.14], the total quotient ring $T(\hat{D})$ of $\hat{D}$ is given by $T(\hat{D}) \cong K_1 \oplus \cdots \oplus K_r$. Let S be an overring of $\hat{D}$. Then, $S \cong \pi_1(S) \oplus \cdots \oplus \pi_r(S)$, where $\pi_i : T(\hat{D}) \to K_i$ is the natural projection. Since $\hat{D}_i$ is a Prüfer domain and $\pi_i(S)$ is an overring of $\hat{D}_i$, we have $\pi_i(S)$ is integrally closed [6, Theorem 26.2] and therefore S is also integrally closed in $T(\hat{D})$. Thus each overring of the ring $\hat{D}$ is integrally closed and hence $\hat{D}$ is a Prüfer ring [8, Theorem 6.2]. □

In [4], Arnold showed that for an SFT Prüfer domain D, $D[\![X_1, \ldots, X_n]\!]$ is not catenarian if and only if $\dim D \geq 2$ and $n \geq 2$. However, the completion $\hat{D}$, which is a quotient ring of some power series ring $D[\![X_1, \ldots, X_n]\!]$, turns out to be catenarian as is shown in the following corollary.

Corollary 16. *If D is a finite-dimensional SFT Prüfer domain, then $\hat{D}$ is a catenarian ring.*

Proof. Let $\sqrt{I} = P_1 \cap \cdots \cap P_m$, where $P_1, \ldots, P_m$ are the minimal primes of I. As is shown in the proof of Theorem 12, $\hat{D} \cong \bigoplus_{i=1}^{m} \hat{D}_i$, where $\hat{D}_i$ is the P_i-adic completion of D. It suffices to show that each $\hat{D}_i$ is catenarian, which is the case since $\hat{D}_i$ is a Prüfer domain by Theorem 15. □

Definition. A partially ordered set S is called a *tree* if every incomparable two elements do not have an upper bound in S. Two trees are said to be *isomorphic* if there exists between them a bijection which preserves partial orders. For a Prüfer domain D, $\mathrm{Spec}(D)$ is a tree w.r.t. the set-theoretic inclusion.

Definition. For a commutative ring R, $X^i(R)=\{P \mid P\in \mathrm{Spec}(R) \text{ and } \mathrm{ht}(P)=i\}$ and $\mathrm{Spec}^+(R)=\{P \mid P\in \mathrm{Spec}(R) \text{ and } \mathrm{ht}(P)>0\}$.

Corollary 17. *Let D be a finite-dimensional SFT Prüfer domain and let m be the number of minimal prime ideals of I.*

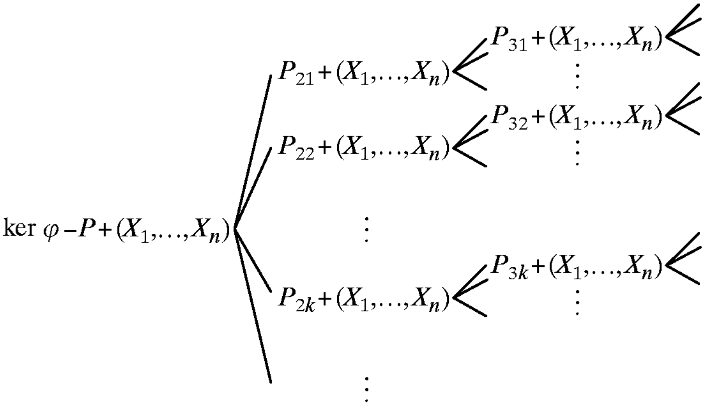
(1) *Both $X^0(\hat{D})$ and $X^1(\hat{D})$ are m-point sets.*
(2) $\mathrm{Spec}^+(\hat{D})=\{Q\hat{D} \mid Q\in \mathrm{Spec}(D)$ *and* $Q\supseteq I\}$.
(3) *For Q_1 and $Q_2\in \mathrm{Spec}^+(\hat{D})$, $Q_1=Q_2 \Leftrightarrow \theta^{-1}(Q_1)=\theta^{-1}(Q_2)$, and $Q_1\supseteq Q_2 \Leftrightarrow \theta^{-1}(Q_1)\supseteq\theta^{-1}(Q_2)$, where $\theta: D\to\hat{D}$ is the canonical ring homomorphism.*
(4) $\mathrm{Spec}^+(\hat{D})\cong \mathrm{Spec}(D/I)$ *as trees.*

