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Direct and inverse polynomial perturbations of hermitian linear functionals

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Abstract

This paper is devoted to the study of direct and inverse (Laurent) polynomial modifications of moment functionals on the unit circle, i.e., associated with hermitian Toeplitz matrices. We present a new approach which allows us to study polynomial modifications of arbitrary degree.

The main objective is the characterization of the quasi-definiteness of the functionals involved in the problem in terms of a difference equation relating the corresponding Schur parameters. The results are presented in the general framework of (non-necessarily quasi-definite) hermitian functionals, so that the maximum number of orthogonal polynomials is characterized by the number of consistent steps of an algorithm based on the referred recurrence for the Schur parameters.

The non-uniqueness of the inverse problem makes it more interesting than the direct one. Due to this reason, special attention is paid to the inverse modification, showing that different approaches are possible depending on the data about the polynomial modification at hand. These different approaches are translated as different kinds of initial conditions for the related inverse algorithm.

Some concrete applications to the study of orthogonal polynomials on the unit circle show the effectiveness of this new approach: an exhaustive and instructive analysis of the functionals coming from a general inverse polynomial perturbation of degree one for the Lebesgue measure; the classification of those pairs of orthogonal polynomials connected by a certain type of linear relation with constant polynomial coefficients; and the determination of those orthogonal polynomials whose associated ones are related to a degree one polynomial modification of the original orthogonality functional.

Keywords: Hermitian functionals; Orthogonal polynomials on the unit circle; Schur parameters; Direct and inverse polynomial modifications; Christoffel and Geronimus transformations

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1. Introduction

The interest in the perturbation theory of moment functionals is, in its origin, linked to the theory of orthogonal polynomials on the real line as a way to generate new families from known ones. However, its importance goes much further, among other things, due to the connection with such an outstanding idea as the Darboux transformation (see [4,21–23,35,43]). This connection has led to unexpectedly fruitful relations with other subjects like, for instance, spectral theory, integrable systems or quantum physics (see for instance [7,30,32,36–38]). It is worth noting that the interest of the Darboux transformation goes beyond the study of the discrete systems underlying the difference equations related to orthogonal polynomials. Indeed, such transformations where initially introduced in [8] to deal with continuous systems governed by differential equations (see also [10,31]).

In the context of orthogonal polynomials on the real line, the Darboux transformation can be interpreted as a way to understand the transformation of the corresponding Jacobi matrix under polynomial perturbations of the related orthogonality functional: the so-called Darboux transformation without parameter is connected to the Christoffel transformation, which multiplies a functional by a polynomial of degree one, while the inverse of this transformation, called the Geronimus transformation, leads to the so-called Darboux transformation with parameter because the "division" by a polynomial is not uniquely defined, giving rise to a one parameter solution. An obvious extension of both transformations consists in allowing an arbitrary degree for the polynomial perturbation.

The intense activity during the last decades around the theory of orthogonal polynomials on the unit circle has stimulated the study of perturbations of hermitian functionals. The possibility to consider perturbations that do not preserve the hermitian character of the functional leads to left and right orthogonal polynomials (see [3]), thus most of the efforts have been concentrated in the analysis of hermitian perturbations as a source of new families of standard orthogonal polynomials (see the recent monograph on orthogonal polynomials on the unit circle [33,34] and the references therein). Then, the preservation of the hermiticity calls for the use of Laurent polynomials as perturbations.

Laurent polynomial perturbations of hermitian functionals have been considered previously (see for instance [1,5,6,9,12,13,11,14–16,18–20,24–27,39]). The most detailed analysis in the literature corresponds to perturbations of degree one, among them the so-called Christoffel and Geronimus transformations on the unit circle due to the analogy with the transformations with the same name on the real line. For instance, in such cases, [26,9,13] study the transformation of the Hessenberg matrix encoding the recurrence of the orthogonal polynomials, following a modus operandi which reminds the Darboux transformation for Jacobi matrices. However, the usual approaches to study general polynomial perturbations on the unit circle have the drawback of being formulated in terms of orthogonal polynomials, kernels and determinants, which makes difficult the practical application, specially for perturbations of high degree.

This paper proposes a new method to study the hermitian modifications obtained when multiplying a hermitian functional by a Laurent polynomial of any degree, as well as the inverse of these modifications, in short, the direct and inverse hermitian polynomial modifications. Contrary to previous methods, this new one focuses the attention on the Schur parameters, i.e., the parameters appearing in the recurrence for the orthogonal polynomials. More precisely, it is based on a recurrence for the Schur parameters of the two functionals involved in the perturbation. This provides an algorithm to generate the Schur parameters of one of the functionals, starting from the Schur parameters of the other one. Furthermore, this recurrence yields a characterization of the maximum number of orthogonal polynomials for one of the functionals, given the number of orthogonal polynomials that the other functional has, something that previous methods cannot handle easily. That is, this approach permits us to study the relation between the quasi-definiteness of a functional and a polynomial modification of any degree with more generality and less computational effort than the methods already existing.

We distinguish between three different but related problems, depending on the data at hand:

- Basic problem: to characterize when two functionals are related by a polynomial perturbation in terms of their Schur parameters.
- Direct problem: to characterize the quasi-definiteness of a direct polynomial modification from the Schur parameters of the original functional.
- Inverse problem: to characterize the quasi-definiteness of an inverse polynomial modification from the Schur parameters of the original functional.

Despite the symmetry between the direct and inverse problems, they have a quite different nature which makes much more interesting the last one. The root of this difference is the fact that, given a functional and a Laurent polynomial, the corresponding polynomial modification is uniquely defined while there are infinitely many functionals whose modification is the given one. This leads to a rich structure in the set of solutions of the inverse problem which, as we will see, is related to another kind of interesting modifications: the addition of Dirac deltas and its derivatives. Hence, any information about inverse polynomial modifications can be translated as a result on perturbations by Dirac deltas.

Furthermore, as an example will show, some special solutions of an inverse problem can act as "attractors" for the asymptotics of the parameters of other solutions. Thus, the analysis of those special solutions provides information about the asymptotics of perturbations by Dirac deltas.

The rich structure of the inverse problem has a double interest due to the fact that our approach, based on a recurrence for the Schur parameters, also yields interesting connections between the study of polynomial modifications and difference equations. Therefore, the asymptotics of the solutions of the inverse problem is closely related to the asymptotics of difference equations.

The content of the paper is structured in the following way: the rest of the introduction summarizes the basic definitions and notations; Section 2 includes the main results about hermitian polynomial modifications, i.e., it is devoted to what we call basic problem; direct and inverse problems are discussed in Section 3, including an exhaustive analysis of an explicit example of inverse problem; and Section 4 shows other applications of the techniques developed in the paper, i.e., a complete classification of the pairs of orthogonal polynomials related by certain type of linear relations with constant polynomial coefficients, and the determination of the orthogonal polynomials whose associated ones come from a polynomial modification of degree one of the original orthogonality functional.

Now we proceed with the conventions for the notation.

In what follows $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ are called respectively the unit circle and the open unit disk on the complex plane. $\mathbb{P} := \mathbb{C}[z]$ is the complex vector space of polynomials with complex coefficients, and $\mathbb{P}_n$ the vector subspace of polynomials whose degree is not greater than n, while $\mathbb{P}_{-1} := \{0\}$ is the trivial subspace. The notation $p \propto q$ for any polynomials $p, q \in \mathbb{P}$ means $p(z) = \lambda q(z)$, $\lambda \in \mathbb{C}^*$. $\Lambda := \mathbb{C}[z, z^{-1}]$ is the complex vector space of Laurent polynomials and, for $m \leq n$, we define the vector subspace $\Lambda_{m,n} := \operatorname{span}\{z^m, z^{m+1}, \ldots, z^n\}$. Given any $f \in \Lambda$ we define $f_*(z) = \overline{f}(z^{-1})$ and, if $p \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$, p^* denotes its reversed polynomial $p^*(z) = z^n p_*(z)$.

Sometimes we use the notation $p^*(z) = z^n p_*(z)$ for polynomials $p \in \mathbb{P}_n$ whose degree can be smaller than n. Then we refer to the $*_n$ operator when it is advisable to avoid misunderstandings.

