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Leverrier–Chebyshev algorithm for the singular pencils

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Abstract

The Leverrier–Chebyshev algorithm is presented. It is an extension of the Leverrier–Fadeev algorithm for simultaneous computation of the adjoint $B(\mu)$ and determinant $a(\mu)$ of the singular pencil $\mu E - A$, where E is singular, but $\det(\mu E - A) \not\equiv 0$. We express the $B(\mu)$ and $a(\mu)$ relative to a basis of orthogonal Chebyshev polynomial, and as such solve a problem of S. Barnett. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

It is clearly seen that the Leverrier–Fadeev algorithm can be applied to compute the matrix $\lambda I - A$ [1], where I denotes the $n \times n$ unit matrix, and $\det(\lambda I - A) \neq 0$. Let

$$(\lambda I - A)^{-1} = B(\lambda)/a(\lambda), \tag{1}$$

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where

$$B(\lambda) = \mathrm{adj}(\lambda I - A) = \lambda^{n-1} I + \lambda^{n-2}\tilde{B}_1 + \cdots + \lambda \tilde{B}_{n-2} + \tilde{B}_{n-1} \tag{2}$$

and

$$a(\lambda) = \det(\lambda I - A) = \lambda^n + \tilde{a}_1 \lambda^{n-1} + \cdots + \tilde{a}_{n-1}\lambda + \tilde{a}_n. \tag{3}$$

Then the coefficients a_k and the matrices B_k can be found alternatively from

$$\tilde{a}_1 = -\mathrm{tr}\, A, \quad \tilde{a}_k = -\frac{1}{k}\mathrm{tr}\,(A\tilde{B}_{k-1}), \quad k = 2, 3, \ldots, n, \tag{4}$$

$$\tilde{B}_1 = A + \tilde{a}_1 I, \quad \tilde{B}_k = \tilde{a}_k I + A\tilde{B}_{k-1}, \quad k = 2, 3, \ldots, n-1, \tag{5}$$

where tr denotes the trace.

In [2], the Leverrier–Fadeev algorithm is extended to the case when both $B(\lambda)$ and $a(\lambda)$ are expressed relative to a basis $\{P_i(\lambda)\}$ of orthogonal polynomials. These can be defined by the standard three-term recurrence relation [3]

$$P_i(\lambda) = (\alpha_i \lambda + \beta_i) P_{i-1}(\lambda) - \gamma_i P_{i-2}(\lambda), \quad i \geqslant 2, \tag{6}$$

with $P_0(\lambda) = 1,\ P_1(\lambda) = \alpha_1 \lambda + \beta_1$.

Expressions (2) and (3) are replaced by the generalized polynomial forms

$$B(\lambda) = \mathrm{adj}(\lambda I - A) = \frac{P_{n-1}(\lambda)B_0 + P_{n-2}(\lambda)B_1 + \cdots + P_0(\lambda)B_{n-1}}{\alpha_1 \cdots \alpha_{n-1}}, \tag{7}$$

$$a(\lambda) = \det(\lambda I - A) = \frac{P_n(\lambda) + a_1 P_{n-1}(\lambda) + \cdots + a_n P_0(\lambda)}{\alpha_1 \cdots \alpha_n}, \tag{8}$$

where the scaling factors are necessary because the coefficient of λ^n in $P_n(\lambda)$ is $\alpha_1 \alpha_2 \cdots \alpha_n$,

$$B_0 = I, \quad B_1 = \alpha_{n-1}\left(\frac{a_1}{\alpha_n} I + \frac{\beta_n}{\alpha_n} B_0 + AB_0\right), \tag{9}$$

$$B_k = \alpha_{n-k}\left(\frac{a_k}{\alpha_n} I - \frac{\gamma_{n-k+2}}{\alpha_{n-k+2}} B_{k-2} + \frac{\beta_{n-k+1}}{\alpha_{n-k+1}} B_{k-1} + AB_{k-1}\right), \quad k = 2, 3, \ldots, n-1 \tag{10}$$

and

$$a_1 = -\alpha_n\left(\mathrm{tr}\, A + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}\right). \tag{11}$$

The other coeffients in (8) can be obtained by

$$\frac{\mathrm{d}a(\lambda)}{\mathrm{d}\lambda} = \mathrm{tr}\, B(\lambda). \tag{12}$$

In this paper, an algorithm for simultaneous computing the adjoint $B(\mu)$ and the determinant $a(\mu)$ of the singular pencil $\mu E - A$ is given, where E is singular, but

$\det(\mu E - A) \not\equiv 0$, which arises in singular linear control problem [6,7]. The adjoint $B(\mu)$ and the determinant $a(\mu)$ are expressed with respect to a basis of Chebyshev orthogonal polynomial.

