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The Numerical Solution of One-Dimensional Thermally Expandable Flows*

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The equations of a thermally expandable fluid provide a simple model of two-phase flow in a nuclear reactor coolant channel. In this paper we consider these equations with a general lag-type boundary condition to account for the closed loop effects of the model. We propose finite-difference equations for their numerical solution and, under suitable conditions, prove convergence of the finite-difference approximations. A numerical example is given.

1. INTRODUCTION

The time-dependent behavior of a two-phase fluid plays an important role in the design of such energy exchange mechanisms as nuclear reactors, boilers, and heat exchangers. However, the equations which describe this behavior are, even in one spatial dimension, quite complicated [7], and therefore difficult to investigate. The theoretical difficulties arise because each phase or component of the mixture can have its own velocity and temperature, and the system of differential equations that describes the flow field must accommodate this phenomenon.

A more tractable problem occurs when the flow is *homogeneous*, that is, when the two phases are perfectly mixed. In this case the relevant one-dimensional equations are the usual equations of fluid dynamics (see, for example, [6, p. 290]). A further simplification results if it is assumed that the pressure throughout the flow field exhibits only small variations about a known mean or *system pressure* p^*. This leads to the equations of a *thermally expandable* fluid flow in a one-dimensional channel or pipe [4, 5].

$$\rho_t + w_x = 0, \tag{1}$$

$$\rho h_t + w h_x = \sigma, \tag{2}$$

$$\rho = f(p^*, h). \tag{3}$$

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Here, $\rho(x, t)$, $h(x, t)$, and $w(x, t)$ denote the density, enthalpy, and mass flux of the fluid, and $\sigma(x, t)$ is a prescribed heat flux. Equations (1) and (2) are the *one-dimensional* continuity and energy equations, while (3) is the equation of state. Since p^* is presumed to be known, and f is assumed to be sufficiently differentiable, Eqs. (1)–(3) determine in principle the unknowns ρ, h, and w.

2. Initial and Boundary Conditions

Solutions of (1)–(3) are generally sought on a space-time rectangle

$$Q = \{(x, t) \mid (x, t) \in [0, a] \times [0, T]\}.$$

That is, we seek to determine the dependent variables h and w in a pipe of length a over the time interval $[0, T]$. The initial conditions are

$$w(x, 0) = w_I(x), \tag{4}$$

$$h(x, 0) = h_I(x), \qquad x \in [0, a]. \tag{5}$$

As we shall see later the x-axis is a characteristic of the system, hence w_I and h_I must satisfy an appropriate characteristic relationship, that is, they cannot be prescribed arbitrarily on the x-axis.

With regard to the boundary conditions, we attempt to represent the fact that the channel is only a part of a larger closed system. For example, in a nuclear reactor the remaining components include the plena, primary side piping, steam generator tubes, etc. We view this part of the system as a "black box" whose effect is to provide feedback to the channel by relating the conditions of the channel entrance to those at its exit at some previous time. Indeed, we define the boundary conditions as follows:

$$w(0, t) = w_L(t) \qquad \text{if} \quad 0 \leqslant t < \tau, \tag{6}$$

$$w(0, t) = S_0(w(a, t - \tau)) \qquad \text{if} \quad \tau \leqslant t \leqslant T, \tag{7}$$

$$\left.\begin{aligned} h(0, t) &= h_L(t) && \text{if} \quad 0 \leqslant t < \tau \quad \text{and} \quad w(0, t) \geqslant 0, \\ h(0, t) &= S_1(h(a, t - \tau)) && \text{if} \quad \tau \leqslant t \leqslant T \quad \text{and} \quad w(0, t) \geqslant 0, \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} h(a, t) &= h_R(t) && \text{if} \quad 0 \leqslant t < \tau \quad \text{and} \quad w(a, t) < 0, \\ h(a, t) &= S_2(h(a, t - \varepsilon)) && \text{if} \quad \tau \leqslant t \leqslant T \quad \text{and} \quad w(a, t) < 0, \end{aligned}\right\} \tag{9}$$

where S_0, S_1, and S_2 are known differentiable functions.

The boundary conditions (8) and (9) indicate that enthalpy is a quantity which is transported with the fluid. Hence, if the direction of the flow reverses at the left endpoint, (8) does not apply since the enthalpy there will be determined by the differential equations. On the other hand, if this condition occurs at the right endpoint, then the enthalpy must be supplied by an auxiliary condition such as (9). In (9), ε is a fixed small number, since we assume that h does not change much when the flow reverses.