Any hermitian linear functional v on Λ ($v[z^{-n}] = \overline{v[z^n]}$, $n = 0, 1, \ldots$) defines a sesquilinear functional $(\cdot,\cdot)_v \colon \Lambda \times \Lambda \longrightarrow \mathbb{C}$ by

$$(f, g)_v := v[f_* g], \quad f, g \in \Lambda.$$

The sequence $(p_n)_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to the hermitian linear functional v if

(i) $p_n \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$,
(ii) $(p_n, p_m)_v = l_n \delta_{n,m}$, $l_n \neq 0$,

and when such a sequence exists v is called a quasi-definite functional. If $v[1] \neq 0$ we can ensure only the existence of a finite segment of orthogonal polynomials, i.e., a finite set $(p_k)_{k=0}^n$ of polynomials satisfying (i) and (ii). When v has a finite segment of orthogonal polynomials $(p_k)_{k=0}^n$ of length $n + 1$ we say that v is quasi-definite on $\mathbb{P}_n$.

In the positive definite case ($l_n > 0$, $n = 0, 1, \ldots$) there exists a positive measure μ supported on $\mathbb{T}$ providing an integral representation for the functional v,

$$v[f] = \int_{\mathbb{T}} f(z)\mathrm{d}\mu(z), \quad f \in \Lambda.$$

Due to this reason a sequence $(p_n)_{n\geq 0}$ satisfying (i) and (ii) is called a sequence of orthogonal polynomials on the unit circle, even in the general quasi-definite case. If $l_n = \pm 1$ for all n, (p_n) is called a sequence of orthonormal polynomials on the unit circle. We denote by $(\hat{p}_n)_{n\geq 0}$ the orthonormal polynomials with positive leading coefficients.

In what follows $(\psi_n)_{n\geq 0}$ denotes the sequence of monic orthogonal polynomials (MOP) with respect to a hermitian functional v. Two hermitian linear functionals v_1, v_2 have a common finite segment $(\psi_j)_{j=0}^n$ of MOP iff there exists $\lambda \in \mathbb{R}^*$ such that $v_1[f] = \lambda v_2[f]$ for any $f \in \Lambda_{-n,n}$, although requiring this condition to hold only for any $f \in \mathbb{P}_n$ is enough due to the hermiticity. In this case we say that v_1 and v_2 are equivalent in $\mathbb{P}_n$ or, in a more symbolic way, $v_1 \equiv v_2$ in $\mathbb{P}_n$. If this holds for any n, we simply say that v_1 and v_2 are equivalent and we write $v_1 \equiv v_2$.

A sequence (ψ_n) is a sequence of MOP on the unit circle iff it satisfies the recurrence relation (see [40,17,33])

$$\psi_n(z) = z\psi_{n-1}(z) + \psi_n(0)\psi_{n-1}^*(z), \quad n = 1, 2 \ldots, \tag{1}$$

with $\psi_0(z) = 1$ and $|\psi_n(0)| \neq 1$ for $n \geq 1$. Applying the $*_n$ operator to the above recurrence we get the equivalent one

$$\psi_n^*(z) = \overline{\psi_n(0)} z\psi_{n-1}(z) + \psi_{n-1}^*(z), \quad n = 1, 2 \ldots. \tag{2}$$

The values $\psi_n(0)$ are called the Schur parameters or reflection coefficients of the hermitian linear functional v.

A direct consequence of the above recurrence relations is the fact ψ_n and ψ_n^* have no common roots. Indeed, in the positive definite case the roots of ψ_n lie on $\mathbb{D}$, while the roots of ψ_n^* are in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

A straightforward computation yields

$$1 - |\psi_n(0)|^2 = \frac{\varepsilon_n}{\varepsilon_{n-1}}, \quad n = 1, 2, \ldots,$$

where $\varepsilon_n := (\psi_n, \psi_n)_v = v[\psi_n z^{-n}]$ relates $\hat{p}_n$ and ψ_n by $\hat{p}_n = |\varepsilon_n|^{-\frac{1}{2}}\psi_n$. When v is positive definite $\varepsilon_n = \|\psi_n\|^2_{L^2(\mu)} > 0$ for $n \geq 0$, which means that $|\psi_n(0)| < 1$ for $n \geq 1$.

2. Hermitian polynomial modifications

We are interested in those (Laurent) polynomial modifications of hermitian functionals which preserve their hermitian character, in short, the hermitian polynomial modifications of hermitian functionals. If v is a linear functional on Λ and $L \in \Lambda$ the modified functional vL is defined by

$$vL[f] := v[Lf], \quad f \in \Lambda.$$

The modified functional vL is hermitian for every hermitian v iff $L_* = L$, which is equivalent to state that $L = P + P_*$ with $P \in \mathbb{P}$ (see [2]). Such a polynomial P can be uniquely determined by L simply requiring $P(0) \in \mathbb{R}$, a convention that we will assume in what follows. We will refer to $\deg P$ as the degree of the polynomial modification, which we will consider greater than or equal to one, and L will be called a hermitian Laurent polynomial of degree $\deg L = \deg P$.

Another way to characterize a hermitian polynomial modification is through the polynomial $A = z^{\deg P} L$ of degree $2 \deg P$. The condition $L_* = L$ means that A is self-reciprocal, i.e., $A^* = A$. Thus the hermitian polynomial modifications are related to the self-reciprocal polynomials of even degree.

The set of roots of a self-reciprocal polynomial, counting the multiplicity, is invariant under the transformation $\zeta \to 1/\overline{\zeta}$. That is, their roots lie on the unit circle or appear in symmetric pairs $\zeta, 1/\overline{\zeta}$. Indeed, this property characterizes the self-reciprocal polynomials up to numerical factors. This implies that any self-reciprocal polynomial of even degree factorizes into a product of self-reciprocal polynomials of degree 2. As a consequence, an arbitrary hermitian polynomial modification is a composition of elementary ones of degree 1, i.e., if $L = P + P_*$ with $\deg P = r$, then $L = L_1 L_2 \cdots L_r$ with $L_k = P_k + P_{k*}$ and $\deg P_k = 1$.

Sometimes we will deal with polynomials $A \in \mathbb{P}_n$ whose degree is not necessarily n but such that $A^{*_n} = A$. In this case we will say that A is self-reciprocal in $\mathbb{P}_n$ to avoid misunderstandings. Such a polynomial has the general form $A(z) = z^s B(z)$ where B is strictly self-reciprocal. Thus, a self-reciprocal polynomial in $\mathbb{P}_n$ is actually self-reciprocal iff it has no zeros at the origin.

Given a hermitian functional v and a Laurent polynomial $L = P + P_*$, our purpose is to obtain relations between the MOP and Schur parameters associated with the functionals v and vL. Multiplying L by a non-null real factor gives rise to a hermitian functional which is equivalent to vL and, hence, with the same MOP and Schur parameters as vL. Therefore, concerning our aim, the Laurent polynomial L, as well as the polynomials P and A, are defined up to non-null real factors.

The following general result will be useful to achieve our objective. In what follows we denote by $S^{\perp_n}$ the orthogonal complement in $\mathbb{P}_n$ of a subspace $S \subset \mathbb{P}_n$.

Lemma 2.1 (*See [41]*)**.** *Let v be a hermitian functional such that the corresponding n-th MOP ψ_n exists. Then, $\mathfrak{B} = \{z^k\psi_n\}_{k=0}^{r} \cup \{z^k\psi_n^*\}_{k=0}^{r-1}$ is a basis of $(z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$ for $n \geq r \geq 1$, and a generator system of $\mathbb{P}_{n+r}$ for $r > n \geq 0$.*

Sketch of the Proof. If $n \geq r \geq 1$, the orthogonality of ψ_n ensures that $\mathfrak{B} \subset (z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$. Besides, $\Omega \in \operatorname{span}\mathfrak{B}$ iff $\Omega = C\psi_n + D\psi_n^*$, $C \in \mathbb{P}_r$, $D \in \mathbb{P}_{r-1}$. Furthermore, this decomposition is unique because $\gcd(\psi_n, \psi_n^*) = 1$, which proves the linear independence of $\mathfrak{B}$. Then, the first result follows from the fact that $\sharp\mathfrak{B} = 2r + 1 = \dim(z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$.