Proof. (1) and (2) First we prove the corollary for the case when I is a prime ideal P. Since D is an SFT-ring, there exists a finitely generated ideal $J=(a_1,\dots,a_n)$ such that $\sqrt{J}=P$. The J-adic completion of D is isomorphic to the P-adic completion of D. Let $\varphi: D[\![X_1,\dots,X_n]\!]\to\hat{D}$ be the canonical epimorphism. We will give a complete description of $\mathrm{Spec}(\hat{D})$: (0) is the minimal prime ideal, the prime ideal $\overline{P+(X_1,\dots,X_n)}$ is the unique height 1 prime ideal of $\hat{D}$, and the other prime ideals are precisely the set $\{\overline{P'+(X_1,\dots,X_n)} \mid P'$ is a prime ideal of D such that $P'\supset P\}$. Let $Q_0\subset Q_1$ be prime ideals in $D[\![X_1,\dots,X_n]\!]$ such that $\ker\varphi=Q_0$ (see Theorem 15.). As in the proof of Lemma 11(1), one can easily show that $Q_1\cap D\supsetneq \mathscr{B}(P)$. By Lemma 10, P and $Q_1\cap D$ are comparable. Hence, $P\subseteq Q_1\cap D$. For otherwise $\mathscr{B}(P)\subsetneq Q_1\cap D\subsetneq P$, a contradiction. So $P+(X_1,\dots,X_n)\subseteq Q_1$ and hence $Q_1=P_1+(X_1,\dots,X_n)$ for a prime ideal P_1 of D such that $P_1\supseteq P$.

Back to the general case, let $\sqrt{I}=P_1\cap\cdots\cap P_m$, where $P_1,\dots,P_m$ are minimal prime ideals of I. As in the proof of Theorem 12, $\hat{D}\cong\hat{D}_1\oplus\cdots\oplus\hat{D}_m$. A nonminimal prime ideal of $\hat{D}$ is of the form $\hat{D}_1\oplus\cdots\oplus Q_i'\oplus\cdots\oplus\hat{D}_m$, where Q_i' is a nonzero prime ideal of $\hat{D}_i$. By the special case, we have $Q_i'=Q_i\hat{D}_i$, where $Q_i\in\mathrm{Spec}(D)$ and $Q_i\supseteq P_i$. Now $\hat{D}_1\oplus\cdots\oplus Q_i'\oplus\cdots\oplus\hat{D}_m=\hat{D}_1\oplus\cdots\oplus Q_i\hat{D}_i\oplus\cdots\oplus\hat{D}_m=Q_i(\hat{D}_1\oplus\cdots\oplus\hat{D}_i\oplus\cdots\oplus\hat{D}_m)=Q_i\hat{D}$. (Note that if a prime ideal Q of D contains P_i, then $Q\not\supseteq P_j$ for any $j\neq i$ since P_i and P_j, $i\neq j$, are incomparable. So $Q\hat{D}_j=(1)$. Moreover, if a prime ideal Q of D contains $\sqrt{I}=P_1\cap\cdots\cap P_m$, then Q contains exactly one P_i since D is a Prüfer domain.) Statements (1) and (2) are completed. It is routine to check (3) and (4). $\square$

Remark. In the local case, $\mathrm{Spec}(\hat{V})\cong\mathrm{Spec}(V/\bigcap_{n=1}^{\infty}I^n)$ (see Theorems 7 or 8). However, $\mathrm{Spec}(\hat{D})\ncong\mathrm{Spec}(D/\bigcap_{n=1}^{\infty}I^n)$ if there are prime ideals of D (other than $P_1,\dots,P_m$) that are just above $\mathscr{B}(P_1),\dots,\mathscr{B}(P_m)$. Moreover, $\mathrm{Spec}(\hat{D})\not\hookrightarrow\mathrm{Spec}(D/\bigcap_{n=1}^{\infty}I^n)$ if it happens that $\mathscr{B}(P_i)=\mathscr{B}(P_j)$ for distinct $i\neq j$, which would force $|X^0(D/\bigcap_{n=1}^{\infty}I^n)|<m=|X^0(\hat{D})|$.

We give a pictogram of the spectrum of $\hat{D}$ in terms of prime ideals in the power series, where $\hat{D}$ is the completion of D w.r.t. the prime ideal P:



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