For sufficiently smooth auxiliary data, the existence, uniqueness, and regularity of a solution of the initial-boundary value problem (1)–(9) are proved in [1]. In this paper we consider the numerical solution of (1)–(9) by finite differences. This is not as straightforward as it may seem, since the quasi-linear system (1)–(2) has the line $t = \text{const}$ as a characteristic. Hence, it is a degenerate hyperbolic system in the sense that signals propagate along a characteristic at infinite speed. Moreover, the Courant–Friedrichs–Lewy stability condition [3] cannot be satisfied unless some degree of implicitness is introduced into the difference equations. This, combined with the lag-type boundary conditions, makes the problem mathematically interesting.

In the next section we propose difference equations which are modifications of those given by Courant, Isaacson, and Rees [2]. Since the original Courant–Isaacson–Rees equation are of an explicit nature, the modifications are necessary to ensure stability. Sections 4 and 5 contain the main mathematical results of the paper, theorems on the convergence of the difference equation's solution to that of (1)–(9). The results of some numerical calculations, which demonstrate the effects of the lag conditions, are given in the final section.

3. Difference Equations

The difference equations proceed from the characteristic normal forms of (1) and (2). Because of the simple structure of (1) and (2), these forms may be obtained in an elementary manner. In fact, if $\rho \geqslant \rho_0 > 0$, (2) may be written as

$$h_t + \frac{w}{\rho} h_x - \frac{\sigma}{\rho} = 0, \tag{10}$$

which is the normal form corresponding to the characteristic defined by $dx/dt = w/\rho$. Furthermore, if we write (1) as

$$\rho_h h_t + w_x = 0, \tag{11}$$

then h_t may be eliminated from (11) and (2). The result is

$$\rho_h w h_x - \rho w_x - \rho_h \sigma = 0, \tag{12}$$

which we recognize as the normal form corresponding to the characteristic $dt/dx = 0$, that is, the line $t = \text{const}$.

Let $\Delta x = a/J$ and $\Delta t = T/N$, where J and N are positive integers. We denote the finite-difference approximations of

$$w(j\Delta x, n\Delta t), \qquad h(j\Delta x, n\Delta t), \qquad j = 0, 1, \ldots, J, \quad n = 0, 1, \ldots, N,$$

by W_j^n and H_j^n, respectively. These are determined by the difference equations for Eq. (10) and (12)

$$R_j^n \frac{H_j^{n+1} - H_j^n}{\Delta t} + \left(\frac{W_j^n}{\Delta x}\right) \begin{bmatrix} H_j^n - H_{j-1}^n \\ H_{j+1}^n - H_j^n \end{bmatrix} - \sigma_j^n = 0, \tag{13}$$

$$R_{j-1}'^n W_{j-1}^n \frac{H_j^{n+1} - H_{j-1}^{n+1}}{\Delta x} - R_{j-1}^n \frac{W_j^{n+1} - W_{j-1}^{n+1}}{\Delta x} - R_{j-1}'^n \sigma_j^n = 0. \tag{14}$$

Here we have introduced the quantities $R_j^n = f(p^*, H_j^n)$ and $R_h'^n = f_h(p^*, H_j^n)$.

Furthermore, in (13) the upper or lower line of the bracketed term is to apply accordingly as W_j^n is nonnegative or not. Equation (14) holds for $j = 1, \ldots, J$, where W_0^{n+1} is determined by the boundary condition. Equation (13) holds for all values of j for which its subscripts lie within the interval $[0, J]$, that is, for $j = 1, \ldots, J-1$ and $j = 0$ if $W_0^n < 0$, $j = J$ if $W_J^n > 0$. In the case $W_0^n > 0$ or $W_J^n < 0$, (13) is replaced by the appropriate boundary condition.

There is a stability condition associated with (13), namely,

$$\theta \equiv \frac{\Delta t}{\Delta x} < \frac{R_j^n}{|W_j^n|}. \tag{15}$$

In the next section we wil show that under certain smoothness assumptions, a constant θ can be chosen which is sufficiently small so as to satisfy (15) for $0 \leqslant j \leqslant J$, $0 \leqslant n \leqslant N$.