Suppose now that $r > n \geq 0$. From the previous result we know that $\{z^k\psi_n\}_{k=0}^{n} \cup \{z^k\psi_n^*\}_{k=0}^{n-1}$ is a basis of $\mathbb{P}_{2n}$. Hence, $\{z^k\psi_n\}_{k=0}^{r} \cup \{z^k\psi_n^*\}_{k=0}^{n-1}$ is a linear independent subset of $\mathbb{P}_{n+r}$ with $n+r+1$ elements, thus it is a basis of $\mathbb{P}_{n+r}$, which proves the second result. □

Our interest in the previous lemma is the following direct consequence.

Corollary 2.2. *Let v be a hermitian functional such that the corresponding n-th MOP ψ_n exists. Then, every polynomial $\Omega \in (z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$ has a unique decomposition $\Omega = C\psi_n + D\psi_n^*$, $C \in \mathbb{P}_r$, $D \in \mathbb{P}_{r-1}$, for $n \geq r \geq 1$, and every polynomial $\Omega \in \mathbb{P}_{n+r}$ has infinitely many such decompositions for $r > n \geq 0$.*

Remark 2.3. It is worth remarking the case $n = r$ in the above corollary, which says that every polynomial $\Omega \in \mathbb{P}_{2r}$ admits a unique decomposition $\Omega = C\psi_r + D\psi_r^*$, $C \in \mathbb{P}_r$, $D \in \mathbb{P}_{r-1}$.

The next theorem is the starting point for our approach to the study of hermitian polynomial modifications of hermitian functionals.

Theorem 2.4. *Let u, v be hermitian functionals with finite segments of MOP $(\varphi_j)_{j=0}^{n}$, $(\psi_j)_{j=0}^{n+r}$ respectively, and let $L = P + P_* = z^{-r}A$ with P a polynomial of degree r. Then, the following statements are equivalent:*

(i) *$u \equiv vL$ in $\mathbb{P}_n$.*

(ii) *There exist $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ with $C_j(0) \neq 0$ such that*

$$A\varphi_j = C_j\psi_{j+r} + D_j\psi_{j+r}^*, \quad j = 0, \ldots, n. \tag{3}$$

(iii) *There exist $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ with $C_j(0) \neq 0$ such that*

$$A\varphi_j^* = zD_j^*\psi_{j+r} + C_j^*\psi_{j+r}^*, \quad D_j^* = D_j^{*_{r-1}},\ j = 0, \ldots, n. \tag{4}$$

The polynomials $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ satisfying (3) or (4) are unique, $\deg C_j = r$, $C_j(0) \in \mathbb{R}$ *and* $C_j^*(0) = A(0)$.

Proof. The equivalence between (ii) and (iii) follows from the use of the $*_{2r+j}$ operator and the fact that A is a self-reciprocal polynomial of degree $2r$. Also, assuming (ii) we get $\deg C_j = r$ because $\deg(D_j\psi_{j+r}^*) < \deg(A\varphi_j) = 2r + j$, and the equality $(\varphi_j, \varphi_j)_u = u[\varphi_j z^{-j}] = C_j(0)\varepsilon_{j+r}$ implies $C_j(0) \in \mathbb{R}$. On the other hand, evaluating (4) at $z = 0$ we find that $C_j^*(0) = A(0)$. It only remains to prove the equivalence between (i) and (ii) and the uniqueness of decomposition (3).

Suppose (i), i.e., $u[f] = \lambda vL[f]$, $\lambda \in \mathbb{R}^*$, for any $f \in \Lambda_{-n,n}$. The orthogonality of $(\varphi_j)_{j=0}^{n}$ with respect to u gives

$$0 = u[\varphi_j z^{-k}] = \lambda v[A\varphi_j z^{-(k+r)}], \quad r \leq k + r \leq j + r - 1,$$

which means that $A\varphi_j \in (z^r\mathbb{P}_{j-1})^{\perp_{2r+j}}$ with respect to v. Using Corollary 2.2 we get (3) and the uniqueness of the polynomials C_j, D_j.

On the other hand, if $(\varphi_j)_{j=0}^{n}$, $(\psi_j)_{j=0}^{n+r}$ satisfy (3), the orthogonality of $(\psi_j)_{j=0}^{n+r}$ with respect to v yields

$$vL[\varphi_j z^{-k}] = v[A\varphi_j z^{-(k+r)}] = v[(C_j\psi_{j+r} + D_j\psi_{j+r}^*)z^{-(k+r)}] = 0$$

for $0 \leq k \leq j - 1$ and

$$vL[\varphi_j z^{-j}] = v[A\varphi_j z^{-(j+r)}] = v[(C_j\psi_{j+r} + D_j\psi_{j+r}^*)z^{-(j+r)}] = C_j(0)\varepsilon_{j+r}.$$

So, $C_j(0) \neq 0$ for $j = 0, \ldots, n$ iff $(\varphi_j)_{j=0}^n$ is a finite segment of MOP with respect to vL, which means that $u \equiv vL$ in $\mathbb{P}_n$. $\square$

Equality (4) is true taking $D_j^* = D_j^{*r-1}$, no matter whether D_j has degree $r-1$ or not. In what follows we will assume this convention for the polynomials D_j.

Remark 2.5. The functional u has a finite segment of MOP of length (at least) one iff $u[1] \neq 0$. Therefore, Theorem 2.4 ensures that the condition $v[L] \neq 0$ is equivalent to the existence of a (unique) decomposition

$$A = C_0\psi_r + D_0\psi_r^*, \quad C_0 \in \mathbb{P}_r,\ D_0 \in \mathbb{P}_{r-1}, \tag{5}$$

with $C_0(0) \neq 0$. However, Remark 2.3 says even more: no matter the value of $v[L]$, there is always a unique decomposition like (5). The equality $v[L] = C_0(0)\varepsilon_r$ implies that $v[L] \neq 0$ is only responsible of $C_0(0) \neq 0$.

The above theorem has the following consequence for quasi-definite functionals.

Corollary 2.6. *Let u, v be quasi-definite functionals with sequences of MOP (φ_n), (ψ_n) respectively, and let $L = P + P_* = z^{-r}A$ with P a polynomial of degree r. Then, $u \equiv vL$ iff there exist polynomials $C_n \in \mathbb{P}_r$, $D_n \in \mathbb{P}_{r-1}$ with $C_n(0) \neq 0$ such that*

$$A\varphi_n = C_n\psi_{n+r} + D_n\psi_{n+r}^*, \quad n \geq 0, \tag{6}$$

or equivalently

$$A\varphi_n^* = zD_n^*\psi_{n+r} + C_n^*\psi_{n+r}^*, \quad n \geq 0.$$

For convenience, in what follows we will use a matrix notation and we will adopt some definitions and conventions that will be used in the rest of the paper. If L is a hermitian Laurent polynomial of degree r, P and A are the polynomials given by $L = P + P_* = z^{-r}A$, $P(0) \in \mathbb{R}$. We denote by φ_j and ψ_j the j-th MOP with respect to the hermitian functionals u and v respectively. Also,

$$a_j = \varphi_j(0), \qquad b_j = \psi_j(0), \qquad e_j = (\varphi_j, \varphi_j)_u, \qquad \varepsilon_j = (\psi_j, \psi_j)_v,$$

$$\Phi_j = \begin{pmatrix} \varphi_j \\ \varphi_j^* \end{pmatrix}, \qquad \mathcal{S}_j = \begin{pmatrix} z & a_j \\ z\bar{a}_j & 1 \end{pmatrix}, \qquad \mathcal{A}_j = \begin{pmatrix} 1 & a_j \\ \bar{a}_j & 1 \end{pmatrix},$$

$$\Psi_j = \begin{pmatrix} \psi_j \\ \psi_j^* \end{pmatrix}, \qquad \mathcal{T}_j = \begin{pmatrix} z & b_j \\ z\bar{b}_j & 1 \end{pmatrix}, \qquad \mathcal{B}_j = \begin{pmatrix} 1 & b_j \\ \bar{b}_j & 1 \end{pmatrix},$$

$$\mathcal{C}_j = \begin{pmatrix} C_j & D_j \\ zD_j^* & C_j^* \end{pmatrix}, \qquad \tilde{\mathcal{C}}_j = \begin{pmatrix} C_j & zD_j \\ D_j^* & C_j^* \end{pmatrix}.$$

The matrices $\mathcal{S}_j$ and $\mathcal{T}_j$, known as *transfer matrices*, permit us to write recurrence relations (1) and (2) for (φ_n) and (ψ_n) in the compact form

$$\Phi_j = \mathcal{S}_j\Phi_{j-1}, \qquad \Psi_j = \mathcal{T}_j\Psi_{j-1}, \tag{7}$$

while the matrices $\mathcal{C}_j$ make possible to combine (3) and (4) into

$$A\Phi_j = \mathcal{C}_j\Psi_{j+r}.$$

The structure of the matrices $\mathcal{C}_j$ is worth to be remarked.