2. General orthogonal basis

Identities (9)–(12) remain valid if A is replaced by a matrix $A(\mu)$, and likewise a_k by $a_k(\mu)$ and B_k by $B_k(\mu)$ for some indeterminate μ. i.e.,

$$\tilde{a}(\lambda) = \det(\lambda I - A(\mu)) = \frac{P_n(\lambda) + a_1(\mu)P_{n-1}(\lambda) + \cdots + a_n(\mu)P_0(\lambda)}{\alpha_1 \cdots \alpha_n}, \tag{13}$$

$$\begin{aligned}\tilde{B}(\lambda) &= \operatorname{adj}(\lambda I - A(\mu)) \\ &= \frac{P_{n-1}(\lambda)B_0(\mu) + P_{n-2}(\lambda)B_1(\mu) + \cdots + P_0(\lambda)B_{n-1}(\mu)}{\alpha_1 \cdots \alpha_{n-1}},\end{aligned} \tag{14}$$

$$B_0(\mu) = I, \quad B_1(\mu) = \alpha_{n-1}\left(\frac{a_1(\mu)}{\alpha_n} I + \frac{\beta_n}{\alpha_n} B_0(\mu) + A(\mu)B_0(\mu)\right), \tag{15}$$

$$\begin{aligned}B_k(\mu) = \alpha_{n-k}\Bigg(&\frac{a_k(\mu)}{\alpha_n} I - \frac{\gamma_{n-k+2}}{\alpha_{n-k+2}} B_{k-2}(\mu) + \frac{\beta_{n-k+1}}{\alpha_{n-k+1}} B_{k-1}(\mu) \\ &+ A(\mu)B_{k-1}(\mu)\Bigg), \quad k = 2, 3, \ldots, n-1,\end{aligned} \tag{16}$$

and

$$a_1(\mu) = -\alpha_n\left(\operatorname{tr} A(\mu) + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}\right), \tag{17}$$

$$\frac{\mathrm{d}\tilde{a}(\lambda)}{\mathrm{d}\lambda} = \operatorname{tr}(\tilde{B}(\lambda)). \tag{18}$$

The case of interest is $A(\mu) = -\mu E + A$, where E is singular but $\det(-\mu E + A) \not\equiv 0$. It follows from (13) and (14), we have

$$\begin{aligned}a(\mu) &= \det(\mu E - A) \\ &= \tilde{a}(0) \\ &= \frac{P_n(0) + a_1(\mu)P_{n-1}(0) + \cdots + a_n(\mu)P_0(0)}{\alpha_1 \cdots \alpha_n},\end{aligned} \tag{19}$$

$$\begin{aligned}B(\mu) &= \operatorname{adj}(\mu E - A) \\ &= \tilde{B}(0) \\ &= \frac{P_{n-1}(0)B_0(\mu) + P_{n-2}(0)B_1(\mu) + \cdots + P_0(0)B_{n-1}(\mu)}{\alpha_1 \cdots \alpha_{n-1}}\end{aligned} \tag{20}$$

and

$$\frac{\mathrm{d}a(\mu)}{\mathrm{d}\mu} = \mathrm{tr}\,(B(\mu)). \tag{21}$$

In principle, we can solve this problem, but in practice it is too complicated for the expressions which are obtained for a general basis of orthogonal polynomial.

3. Leverrier–Chebyshev algorithm

We consider the second kind of Chebyshev polynomials defined by

$$S_1(\mu) = \mu, \quad S_i(\mu) = \mu S_{i-1}(\mu) - S_{i-2}(\mu). \tag{22}$$

Since $\alpha_i = 1,\ \beta_i = 0,\ \gamma_i = 1\ \forall i$, Eq. (16) reduces to

$$B_k(\mu) = a_k(\mu)I - B_{k-2}(\mu) + A(\mu)B_{k-1}(\mu), \quad k \geqslant 2. \tag{23}$$

Taking the trace of both side of (23), and using the method in [2], we have

$$-ka_k(\mu) = \mathrm{tr}\,(A(\mu)B_{k-1}(\mu)) - 2(n-k+1)(a_{k-2}(\mu) + a_{k-4}(\mu) + \cdots), \quad k \geqslant 2. \tag{24}$$

Since the degree of $S_k(\mu)$ is k, then it is seen from (23) and (24) that the degree of the polynomial matrix $B_k(\mu),\ k = 1, 2, \ldots, n-1$, and of the polynomial quantity $a_k(\mu),\ k = 1, 2, \ldots, n$, is at most equal to k. Hence $B_i(\mu)$ and $a_i(\mu)$ can be written as

$$B_i(\mu) = \sum_{k=0}^{i} B_{i,k} S_k(\mu), \quad i = 1, 2, \ldots, n-1, \tag{25}$$

$$a_i(\mu) = \sum_{k=0}^{i} a_{i,k} S_k(\mu), \quad i = 1, 2, \ldots, n. \tag{26}$$