Since we now have a discrete problem, the boundary conditions have to be modified. We assume that $\tau/\Delta t$ and $\varepsilon/\Delta t$ are integers l and m. This assumption is convenient for treating the boundary conditions since we can use the computed H and W at time $(n-l)\Delta t$ to find the boundary values of H and W at time $n\Delta t$. We lose no generality, since it can be shown that if w and h are Lipschitz continuous on the boundary, and if we use interpolation to set the boundary conditions, then the order of the error does not change. Now for $n\Delta t > \tau$, (7), (8), and (9) become

$$\begin{aligned} W_0^n &= S_0(W_J^{n-l}), \\ H_0^n &= S_1(H_J^{n-l}) \qquad \text{if} \quad W_0^{n-1} \geqslant 0, \\ H_J^n &= S_2(H_J^{n-m}) \qquad \text{if} \quad W_J^{n-1} < 0. \end{aligned}$$

We remark that if the flow reverses at the right end of the pipe, we can no longer ensure the continuity of the boundary conditions and therefore our existence proof does not hold. Condition (9) represents one simple way to treat such flow reversal and could be replaced by a more realistic ad hoc model.

The algorithm is as follows:

(i) Use Eq. (13) to compute the enthalpy at time $t+\Delta t$, integrating from left to right or from right to left (or both), if at time t the flow on the left is positive or the flow on the right is negative (or both).

(ii) Use Eq. (14) to compute the flow W at time $t+\Delta t$ integrating from left to right, using our knowledge of H from step (i).

One more point that should be mentioned concerns the starting of the algorithm. The initial conditions should satisfy the differential equations, and the initial and boundary conditions resulting from the lag condition should be differentiable. This in general cannot be obtained a priori. One can obtain continuity and differentiability on the boundary $x=0$ as follows. We specify a smooth $w(0,t)$, $h(0,t)$ on $0<t<\tau-\varepsilon$, where ε is a small positive number. In the transition region $\tau-\varepsilon<t<\tau$ we define $w(0,t)$, $h(0,t)$ by Hermite interpolation, using obvious numerical estimates of $S_0(w(a,0))$, $S_1(h(a,0))$ and $S_{0t}(w(a,0))$, $S_{1t}(h(a,0))$ to set the values of $w(0,\tau)$, $h(0,\tau)$ and $w_t(0,\tau)$, $h_t(0,\tau)$. Once we have a solution for $t>\tau$, the lag condition produces a differential boundary condition, because the solution of the differential equations has this property.

4. Convergence on a Restricted Domain

Let $p_j^n=W_j^n-w_j^n$ and $q_j^n=H_j^n-h_j^n$ denote the discretization errors. Suppose that in Q, $\rho>2\rho_0>0$, w, h, $\rho<M-1$ for some positive constants $\rho_0<1$ and M. Let $\lambda>1$ be a Lipschitz constant for S_0, S_1, S_2 and $f(h)$, and let $\theta=\Delta t/\Delta x$ denote the constant mesh ratio.

Theorem 1. *Assume that*

(i) $\theta<\rho_0/M$.

(ii) *f_h is Lipschitz continuous and bounded on* $(-\infty,\infty)$.

(iii) *h_x and w_x are x-Lipschitz continuous (i.e., Lipschitz continuous in x only), h, w and σ are t-Lipschitz continuous on Q.*

Then for Δx, Δt sufficiently small, θ satisfies the stability condition (15). *Furthermore, there are α and γ such that if $a<\alpha$ and $T<\gamma$, then p_j^n, q_j^n are $O(\Delta x)$ for $0\leqslant n\leqslant N$, $0\leqslant j\leqslant J$. In fact, let $B=4\max(|f_hw/\rho|,\lambda)+1$ in Q.*

Let k be the integral part of T/τ, and let $K = \sum_{i=1}^{k+1} (4B)^i$. Then $q_j^n < \Delta t$, $p_j^n < K\Delta t$, $j = 0, \ldots, J$, $n = 0, \ldots, N$ in Q provided that Δt is small enough.