Definition 2.7. A polynomial matrix $\mathcal{C} = \begin{pmatrix} C_1 & D_1 \\ D_2 & C_2 \end{pmatrix}$, $C_i, D_i \in \mathbb{P}_r$, satisfying $\mathcal{C}^{*_r} = J\mathcal{C}J$ with $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ will be called a *J-self-reciprocal matrix* in $\mathbb{P}_r$. This is equivalent to state that $C_2 = C_1^{*_r}$ and $D_2 = D_1^{*_r}$.

We denote by $\mathbb{J}_r$ the set of J-self-reciprocal matrices in $\mathbb{P}_r$ such that $C_2(0) \neq 0$ and $D_2(0) = 0$. These conditions mean that $\deg C_1 = r$ and $\deg D_1 \leq r - 1$, thus the general form of a polynomial matrix $\mathcal{C} \in \mathbb{J}_r$ is

$$\mathcal{C} = \begin{pmatrix} C & D \\ zD^* & C^* \end{pmatrix}, \qquad \deg C = r, \qquad \deg D \leq r - 1, \tag{8}$$

where here and below we assume that $D^* = D^{*_{r-1}}$.

Given a polynomial matrix $\mathcal{C} \in \mathbb{J}_r$ like (8) we will denote

$$\tilde{\mathcal{C}} = \begin{pmatrix} C & zD \\ D^* & C^* \end{pmatrix},$$

which is J-self-reciprocal too, but in general does not necessarily belong to $\mathbb{J}_r$ because zD can have degree r.

The determinant of a J-self-reciprocal matrix $\mathcal{C}$ in $\mathbb{P}_r$ is a self-reciprocal polynomial in $\mathbb{P}_{2r}$. When $\det\mathcal{C}$ has degree $2r$ we will say that $\mathcal{C}$ is a *regular J-self-reciprocal matrix*. This is equivalent to $\det\mathcal{C}(0) \neq 0$, which in case of $\mathcal{C} \in \mathbb{J}_r$ means simply $C(0) \neq 0$. We will denote by $\mathbb{J}_r^{\text{reg}}$ the subset of regular J-self-reciprocal matrices of $\mathbb{J}_r$.

The next result about J-self-reciprocal matrices will be useful later on.

Lemma 2.8. *Let $\mathcal{S} = \begin{pmatrix} z & a \\ z\overline{a} & 1 \end{pmatrix}$, $\mathcal{T} = \begin{pmatrix} z & b \\ z\overline{b} & 1 \end{pmatrix}$ with $a, b \in \mathbb{C}$.*

(i) *If $|a| \neq 1$, $\mathcal{C} \in \mathbb{J}_r$, the equation $\mathcal{C}\mathcal{T} = \mathcal{S}\hat{\mathcal{C}}$ defines a matrix $\hat{\mathcal{C}} \in \mathbb{J}_r$ iff*

$$aC^*(0) = bC(0) + D(0).$$

In this case $\hat{\mathcal{C}} \in \mathbb{J}_r^{\text{reg}} \Leftrightarrow |b| \neq 1, \mathcal{C} \in \mathbb{J}_r^{\text{reg}}$.

(ii) *If $|b| \neq 1$, $\mathcal{C} \in \mathbb{J}_r$, the equation $\hat{\mathcal{C}}\mathcal{T} = \mathcal{S}\mathcal{C}$ defines a matrix $\hat{\mathcal{C}} \in \mathbb{J}_r$ iff*

$$a\overline{C(0)} = b\overline{C^*(0)} - \overline{D^*(0)}.$$

In this case $\hat{\mathcal{C}} \in \mathbb{J}_r^{\text{reg}} \Leftrightarrow |a| \neq 1, \mathcal{C} \in \mathbb{J}_r^{\text{reg}}$.

Proof. If $|a| \neq 1$ the equation $\mathcal{C}\mathcal{T} = \mathcal{S}\hat{\mathcal{C}}$ can be written as

$$\hat{\mathcal{C}} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{X} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{X} = \frac{1}{1 - |a|^2} \begin{pmatrix} 1 & -a \\ -\overline{a} & 1 \end{pmatrix} \mathcal{C} \begin{pmatrix} 1 & b \\ \overline{b} & 1 \end{pmatrix}.$$

Let $\mathcal{C} \in \mathbb{J}_r$. Then $\mathcal{X}$ is a J-self-reciprocal matrix in $\mathbb{P}_r$, i.e., $\mathcal{X} = \begin{pmatrix} X & Y \\ Y^{*_r} & X^{*_r} \end{pmatrix}$ with $X, Y \in \mathbb{P}_r$. Therefore, $\hat{\mathcal{C}}$ is a polynomial matrix iff $Y(0) = 0$, which yields the relation between a and b given in (i). In such a case $Y = z\hat{Y}$, $\hat{Y} \in \mathbb{P}_{r-1}$, and $X^{*_r}(0) = C^*(0) \neq 0$, thus $\hat{\mathcal{C}} = \begin{pmatrix} X & \hat{Y} \\ z\hat{Y}^{*_{r-1}} & X^{*_r} \end{pmatrix} \in \mathbb{J}_r$. Also, $X(0) = C(0)(1 - |b|^2)/(1 - |a|^2)$, hence $\hat{\mathcal{C}} \in \mathbb{J}_r^{\text{reg}} \Leftrightarrow |b| \neq 1, \mathcal{C} \in \mathbb{J}_r^{\text{reg}}$.

On the other hand, if $|b| \neq 1$ the equation $\hat{\mathcal{C}}\mathcal{T} = \mathcal{S}\mathcal{C}$ reads as

$$\hat{\mathcal{C}} = \frac{1}{1 - |b|^2} \begin{pmatrix} 1 & a \\ \overline{a} & 1 \end{pmatrix} \tilde{\mathcal{C}} \begin{pmatrix} 1 & -b \\ -\overline{b} & 1 \end{pmatrix}.$$

Suppose that $\mathcal{C} \in \mathbb{J}_r$. Then $\hat{\mathcal{C}}$ is a J-self-reciprocal matrix in $\mathbb{P}_r$, hence $\hat{\mathcal{C}} = \begin{pmatrix} X & Y \\ Y^{*_r} & X^{*_r} \end{pmatrix}$ with $X, Y \in \mathbb{P}_r$. The relation between a and b given in (ii) is equivalent to $Y^{*_r}(0) = 0$, and also gives $X^{*_r}(0) = C^*(0) \neq 0$, $X(0) = C(0)(1-|a|^2)/(1-|b|^2) \neq 0$, so $\hat{\mathcal{C}} \in \mathbb{J}_r$ and $\hat{\mathcal{C}} \in \mathbb{J}_r^{\text{reg}} \Leftrightarrow |a| \neq 1, \mathcal{C} \in \mathbb{J}_r^{\text{reg}}$. $\square$

The goal of the rest of the section is to present a more economical and effective approach than the ones already existing in the literature (see for instance [18–20]) to study the relation $u \equiv vL$ for any degree of L. This new point of view avoids the calculation of determinants and MOP related to u and v, requiring only the knowledge of the corresponding Schur parameters and the Laurent polynomial L. More precisely, we will characterize the relation $u \equiv vL$ through a matrix difference equation for the Schur parameters involving J-self-reciprocal matrices.

The first step to formulate this new approach is to translate the relations between the MOP (φ_n) and (ψ_n) into relations between the corresponding Schur parameters. The following result will be useful for this purpose.