Substituting (25) and (26) in the recursive relations $a_k(\mu),\ k = 1, 2, \ldots, n$, and $B_k(\mu),\ k = 1, 2, \ldots, n-1$, in (24), and we use (22) to get

$$\mu S_{i-1}(\mu) = S_i(\mu) + S_{i-2}(\mu), \quad i \geqslant 2, \tag{27}$$

then we obtain the following general relations by equating the coefficients of the orthogonal polynomials $S_i(\mu),\ i = 0, 1, \ldots, k$, on the two sides of each equation.

Leverrier–Chebyshev algorithm. The coefficients $a_{i,k}$ in (26) and $B_{i,k}$ in (25), when the basis is the set of Chebyshev polynomials, are sequentially given by

$$
\begin{aligned}
a_{1,0} &= -\mathrm{tr}\,(A),\\
a_{1,1} &= \mathrm{tr}\, E,\\
a_{i,0} &= -\tfrac{1}{i}\mathrm{tr}\left[AB_{i-1,0} - EB_{i-1,1}\right] + \tfrac{2}{i}(n-i+1)\left[a_{i-2,0} + a_{i-4,0} + \cdots\right],\\
a_{i,k} &= -\tfrac{1}{i}\mathrm{tr}\left[AB_{i-1,k} - EB_{i-1,k-1} - EB_{i-1,k+1}\right]\\
&\qquad +\tfrac{2}{i}(n-i+1)\left[a_{i-2,k} + a_{i-4,k} + \cdots\right], \quad k = 1,2,\ldots,i-2,\\
a_{i,i-1} &= -\tfrac{1}{i}\mathrm{tr}\left[AB_{i-1,i-1} - EB_{i-1,i-2}\right],\\
a_{i,i} &= \tfrac{1}{i}\mathrm{tr}\left[EB_{i-1,i-1}\right] \quad i \geqslant 2.
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
B_{1,0} &= a_{1,0}I + A,\\
B_{1,1} &= a_{1,1}I - E,\\
B_{i,0} &= a_{i,0}I + B_{i-2,0} + AB_{i-1,0} - EB_{i-1,1},\\
B_{i,k} &= a_{i,k} + B_{i-2,k} - EB_{i-1,k-1} - EB_{i-1,k+1} + AB_{i-1,k},\\
&\quad k = 1,2,\ldots,i-2,\\
B_{i,i-1} &= a_{i,i-1}I - EB_{i-1,i-2} + AB_{i-1,i-1},\\
B_{i,i} &= a_{i,i}I - EB_{i-1,i-1}, \quad i \geqslant 2.
\end{aligned} \tag{29}
$$

4. Example

Consider

$$
A = \begin{pmatrix} 1 & -4 & -1 & -4\\ 2 & 0 & 5 & -4\\ -1 & 1 & -2 & 3\\ -1 & 4 & -1 & 6 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

By definition, we know

$$
a(\mu) = \det(\mu E - A) = 19\mu^2 - 15\mu + 2
$$

and

$$
\begin{aligned}
B(\mu) &= \mathrm{adj}(\mu E - A)\\
&= \begin{pmatrix} 11\mu - 2 & -4\mu^2 - \mu + 2 & -20\mu + 8 & 16\mu - 4\\ 2\mu^2 - 9\mu + 1 & \mu^3 - 5\mu^2 - 10\mu + 8 & 5\mu^2 - 33\mu + 22 & -4\mu^2 + 3\mu - 5\\ 2\mu & \mu^2 + 9\mu - 6 & 24\mu - 16 & -4\mu + 4\\ 8\mu - 1 & 4\mu^2 + 7\mu - 6 & 20\mu - 16 & 3\mu + 3 \end{pmatrix}.
\end{aligned}
$$

Now we apply Leverrier–Chebyshev Algorithm in Section 3. From (28) and (29),

$$a_{1,0} = -\mathrm{tr}\,A = -5, \quad a_{1,1} = \mathrm{tr}\,E = 3,$$

which give

$$B_{1,0} = a_{1,0}I + A = \begin{pmatrix} -4 & -4 & -1 & -4 \\ 2 & -5 & 5 & -4 \\ 1 & 1 & -7 & 3 \\ 1 & 4 & -1 & 1 \end{pmatrix},$$

$$B_{1,1} = a_{1,1}I - E = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

From (28),

$$a_{2,0} = -\tfrac{1}{2}\mathrm{tr}\,(AB_{1,0} - EB_{1,1}) + 3a_{0,0} = 15,$$
$$a_{2,1} = -\tfrac{1}{2}\mathrm{tr}\,(AB_{1,1} - EB_{1,0}) = -10,$$
$$a_{2,2} = \tfrac{1}{2}\mathrm{tr}\,(EB_{1,1}) = 3,$$