Proof. We first derived equations for the errors. This derivation requires a choice of sign for W_j^n in (13). Following our assumptions, we take W_j^n to be positive. (However, the results are similar for negative W.) We use (ii) along with Eqs. (10) and (13) to deduce that

$$\begin{aligned}
&\frac{R_j^n(h_j^{n+1} - h_j^n)}{\Delta t} + \frac{W_j^n(h_j^n - h_{j-1}^n)}{\Delta x} - \sigma_j^n \\
&\quad = \frac{[f(h_j^n) + O(q_j^n)](h_j^{n+1} - h_j^n)}{\Delta t} + \frac{(w_j^n + p_j^n)(h_j^n - h_{j-1}^n)}{\Delta x} - \sigma_j^n \\
&\quad = [f(h_j^n) + O(q_j^n)][(h_t)_j^n + O(\Delta t)] + (w_j^n + p_j^n)[(h_x)_j^n + O(\Delta x)] - \sigma_j^n \\
&\quad = O(\Delta x + \Delta t + q_j^n + p_j^n).
\end{aligned} \tag{16}$$

$$\begin{aligned}
&R'^n_{j-1} W_{j-1}^n \left(\frac{h_j^{n+1} - h_{j-1}^{n+1}}{\Delta x}\right) - R_{j-1}^n \left(\frac{w_j^{n+1} - w_{j-1}^{n+1}}{\Delta x}\right) - \sigma_{j-1}^n R'^n_{j-1} \\
&\quad = \{f_h(h_{j-1}^n)\, w_{j-1}^n + f_h(H_{j-1}^n)\, p_{j-1}^n + w_{j-1}^n [f_h(H_{j-1}^n) - f_h(h_{j-1}^n)]\} \\
&\qquad \times \left(\frac{h_j^{n+1} - h_{j-1}^{n+1}}{\Delta x}\right) - [f(h_{j-1}^n) + O(q_{j-1}^n)] \left(\frac{w_j^{n+1} - w_{j-1}^{n+1}}{\Delta x}\right) \\
&\qquad - \sigma_{j-1}^n [f_h(h_{j-1}^n) + O(q_{j-1}^n)] \\
&\quad = [f_h(h_{j-1}^n)\, w_{j-1}^n + O(p_{j-1}^n + q_{j-1}^n)][(h_x)_{j-1}^n + O(\Delta t + \Delta x)] \\
&\qquad - [f(h_{j-1}^n) + O(q_{j-1}^n)][(w_x)_{j-1}^n + O(\Delta t + \Delta x)] \\
&\qquad - \sigma_{j-1}^n [f_h(h_{j-1}^n) + O(q_{j-1}^n)] \\
&\quad = O(\Delta t + \Delta x + p_{j-1}^n + q_{j-1}^n).
\end{aligned} \tag{17}$$

Subtraction of (16) from (13) and (17) from (14) then yields

$$\begin{aligned}
&\frac{R_j^n(q_j^{n+1} - q_j^n)}{\Delta t} + \frac{W_j^n(q_j^n - q_{j-1}^n)}{\Delta x} \\
&\quad = O(\Delta x + \Delta t + p_j^n + q_j^n),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\frac{R'^n_{j-1} W_{j-1}^n(q_j^{n+1} - q_{j-1}^{n+1})}{\Delta x} - \frac{R_{j-1}^n(p_j^{n+1} - p_{j-1}^{n+1})}{\Delta x} \\
&\quad = O(\Delta x + \Delta t + p_{j-1}^n + q_{j-1}^n).
\end{aligned} \tag{19}$$

We proceed with an induction argument. Define p^n, q^n, W^n as $p^n = \max_{0 \le l \le n, 0 \le j \le J} |p_j^l|$, etc., and assume that $q^n < \Delta t$, $p^n < K\Delta t$,

$W^n < M$, $R_j^n > \rho_0$, $j = 1,\dots,J$. Since we match the initial condition exactly, these inequalities hold for $n = 0$. We show that they are preserved at time step $n+1$. We shall use the letter C as a generic for the constants associated with any order relation.

If (18) is solved for $q_j^{n+1}, j \geqslant 1$, there results

$$q_j^{n+1} = q_j^n \left(1 - \frac{\theta W_j^n}{R_j^n}\right) + \frac{q_{j-1}^n \theta W_j^n}{R_j^n} + \left(\frac{\Delta t}{R_j^n}\right) O(\Delta t + \Delta x + p_j^n + q_j^n). \tag{19a}$$

However, $W_j^n > 0$ and $\theta < \rho_0/M$ imply that the coefficients of q_j^n and q_{j-1}^n in this equation are nonnegative. Assume first that $n+1 \leqslant l$, so the lag condition is not used. Then

$$|q_j^{n+1}| \leqslant q^n + (\Delta t/R_j^n)\, O(\Delta t + \Delta x + p_j^n + q_j^n) = q^n + \Delta t\, O(\Delta t), \qquad j \geqslant 1,$$

and so, since $q_0^{n+1} = 0$,

$$q^{n+1} \leqslant q^n + C(\Delta t)^2 \leqslant Cl(\Delta t)^2 = C\tau\Delta t \leqslant C\gamma\Delta t. \tag{20}$$

Therefore $|q_j^{n+1}| \leqslant \Delta t$, provided that $\gamma < 1/C$.