Lemma 2.9. *Let P, Q be relatively prime polynomials with* $\deg Q \leq \deg P$. *If the polynomial matrices*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

satisfy $\deg(M_2 - N_2), \deg(M_4 - N_4) < \deg P$, *then*

$$M \begin{pmatrix} P \\ Q \end{pmatrix} = N \begin{pmatrix} P \\ Q \end{pmatrix} \Leftrightarrow M = N.$$

Proof. $M_1 P + M_2 Q = N_1 P + N_2 Q$, thus $(M_1 - N_1)P = (N_2 - M_2)Q$. Since $\gcd(P, Q) = 1$, necessarily P divides $M_2 - N_2$, which implies $M_2 - N_2 = 0$ because $\deg(M_2 - N_2) < \deg P$. Therefore $M_1 - N_1 = 0$ too. Analogously $M_3 - N_3 = M_4 - N_4 = 0$. $\square$

The next result is the matrix form of Theorem 2.4, together with a stronger result and some properties of the polynomial matrices $\mathcal{C}_j$, including the first relations between the Schur parameters (a_n) and (b_n).

Theorem 2.10. *Let u, v be quasi-definite in $\mathbb{P}_n$, $\mathbb{P}_{n+r}$ respectively and let L be a hermitian Laurent polynomial of degree r. Then, the following statements are equivalent:*

(i) *$u \equiv vL$ in $\mathbb{P}_n$.*

(ii) *There exist $\mathcal{C}_0, \ldots, \mathcal{C}_n \in \mathbb{J}_r^{\text{reg}}$ such that*

$$A\Phi_j = \mathcal{C}_j \Psi_{j+r}, \quad j = 0, \ldots, n. \tag{9}$$

(iii) *There exists $\mathcal{C}_n \in \mathbb{J}_r^{\text{reg}}$ such that*

$$A\Phi_n = \mathcal{C}_n \Psi_{n+r}. \tag{10}$$

The matrices $\mathcal{C}_j$ are the only solutions of (9) in $\mathbb{J}_r$, so $\mathcal{C}_0$ is determined by

$$\mathcal{C}_0 \Psi_r = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathcal{C}_0 \in \mathbb{J}_r. \tag{11}$$

Besides, we have the relations

$$\mathcal{C}_j \mathcal{T}_{j+r} = \mathcal{S}_j \mathcal{C}_{j-1}, \quad j = 1, \dots, n, \tag{12}$$

$$\mathcal{C}_j \mathcal{B}_{j+r} = \mathcal{A}_j \tilde{\mathcal{C}}_{j-1}, \quad j = 1, \dots, n, \tag{13}$$

$$\det \mathcal{C}_j = C_j(0) A, \quad j = 0, \dots, n. \tag{14}$$

Proof. Bearing in mind Theorem 2.4, it is enough to prove (iii)⇒(ii)⇒ (12), (13), (14). Suppose that only (iii) holds. Evaluating (10) at $z = 0$ we find $a_n A(0) = b_{n+r} C_n(0) + D_n(0)$ and $C_n^*(0) = A(0)$. Hence, Lemma 2.8(i) ensures the existence of $\mathcal{C}_{n-1} \in \mathbb{J}_r^{\mathrm{reg}}$ satisfying $\mathcal{C}_n \mathcal{T}_{n+r} = \mathcal{S}_n \mathcal{C}_{n-1}$. Then, the equality $A \mathcal{S}_n \Phi_{n-1} = A \Phi_n = \mathcal{C}_n \Psi_{n+r} = \mathcal{C}_n \mathcal{T}_{n+r} \Psi_{n+r-1} = \mathcal{S}_n \mathcal{C}_{n-1} \Psi_{n+r-1}$ shows that $A \Phi_{n-1} = \mathcal{C}_{n-1} \Psi_{n+r-1}$. Iterating this procedure we obtain (ii).

Combining (9) and recurrence relations (7),

$$A \Phi_j = \mathcal{C}_j \Psi_{j+r} = \mathcal{C}_j \mathcal{T}_{j+r} \Psi_{j+r-1}, \qquad A \Phi_j = A \mathcal{S}_j \Phi_{j-1} = \mathcal{S}_j \mathcal{C}_{j-1} \Psi_{j+r-1}.$$

Therefore, $\mathcal{C}_j \mathcal{T}_{j+r} \Psi_{j+r-1} = \mathcal{S}_j \mathcal{C}_{j-1} \Psi_{j+r-1}$, or equivalently

$$\mathcal{C}_j \mathcal{B}_{j+r} \begin{pmatrix} z\psi_{j+r-1} \\ \psi_{j+r-1}^* \end{pmatrix} = \mathcal{A}_j \tilde{\mathcal{C}}_{j-1} \begin{pmatrix} z\psi_{j+r-1} \\ \psi_{j+r-1}^* \end{pmatrix}.$$

Taking into account that $z\psi_j$, ψ_j^* are relatively prime and $\deg C_j = r$, $\deg D_j \le r - 1$, relations (12) and (13) follow from Lemma 2.9.

To prove (14) notice that $A = C_0 \psi_r + D_0 \psi_r^* = C_0^* \psi_r^* + z D_0^* \psi_r$, hence we have the equality $(C_0 - z D_0^*) \psi_r = (C_0^* - D_0) \psi_r^*$. Since ψ_r, ψ_r^* are relatively prime this implies $C_0(0) \psi_r = C_0^* - D_0$ and $C_0(0) \psi_r^* = C_0 - z D_0^*$. So,

$$C_0(0) A = C_0(0)(C_0 \psi_r + D_0 \psi_r^*) = C_0 C_0^* - z D_0 D_0^* = \det \mathcal{C}_0.$$

Besides, from (12) we find that $\det \mathcal{C}_j \propto \det \mathcal{C}_0$ for $j = 1, \dots, n$. Evaluating at $z = 0$ we finally obtain $\det \mathcal{C}_j = (C_j(0)/C_0(0)) \det \mathcal{C}_0 = C_j(0) A$. □

The equivalence (i)⇔(iii) of the previous theorem means that the last condition ($j = n$) in (3) or (4) suffices for the equivalence in Theorem 2.4.

There also exist inverse relations between the finite segments of MOP $(\varphi_j)_{j=0}^n$ and $(\psi_j)_{j=0}^{n+r}$. The polynomial matrix coefficients of these inverse relations are not independent of the polynomial matrix coefficients $\mathcal{C}_j$ of the direct relations. Indeed, both polynomial matrix coefficients are essentially adjoints of each other, understanding the adjoint of a 2×2 matrix $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ as the matrix $\mathrm{Adj}(M) = \begin{pmatrix} M_4 & -M_2 \\ -M_3 & M_1 \end{pmatrix}$. Thus, given a 2×2 polynomial matrix M in $\mathbb{P}_r$, $\mathrm{Adj}(M)$ is a 2×2 polynomial matrix in $\mathbb{P}_r$ satisfying

$$\mathrm{Adj}(M) M = (\det M) I,$$

where I is the identity matrix of the same size as M.

Theorem 2.11. *If u, v are quasi-definite in $\mathbb{P}_n$, $\mathbb{P}_{n+r}$ respectively, the following statements are equivalent:*

(i) *$u \equiv vL$ in $\mathbb{P}_n$ for some hermitian Laurent polynomial L of degree r.*
(ii) *There exist $\mathcal{X}_r, \dots, \mathcal{X}_n \in \mathbb{J}_r^{\mathrm{reg}}$ such that*

$$\Psi_{j+r} = \mathcal{X}_j \Phi_j, \quad j = r, \dots, n. \tag{15}$$

(iii) *There exists* $\mathcal{X}_n \in \mathbb{J}_r^{\mathrm{reg}}$ *such that*

$$\Psi_{n+r} = \mathcal{X}_n \Phi_n. \tag{16}$$

The matrices $\mathcal{X}_j$ *are the only solutions of* (15) *in* $\mathbb{J}_r$*, so* $\mathcal{X}_r$ *is determined by*

$$\mathcal{X}_r \Phi_r = \Psi_{2r}, \quad \mathcal{X}_r \in \mathbb{J}_r. \tag{17}$$

Besides, we have the relations

$$\mathcal{T}_{j+r}\mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = r+1, \dots, n \tag{18}$$

$$\mathcal{B}_{j+r}\tilde{\mathcal{X}}_{j-1} = \mathcal{X}_j \mathcal{A}_j, \quad j = r+1, \dots, n \tag{19}$$

$$\det \mathcal{X}_j \propto A, \quad j = r, \dots, n \tag{20}$$

$$\mathcal{C}_j \mathcal{X}_j = A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = r, \dots, n \tag{21}$$

$$\mathcal{X}_j = \frac{1}{C_j(0)} \mathrm{Adj}(\mathcal{C}_j), \quad j = r, \dots, n. \tag{22}$$