and hence from (29) we have

$$B_{2,0} = a_{2,0}I - B_{0,0} + AB_{1,0} - EB_{1,1} = \begin{pmatrix} 5 & -1 & -10 & 5 \\ -9 & -5 & -33 & 3 \\ 5 & 9 & 29 & -3 \\ 7 & 7 & 22 & 3 \end{pmatrix},$$

$$B_{2,1} = a_{2,1}I - EB_{1,0} + AB_{1,1} = \begin{pmatrix} -4 & -8 & -1 & -4 \\ 4 & -10 & 10 & -8 \\ -1 & 2 & -7 & 3 \\ 1 & 8 & -1 & 1 \end{pmatrix},$$

$$B_{2,2} = a_{2,2}I - EB_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying (28), this leads to

$$a_{3,0} = -\tfrac{1}{3}\mathrm{tr}\,(AB_{2,0} - EB_{2,1}) + \tfrac{4}{3}a_{1,0} = -22,$$
$$a_{3,1} = -\tfrac{1}{3}\mathrm{tr}\,(AB_{2,1} - EB_{2,0} - EB_{2,2}) + \tfrac{4}{3}a_{1,1} = 36,$$
$$a_{3,2} = -\tfrac{1}{3}\mathrm{tr}\,(AB_{2,2} - EB_{2,1}) = -5,$$
$$a_{3,3} = \tfrac{1}{3}\mathrm{tr}\,(EB_{2,2}) = 1.$$

From (29) we obtain

$$B_{3,0} = a_{3,0}I - B_{1,0} + AB_{2,0} - EB_{2,1} = \begin{pmatrix} -6 & -6 & 7 & -8 \\ 5 & -2 & 32 & -13 \\ -1 & -4 & -23 & 7 \\ 2 & 2 & -17 & 4 \end{pmatrix},$$

$$B_{3,1} = a_{3,1}I - B_{1,1} - EB_{2,0} - EB_{2,2} + AB_{2,1} = \begin{pmatrix} 13 & -1 & -20 & 16 \\ -9 & -5 & -33 & 3 \\ 2 & 9 & 26 & -4 \\ 8 & 7 & 20 & 5 \end{pmatrix},$$

$$B_{3,2} = a_{3,2}I - EB_{2,1} + AB_{2,2} = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 2 & -5 & 5 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix},$$

$$B_{3,3} = a_{3,3}I - EB_{2,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally,

$$\begin{aligned} a_{4,0} &= -\tfrac{1}{4}\operatorname{tr}(AB_{3,0} - EB_{3,1}) + \tfrac{1}{2}(a_{2,0} + a_{0,0}) = 35, \\ a_{4,1} &= -\tfrac{1}{4}\operatorname{tr}(AB_{3,1} - EB_{3,0} - EB_{3,2}) + \tfrac{1}{2}a_{2,1} = -25, \\ a_{4,2} &= -\tfrac{1}{4}\operatorname{tr}(AB_{3,2} - EB_{3,1} - EB_{3,3}) + \tfrac{1}{2}a_{2,2} = 22, \\ a_{4,3} &= -\tfrac{1}{4}\operatorname{tr}(AB_{3,3} - EB_{3,2}) = 0, \\ a_{4,4} &= \tfrac{1}{4}\operatorname{tr}(EB_{3,3}) = 0, \end{aligned}$$

Thus, the adjoint $B(\mu)$ and the determinant $a(\mu)$ of $\mu E - A$ is given by (20) and (19) as

$$\begin{aligned} a(\mu) &= S_4(0) + a_1(\mu)S_3(0) + a_2(\mu)S_2(0) + a_3(\mu)S_1(0) + a_4(\mu)S_0(0) \\ &= 1 - a_2(\mu) + a_4(\mu) \\ &= 21S_0(\mu) - 15S_1(\mu) + 19S_2(\mu). \end{aligned}$$

$$\begin{aligned} B(\mu) &= S_3(0)B_0(\mu) + S_2(0)B_1(\mu) + S_1(0)B_2(\mu) + S_0(0)B_3(\mu) \\ &= -B_1(\mu) + B_3(\mu) \\ &= (B_{3,0} - B_{1,0})S_0(\mu) + (B_{3,1} - B_{1,1})S_1(\mu) + B_{3,2}S_2(\mu) + B_{3,3}S_3(\mu). \end{aligned}$$

Note. Both $B(\mu)$ and $a(\mu)$ can be expressed relative to a basis of Hermite, Legendre or Laguerre polynomials too, it is omitted here.

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