Now suppose that $n+1 > l$. In this case we claim that for any $0 \leqslant i \leqslant l$,

$$q^{\alpha+i} \leqslant \lambda q^\alpha + i\Delta t\, O(\Delta t), \tag{20a}$$

where $\alpha \equiv n + 1 - l$.

For $i = 0$, (20a) is obviously true. Suppose (20a) is true for $i \leqslant l - 1$. Then from (19a) we conclude that (since $j - 1$ might be zero),

$$\begin{aligned} |q_j^{\alpha+i+1}| &\leqslant \max(q^{\alpha+i}, \lambda q^{\alpha+i+1-l}) + \Delta t O(\Delta t + \Delta x + p^{\alpha+1} + q^{\alpha+i}) \\ &\leqslant \max(q^{\alpha+i}, \lambda q^\alpha) + \Delta t O(\Delta t), \qquad j = 1,\dots,J. \end{aligned}$$

However, the inequality also clearly holds for $j = 0$. Therefore, from our hypothesis on i we have

$$\begin{aligned} q^{\alpha+i+1} &\leqslant \max(\lambda q^\alpha + i\Delta t O(\Delta t), \lambda q^\alpha) + \Delta t O(\Delta t) \\ &= \lambda q^\alpha + (i+1)\,\Delta t O(\Delta t), \end{aligned}$$

which establishes (20a) by induction. Setting $i = l$ in (20a), we have

$$\begin{aligned} q^{n+1} &\leqslant \lambda q^{n+1-l} + \tau O(\Delta t) \leqslant \lambda^2 q^{n+1-2l} + \tau O(\Delta t)(1+\lambda) \\ &\leqslant \lambda^k q^{n+1-kl} + \tau O(\Delta t)(1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1}). \end{aligned}$$

But $(n + 1 - kl)\,\Delta t < \tau = l\Delta t$, that is, $n + 1 - kl < l$.

It follows from (20) that

$$q^{n+1} \leqslant C\gamma \Delta t(1 + \lambda + \cdots \lambda^k) = C\gamma \Delta t \frac{\lambda^{k+1} - 1}{\lambda - 1}.$$

Hence, $|q_j^{n+1}| < \Delta t$, if $\gamma < (\lambda - 1)/C(\lambda^{k+1} - 1)$.

To estimate p_j^{n+1}, we solve (19) for p_j^{n+1}, $j \geqslant 1$, to get

$$p_j^{n+1} = p_{j-1}^{n+1} + (R_{j-1}'^n W_{j-1}^n / R_{j-1}^n)(q_j^{n+1} - q_{j-1}^{n+1}) + (1/R_{j-1}^n)\Delta x O(\Delta x + \Delta t + p_{j-1}^n + q_{j-1}^n).$$

Denote $A_j^n = (R_j'^n W_j^n)/R_j^n$ and $a_j^n = (f_h(h_j^n)\, w_j^n)/f(h_j^n)$.

Then

$$\begin{aligned}
p_j^{n+1} &= p_{j-1}^{n+1} + A_{j-1}^n(q_j^{n+1} - q_{j-1}^{n+1}) + (1/R_{j-1}^n)\Delta x O(\Delta x + \Delta t + p_{j-1}^n + q_{j-1}^n) \\
&= p_{j-2}^{n+1} + A_{j-2}^n(q_{j-1}^{n+1} - q_{j-2}^{n+1}) + A_{j-1}^n(q_j^{n+1} - q_{j-1}^{n+1}) \\
&\quad + (1/R_{j-1}^n + 1/R_{j-2}^n)\Delta x O(\Delta x + \Delta t + p^n + q^n) \\
&= p_0^{n+1} + \sum_{i=0}^{j-1} A_i^n(q_{i+1}^{n+1} - q_i^{n+1}) + \sum_{i=0}^{j-1} (1/R_i^n)\Delta x O(\Delta x + \Delta t + p^n + q^n) \\
&= p_0^{n+1} + A_{j-1}^n q_j^{n+1} - A_0^n q_0^{n+1} + \sum_{i=1}^{j-1} q_i^{n+1}(A_{i-1}^n - A_i^n) \\
&\quad + \Delta x \sum_{i=0}^{j-1} (1/R_i^n)\, O(\Delta x + \Delta t + p^n + q^n).
\end{aligned} \tag{21}$$