Proof. If $u \equiv vL$ in $\mathbb{P}_n$, Theorem 2.10 ensures the existence of $\mathcal{C}_j \in \mathbb{J}_r^{\mathrm{reg}}$ such that $A\Phi_j = \mathcal{C}_j \Psi_{j+r}$ for $j = 0, \dots, n$. Multiplying this identity on the left by $\mathrm{Adj}(\mathcal{C}_j)$ and taking into account (14) we find that $\Psi_{j+r} = \mathcal{X}_j \Phi_j$ for $j = 0, \dots, n$, where $\mathcal{X}_j = \mathrm{Adj}(\mathcal{C}_j)/C_j(0) \in \mathbb{J}_r^{\mathrm{reg}}$. Then, (18)–(21) are a direct consequence of (12)–(14). The uniqueness of $\mathcal{X}_j \in \mathbb{J}_r$ for $j \geq r$ follows from Corollary 2.2 and the fact that (15) is equivalent to $\psi_{j+r} = X_j\varphi_j + Y_j\varphi_j^*$, where $X_j \in \mathbb{P}_r$, $Y_j \in \mathbb{P}_{r-1}$ are the polynomials appearing in $\mathcal{X}_j = \begin{pmatrix} X_j & Y_j \\ zY_j^* & X_j^* \end{pmatrix}$.

It only remains to prove (iii)⇒(i). Multiplying (16) on the left by $\mathcal{C}_n = \mathrm{Adj}(\mathcal{X}_n) \in \mathbb{J}_r^{\mathrm{reg}}$ we obtain $A\Phi_n = \mathcal{C}_n \Psi_{n+r}$ where $A = \det \mathcal{X}_n$ is a self-reciprocal polynomial of degree $2r$. This proves that $u \equiv vAz^{-r}$ due to Theorem 2.10. □

Concerning the polynomial matrix coefficients $\mathcal{X}_j \in \mathbb{J}_r^{\mathrm{reg}}$ of the inverse relations, when it is necessary we will use the explicit notation

$$\mathcal{X}_j = \begin{pmatrix} X_j & Y_j \\ zY_j^* & X_j^* \end{pmatrix}, \qquad \deg X_j = r, \qquad \deg Y_j = r-1,$$

so that $\Psi_{j+r} = \mathcal{X}_j \Phi_j$ is equivalent to $\psi_{j+r} = X_j\varphi_j + Y_j\varphi_j^*$. This shows that X_j is monic. Besides, from (22) we have the relations $X_j = C_j^*/C_j(0)$, $Y_j = -D_j/C_j(0)$.

The proof of the previous theorem shows that, when $u \equiv vL$ in $\mathbb{P}_n$ for some hermitian Laurent polynomial L of degree r,

$$\Psi_{j+r} = \mathcal{X}_j \Phi_j, \quad \mathcal{X}_j \in \mathbb{J}_r, \; j = 0, \dots, n, \tag{23}$$

and not only for $j \geq r$. Indeed, the proof of the theorem implies that (23) has solutions $\mathcal{X}_j \in \mathbb{J}_r^{\mathrm{reg}}$ for $j < r$ too. The only difference is that, contrary to $j \geq r$, (23) does not determine $\mathcal{X}_j$ univocally for $j < r$, as Corollary 2.2 points out. The reason is that $\mathfrak{B} = \{z^k\psi_j\}_{k=0}^r \cup \{z^k\psi_j^*\}_{k=0}^{r-1}$ is linearly independent for $j \geq r$, but not for $j < r$. Actually, when $j < r$, Lemma 2.1 shows that $\mathrm{rank}(\mathfrak{B}) = j + r + 1$, so the solutions $\mathcal{X}_j$ of (23) form an affine subspace of dimension $r - j$.

Among the solutions of (23) for $j < r$ there is a choice of special interest: similar arguments to those at the beginning of the proof of Theorem 2.10 show that Lemma 2.8(i), together with (17), ensures that (18) can be extended in a unique way to $j = 1, \dots, r$, giving rise to particular solutions $\mathcal{X}_0, \dots, \mathcal{X}_{r-1} \in \mathbb{J}_r^{\text{reg}}$ of (23). The choice of $\mathcal{X}_j$ determined by the extension of (18) has the particularity that $\det \mathcal{X}_j$ is independent of j up to numerical factors. Indeed, this property characterizes such a particular choice because different solutions of (23) cannot have proportional determinants: let $\mathcal{X}^{(1)}, \mathcal{X}^{(2)} \in \mathbb{J}_r$ be such that $\Psi_{j+r} = \mathcal{X}^{(k)} \Phi_j$. Then, $(\det \mathcal{X}^{(k)}) \Phi_j = \mathcal{C}^{(k)} \Psi_{j+r}$ with $\mathcal{C}^{(k)} = \operatorname{Adj}(\mathcal{X}^{(k)})$. If $\det \mathcal{X}^{(2)} = \lambda \det \mathcal{X}^{(1)}$, $\lambda \in \mathbb{R}^*$, Lemma 2.9 ensures that $\mathcal{C}^{(2)} = \lambda \mathcal{C}^{(1)}$, thus $\det \mathcal{X}^{(2)} = \lambda^2 \det \mathcal{X}^{(1)}$, which implies $\lambda = 1$, so $\mathcal{X}^{(2)} = \mathcal{X}^{(1)}$.

Properties (12) and (18) are the cornerstone of the main objective of this section: a new characterization of the relation $u \equiv vL$ in terms of a recurrence for the corresponding Schur parameters. Like in the previous characterizations, the J-self-reciprocal matrices play an important role, but now only one MOP of u and v enters in the equivalence, and it appears only in the initial condition for the recurrence. The direct and inverse relations between the MOP of u and v lead to different characterizations, depending on whether the hermitian Laurent polynomial L is fixed or not. Indeed, L appears explicitly only in the initial condition for the direct characterization.

Theorem 2.12. *Let u, v be quasi-definite in $\mathbb{P}_n$, $\mathbb{P}_{n+r}$ respectively and consider an index $m \in \{0, \dots, n\}$.*

(i) *Given a hermitian Laurent polynomial L of degree r, $u \equiv vL$ in $\mathbb{P}_n$ iff there exist $\mathcal{C}_m \in \mathbb{J}_r^{\text{reg}}$ and $\mathcal{C}_{m+1}, \dots, \mathcal{C}_n \in \mathbb{J}_r$ such that*

$$\mathcal{C}_m \Psi_{m+r} = A \Phi_m, \quad \textit{(Direct Initial Condition)}$$
$$\mathcal{C}_j \mathcal{T}_{j+r} = \mathcal{S}_j \mathcal{C}_{j-1}, \quad j = m+1, \dots, n \quad \textit{(Direct Recurrence).}$$

Moreover, $A \Phi_j = \mathcal{C}_j \Psi_{j+r}$, $\mathcal{C}_j \in \mathbb{J}_r^{\text{reg}}$ and $\det \mathcal{C}_j \propto A$ for $j = m, \dots, n$.

(ii) *There is a hermitian Laurent polynomial L of degree r such that $u \equiv vL$ in $\mathbb{P}_n$ iff there exist $\mathcal{X}_m \in \mathbb{J}_r^{\text{reg}}$ and $\mathcal{X}_{m+1}, \dots, \mathcal{X}_n \in \mathbb{J}_r$ such that*

$$\mathcal{X}_m \Phi_m = \Psi_{m+r}, \quad \textit{(Inverse Initial Condition)}$$
$$\mathcal{T}_{j+r} \mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = m+1, \dots, n \quad \textit{(Inverse Recurrence).}$$

Moreover, $\Psi_{j+r} = \mathcal{X}_j \Phi_j$, $\mathcal{X}_j \in \mathbb{J}_r^{\text{reg}}$ and $\det \mathcal{X}_j \propto A$ for $j = m, \dots, n$.