Now $f(h_j^n) > 2\rho_0$ and f_h and w are bounded. Therefore,

$$A_j^n = a_j^n + O(p_j^n + q_j^n).$$

Since f, f_h, w, and h are Lipschitz continuous, $|a_i^n - a_{i-1}^n| = O(\Delta x)$ and hence

$$|A_i^n - A_{i-1}^n| = O(p^n + q^n + \Delta x).$$

Using the last result, and letting $A^n = \max_{0 \leqslant j \leqslant J} |A_j^n|$, it follows from (21) that

$$|p_j^{n+1}| \leqslant |p_0^{n+1}| + 2q^{n+1}A^n + O(p^n + q^n + \Delta x)\sum_{i=1}^{j-1} |q_i^{n+1}| + \Delta x \sum_{i=0}^{j-1} (1/R_i^n)\, O(\Delta x + \Delta t + p^n + q^n).$$

Since $q^{n+1} \leqslant \Delta t$, and $2A^n \leqslant B + O(\Delta t)$, we have

$$\begin{aligned} |p_j^{n+1}| &\leqslant |p_0^{n+1}| + 2A^n\Delta t + a\theta O(p^n + q^n + \Delta x) \\ &\quad + (1/\rho_0)\,\alpha O(\Delta x + \Delta t + p^n + q^n) \\ &\leqslant |p_0^{n+1}| + \alpha C p^n + B\Delta t + \alpha C \Delta t. \end{aligned} \tag{22}$$

Note that (22) holds for $j = 0, \ldots, J$.

Now assume that α is so small that $C\alpha < 1/2$ and use the Lipschitz continuity of the boundary condition to conclude that

$$|p_j^{n+1}| < \lambda p^{n+1-l} + (1/2)\, p^n + 2B\Delta t, \qquad j = 0, \ldots, J.$$

Since by definition, p^n is monotone with respect to n, we conclude that

$$\begin{aligned} p^{n+1} &\leqslant 2B(p^{n+1-l} + \Delta t) + (1/2)(\lambda p^{n-l} + (1/2)\, p^{n-1} + 2B\Delta t) \\ &\leqslant 2B(p^{n+1-l} + \Delta t)[1 + (1/2)] + (1/2)^2\, p^{n-1} \\ &\leqslant 2B(p^{n+1-l} + \Delta t)[1 + 1/2 + \cdots + (1/2)^{l-1}] + (1/2)^l\, p^{n+1-l} \\ &\leqslant 4B(p^{n+1-l} + \Delta t) \\ &\leqslant \Delta t[4B + (4B)^2 + \cdots + (4B)^k] + (4B)^k\, p^i, \end{aligned}$$

where $i < l$. Since $p_0^i = 0$, $p^i \leqslant \frac{1}{2} p^{i-1} + 2B\Delta t$. Continuing the recursion to the initial condition, we find that $p^i < 4B\Delta t$. Hence

$$p^{n+1} < \sum_{i=1}^{k+1} (4B)^i\, \Delta t = K\Delta t. \tag{23}$$

Using these bounds on the errors, it is easy to show that if $\Delta t \leqslant \rho_0/2K$, then W^{n+1}, $H^{n+1} \leqslant M$ and, moreover, $R_j^n > \rho_0$. Thus the stability condition (15) is satisfied and the proof is complete.

5. Extension of the Domain of Convergence

Our proof of convergence applies to a domain which may be a proper subset of the original domain Q. In view of the lag boundary condition, this appears to be a severe restriction. However, we will now indicate how Theorem 1 can be used to establish global convergence on Q.

The idea is to show that the error estimates of Theorem 1 remain valid on a rectangle $R_{im} \subset Q$,

$$R_{im} = \{(x, t) \mid x_i \leqslant x \leqslant x_i + \alpha,\ t_m \leqslant t \leqslant t_m + \gamma\},$$

when the errors on $x = x_i$, and $t = t_m$ are nonzero, but sufficiently small. Since Q may be decomposed into a *finite* number of R_{im} by a fixed "coarse" mesh $0 = x_0 < x_1 \cdots < x_I = a$, $0 = t_0 < t_1 < \cdots < t_M = T$, chosen so that the errors on the $x = x_0$, $t = t_0$ sides of R_{00} are exactly zero, we see that for Δt, Δx sufficiently small, the errors on all subsequent R_{im} can be made arbitrarily small. For brevity we sketch a proof of the error estimates on R_{im} without the lag condition.