Proof. We will prove only (i), the proof of (ii) being similar. In view of Theorem 2.10, it suffices to show that Direct Initial Condition and Direct Recurrence imply $A \Phi_j = \mathcal{C}_j \Psi_{j+r}$, $j = m, \dots, n$ and $\mathcal{C}_j \in \mathbb{J}_r^{\text{reg}}$, $j = m+1, \dots, n$ when $\mathcal{C}_m \in \mathbb{J}_r^{\text{reg}}$. Direct Recurrence yields $(1-|b_{j+r}|^2) \det \mathcal{C}_j = (1-|a_j|^2) \det \mathcal{C}_{j-1}$, thus $\mathcal{C}_m \in \mathbb{J}_r^{\text{reg}}$ implies $\mathcal{C}_j \in \mathbb{J}_r^{\text{reg}}$ for $j = m+1, \dots, n$. Also, Direct Recurrence and Direct Initial Condition combined with recurrence relations (7) lead to $A \Phi_j = A \mathcal{S}_j \cdots \mathcal{S}_{m+1} \Phi_m = \mathcal{S}_j \cdots \mathcal{S}_{m+1} \mathcal{C}_m \Psi_{m+r} = \mathcal{C}_j \mathcal{T}_{j+r} \cdots \mathcal{T}_{m+r+1} \Psi_{m+r} = \mathcal{C}_j \Psi_{j+r}$ for $j = m, \dots, n$. $\square$

Some special cases of the above theorem will be of interest to us. We will summarize them.

Theorem 2.13. *Let u, v be quasi-definite in $\mathbb{P}_n$, $\mathbb{P}_{n+r}$ respectively.*

Direct characterization . *Given a hermitian Laurent polynomial L of degree r, the following statements are equivalent:*

(i) *$u \equiv vL$ in $\mathbb{P}_n$.*

(ii) *There exist $\mathcal{C}_0 \in \mathbb{J}_r^{\text{reg}}$ and $\mathcal{C}_1, \dots, \mathcal{C}_n \in \mathbb{J}_r$ such that*

$$\mathcal{C}_0 \Psi_r = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \textit{(Initial Condition D)}$$
$$\mathcal{C}_j \mathcal{T}_{j+r} = \mathcal{S}_j \mathcal{C}_{j-1}, \quad j = 1, \dots, n \quad \textit{(Recurrence D).}$$

Inverse characterization . *The following statements are equivalent:*

(i) $u \equiv vL$ *in* $\mathbb{P}_n$ *for some hermitian Laurent polynomial* L *of degree* r.

(ii) *There exist* $\mathcal{X}_r \in \mathbb{J}_r^{\mathrm{reg}}$ *and* $\mathcal{X}_{r+1}, \ldots, \mathcal{X}_n \in \mathbb{J}_r$ *such that*

$$\mathcal{X}_r \Phi_r = \Psi_{2r}, \quad \textit{(Initial Condition I1)}$$
$$\mathcal{T}_{j+r}\mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = r+1, \ldots, n \quad \textit{(Recurrence I1).}$$

(iii) *There exist* $\mathcal{X}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ *and* $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathbb{J}_r$ *such that*

$$\mathcal{X}_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Psi_r, \quad \textit{(Initial Condition I2)}$$
$$\mathcal{T}_{j+r}\mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = 1, \ldots, n \quad \textit{(Recurrence I2).}$$

The difference between the inverse characterizations I1 and I2 is that the initial condition determines univocally the initial matrix $\mathcal{X}_r$ for I1 but not the initial matrix $\mathcal{X}_0$ for I2, thus there is a freedom in such initial matrix for I2. We will go back to this point later on.

Theorems 2.10–2.13 have an obvious generalization to the quasi-definite case.

Theorem 2.13 shows that the regularity of $\mathcal{C}_j$, $j \neq 0$, and $\mathcal{X}_j$, $j \neq r$, is a superfluous condition in statement (ii) of Theorems 2.10 and 2.11 respectively. Remember that the regularity of $\mathcal{C}_0$ is equivalent to $v[L] \neq 0$. On the other hand, the regularity conditions for $\mathcal{X}_j$ in Theorems 2.11 and 2.13 can be completely avoided if we do not fix the degree of L. In other words, if $\mathcal{X}_j \in \mathbb{J}_r \setminus \mathbb{J}_r^{\mathrm{reg}}$ then $u \equiv vL$ in $\mathbb{P}_n$ too, but $\deg L < r$, as follows from the following proposition.

Proposition 2.14. *If* $\Psi_{j+r} = \mathcal{X}_j \Phi_j$ *with* $\mathcal{X}_j \in \mathbb{J}_r \setminus \mathbb{J}_r^{\mathrm{reg}}$, *then* $\Psi_{j+r-1} = \hat{\mathcal{X}}_j \Phi_j$ *with* $\hat{\mathcal{X}}_j \in \mathbb{J}_{r-1}$.

Proof. Suppose $\Psi_{j+r} = \mathcal{X}_j \Phi_j$, $\mathcal{X}_j \in \mathbb{J}_r$ with $X_j(0) = 0$. Then $b_{j+r} = Y_j(0)$ and $X_j = z\hat{X}_j$ with $\hat{X}_j$ monic of degree $r-1$. Thus we can write

$$\Psi_{j+r} = \begin{pmatrix} z\hat{X}_j & Y_j \\ zY_j^* & \hat{X}_j^* \end{pmatrix} \Phi_j.$$

From $\Psi_{j+r} = \mathcal{T}_{j+r}\Psi_{j+r-1}$ we get $\Psi_{j+r-1} = \hat{\mathcal{X}}_j \Phi_j$ where

$$\hat{\mathcal{X}}_j = \frac{1}{1-|b_{j+r}|^2} \begin{pmatrix} \hat{X}_j - b_{j+r}Y_j^* & z^{-1}(Y_j - b_{j+r}\hat{X}_j^*) \\ z(Y_j^* - \overline{b}_{j+r}\hat{X}_j) & \hat{X}_j^* - \overline{b}_{j+r}Y_j \end{pmatrix}.$$

Since $Y_j(0) - b_{j+r}\hat{X}_j^*(0) = 0$ and $\hat{X}_j^*(0) - \overline{b}_{j+r}Y_j(0) = 1 - |b_{j+r}|^2 \neq 0$ we conclude that $\hat{\mathcal{X}}_j \in \mathbb{J}_{r-1}$. □

3. Direct and inverse problems

In the previous section, given two hermitian linear functionals u, v and a hermitian Laurent polynomial L, we have studied the relation $u \equiv vL$ obtaining characterizations in terms of linear relations with polynomial coefficients between the corresponding MOP, as well as in terms of a matrix difference equation between the related Schur parameters. In this section we will use these results to answer the following question: Which conditions ensure the quasi-definiteness of $u = vL$ or v once we know that the other functional is quasi-definite?

Indeed we will answer this question in the more general context of quasi-definite functionals in some subspace $\mathbb{P}_n$: we will try to know the minimum length of the finite segments of MOP for one of the functionals assuming that the other functional has a finite segment of MOP with a given length. Like in the previous section, the main goal is to develop techniques for this problem based almost exclusively on the knowledge of the Schur parameters.

The new results will seem quite similar to those of the previous section, however they provide new information: in the previous section we assumed that u and v had finite segments of MOP of certain length and we asked about a characterization of the relation $u \equiv vL$ in some subspace $\mathbb{P}_n$; now we will consider the relation $u = vL$ as a data and we will ask about the length of the finite segments of MOP.

3.1. Direct problem

The direct problem refers to the case where we suppose that a hermitian functional v with a finite segment of MOP $(\psi_j)_{j=0}^m$ and a hermitian polynomial L of degree r are given. Then, we will try to obtain information about the functional $u = vL$ and its finite segments of MOP $(\varphi_j)_{j=0}^n$. Our first result is essentially a reinterpretation of relation (9).