It clearly suffices to assume $x_i = t_m = 0$. Using the notation of Theorem 1, we assume that $|p_0^n|$, $|p_i^0|$, $|q_0^n|$, and $|q_i^0|$ are bounded by δ. Then from (19a) we have

$$q^{n+1} \leqslant q^n + \delta + C(\Delta t)^2 \leqslant (n+2)\,\delta + (n+1)\,C(\Delta t)^2$$
$$< \frac{\gamma}{\Delta t}\,\delta + \gamma C \Delta t.$$

However if $\gamma C < 1$ and $\delta \leqslant ((1-\gamma C)/\gamma)(\Delta t)^2$, then $q^{n+1} \leqslant \Delta t$.

In a similar manner it follows from (22) that when $aC < 1/2$,

$$p^{n+1} \leqslant \delta + \tfrac{1}{2} p^n + (B + \tfrac{1}{2})\,\Delta t$$
$$\leqslant \delta(\tfrac{1}{2})^{n+1} + 2|1 - (\tfrac{1}{2})^{n+1}||\delta + (B + \tfrac{1}{2})\,\Delta t|$$
$$\leqslant \tfrac{3}{2}\delta + (B + \tfrac{1}{2})\,\Delta t \leqslant \tfrac{3}{2}\delta + 2B\Delta t.$$

6. Numerical Experiment

In the following experiment we want to demonstrate the effect of the lag boundary condition and to show, qualitatively, that the scheme suggested in the previous section is convergent.

Let the functions $\rho(h)$ and $\sigma(x, t)$ be defined by

$$\rho(h) = 308 - 0.4h,$$
$$\sigma(x, t) = -132 \exp(-0.15x - 0.05t).$$

Then

$$h(x, t) = 660 + 60 \exp(-0.15x - 0.05t),$$
$$w(x, t) = 8 \exp(0.15x - 0.05t)$$

is an analytic solution for the system (1)–(2). Let a, the length of the channel, be 5 units. Then since

$$\exp(-0.15a - 0.05t) = \exp(-0 - 0.05(t + \tau)),$$

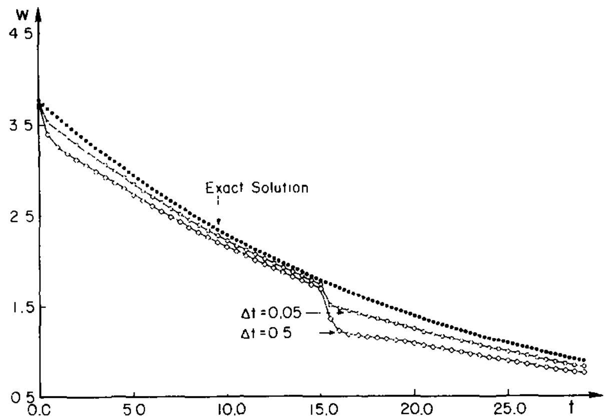

FIG. 1. Exit Mass Velocity Verses Time.

we may choose the lag $\tau = 15$. Consequently, we prescribe exact initial-boundary conditions on $\{(x, 0); 0 \leqslant x \leqslant 5\}$, $\{(0, t); 0 \leqslant t < 15\}$ and determine the boundary condition for $t \geqslant 15$ from the numerical solution at $x = 5$. Since the flow never reverses, we do not need a boundary condition on $\{(a, t); 0 < t < \infty\}$.

Two numerical solutions of the mass velocity are presented in Fig. 1. One solution corresponds to $\Delta x = 0.2$, $\Delta t = 0.5$, and the second to $\Delta x = 0.2$, $\Delta t = 0.05$. We plot these solutions at the outlet of the channel (at $x = 5$) as a function of time. Also shown in the figure is a plot of the exact mass velocity.

The numerical solution obviously appears to converge as Δt decreases. The effect of the lag condition becomes evident at time $t > 15$. At this time the lag condition introduces a numerical error in the boundary condition which was exact until that time. Interestingly enough, the scheme tends to correct itself in this example, and after the perturbation occurs the numerical solution approaches the analytic solution.

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