Theorem 3.1. *Let v be quasi-definite in $\mathbb{P}_{n+r}$ and let L be a hermitian Laurent polynomial of degree r. Then, $u = vL$ is quasi-definite in $\mathbb{P}_n$ iff there exists $\mathcal{C}_j \in \mathbb{J}_r^{\mathrm{reg}}$ such that A divides $\mathcal{C}_j \Psi_{j+r}$ for $j = 0, \ldots, n$.*

Besides, $\det \mathcal{C}_j \propto A$ *and there is a unique choice of $\mathcal{C}_j$ such that $C_j^*(0) = A(0)$. For such a choice the finite segment of MOP with respect to u is given by $A \Phi_j = \mathcal{C}_j \Psi_{j+r}$ for $j = 0, \ldots, n$.*

Proof. First of all notice that, no matter the value of $\lambda_j \in \mathbb{C}^*$, A divides $\mathcal{C}_j \Psi_{j+r}$ iff it divides $\hat{\mathcal{C}}_j \Psi_{j+r}$ with $\hat{\mathcal{C}}_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \bar{\lambda}_j \end{pmatrix} \mathcal{C}_j$, and $\mathcal{C}_j \in \mathbb{J}_r^{\mathrm{reg}}$ iff $\hat{\mathcal{C}}_j \in \mathbb{J}_r^{\mathrm{reg}}$. Therefore, we can suppose without loss of generality that $C_j^*(0) = A(0)$. Then, the divisibility condition is equivalent to $A \Phi_j = \mathcal{C}_j \Psi_{j+r}$ with φ_j a monic polynomial of degree j which, for the moment, has no relation with u. Taking into account Theorem 2.10, to prove the result we only need to see that φ_j is the j-th MOP with respect to u. This follows from the orthogonality conditions of ψ_{j+r} with respect to v, which give

$$u[\varphi_j z^{-k}] = v[(C_j \psi_{j+r} + D_j \psi_{j+r}^*) z^{-(k+r)}] = 0, \quad 0 \le k \le j-1,$$
$$u[\varphi_j z^{-j}] = v[(C_j \psi_{j+r} + D_j \psi_{j+r}^*) z^{-(j+r)}] = C_j(0) \varepsilon_{j+r} \neq 0.$$

The rest of the theorem is a consequence of Theorem 2.10. □

The above results allow us to obtain a necessary and sufficient condition for the quasi-definiteness of the functional $u = vL$ in terms of determinants involving the MOP of v.

Proposition 3.2. *Let v be quasi-definite in $\mathbb{P}_{n+r}$ and let L be a hermitian Laurent polynomial of degree r. Then, $u = vL$ is quasi-definite in $\mathbb{P}_n$ iff* $\det M^{(m)} \neq 0$ *for $m = 0, \ldots, n+1$, where $M^{(m)} = (M_{ij}^{(m)})_{i,j=1}^{2r}$ is the square matrix of order $2r$ given by*

$$M_{ij}^{(m)} = \begin{cases} (z^{j-1} \psi_{m+r})^{(l_i}(\zeta_i), & j = 1, \ldots, r, \\ (z^{j-r-1} \psi_{m+r}^*)^{(l_i}(\zeta_i), & j = r+1, \ldots, 2r, \end{cases} \qquad i = 1, \ldots, 2r,$$

with $\zeta_1, \ldots, \zeta_{2r}$ the roots of A counting the multiplicity and l_i the number of roots ζ_j, $j < i$, such that $\zeta_j = \zeta_i$.

Proof. By Theorem 3.1, to decide the quasi-definiteness of $u = vL$ in $\mathbb{P}_n$, we simply have to analyze the existence of unique polynomials C_m, D_m with $\deg C_m = r$, $\deg D_m \le r-1$, $C_m(0) \neq 0$, $C_m^*(0) = A(0)$, such that A divides $C_m \psi_{m+r} + D_m \psi_{m+r}^*$ for $m = 0, \ldots, n$.

Let us write $C_m(z) = \sum_{k=0}^{r} c_{m,k} z^k$ and $D_m(z) = \sum_{k=0}^{r-1} d_{m,k} z^k$. The condition $C_m^*(0) = A(0)$ only means that $c_{m,r}$ is the leading coefficient of A. Then, the existence of unique polynomials C_m, D_m is equivalent to the existence and uniqueness of the $2r$ coefficients $c_{m,0}, \ldots, c_{m,r-1}$ and $d_{m,0}, \ldots, d_{m,r-1}$, while the condition $C_m(0) \neq 0$ becomes $c_{m,0} \neq 0$.

If $\zeta_1, \ldots, \zeta_{2r}$ denote the $2r$ roots of the polynomial A counting the multiplicity and l_i is the number of roots ζ_j such that $\zeta_j = \zeta_i$ for $j < i$, the divisibility condition is equivalent to the system

$$(C_m \psi_{m+r})^{(l_i)}(\zeta_i) + (D_m \psi_{m+r}^*)^{(l_i)}(\zeta_i) = 0, \quad i = 1, \ldots, 2r.$$

This system has a unique solution in $c_{m,k}, d_{m,k}$, $k = 0, \ldots, r-1$, exactly when $\det M^{(m)} \neq 0$.

It remains to translate the condition $c_{m,0} \neq 0$. The solution for $c_{m,0}$ is proportional to the determinant of a matrix M obtained substituting in $M^{(m)}$ the first column $(\psi_{m+r}^{(l_i)}(\zeta_i))_{i=1}^{2r}$ by $(z^r \psi_{m+r}^{(l_i)}(\zeta_i))_{i=1}^{2r}$. Since (1) and (2) imply the equality $\operatorname{span}\{z^{j+1}\psi_{m+r}, z^j \psi_{m+r}^*\} = \operatorname{span}\{z^j \psi_{m+r+1}, z^j \psi_{m+r+1}^*\}$, $\det M$ vanishes at the same time as $\det M^{(m+1)}$. Hence, $c_{m,0} \neq 0$ is equivalent to $\det M^{(m+1)} \neq 0$. $\square$

The condition given by the above proposition is theoretically interesting but in practice it is not manageable, specially for polynomial perturbations of high degree r due to the need to evaluate determinants of $2r \times 2r$ matrices. Even in case of low degree r, the practical application of the previous result needs the construction of the MOP ψ_j and the evaluation at some points of these MOP and their derivatives.

When $r = 1$ the self-reciprocal polynomial A has two roots ζ_1, ζ_2 such that $\zeta_2 = 1/\overline{\zeta}_1$ or $\zeta_1, \zeta_2 \in \mathbb{T}$, $\zeta_1 \neq \zeta_2$. Obviously, when v is positive definite and $\zeta_2 = 1/\overline{\zeta}_1$ the functional vL is positive definite too. However, in general, v quasi-definite in $\mathbb{P}_{n+r}$ implies vL quasi-definite in $\mathbb{P}_n$ iff (see [39,5,9,1]) $K_m(\zeta_1, 1/\overline{\zeta}_2) \neq 0$ for $m = 1, \ldots, n+1$, where $K_m(z, w) = \sum_{j=0}^{m} \varepsilon_j^{-1} \psi_j(z) \overline{\psi_j(w)}$ is the m-th kernel associated with the MOP (ψ_j).

Nevertheless, it is naive to think that the general situation can be solved by factoring the polynomial A. Consider for instance a positive definite functional v and let $A(z) \propto (z-\zeta_1)(z-\zeta_2)$ with $\zeta_1, \zeta_2 \in \mathbb{T}$, $\zeta_1 \neq \zeta_2$, satisfying $K_m(\zeta_1, 1/\overline{\zeta}_2) = 0$ for some m. Then vL is not quasi-definite but vL^2 is positive definite.

A more practical characterization of the quasi-definiteness of $u = vL$, which avoids the construction of the MOP of v and does not need the calculation of determinants, is given in terms of the recurrence for the Schur parameters.

Theorem 3.3. *Let v be quasi-definite in $\mathbb{P}_{n+r}$ and let L be a hermitian Laurent polynomial of degree r. Then, $u = vL$ is quasi-definite in $\mathbb{P}_n$ iff there exist $a_1, \ldots, a_n \in \mathbb{C}$ and $\mathcal{C}_0, \ldots, \mathcal{C}_n \in \mathbb{J}_r^{\mathrm{reg}}$ such that*

$$\mathcal{C}_0 \Psi_r = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{24}$$

$$\mathcal{C}_j \mathcal{T}_{j+r} = \mathcal{S}_j \mathcal{C}_{j-1}, \quad j = 1, \ldots, n. \tag{25}$$

Besides, $A\Phi_j = \mathcal{C}_j \Psi_{j+r}$, $\det \mathcal{C}_j \propto A$, $j = 0, \ldots, n$, and $a_j = \varphi_j(0) \in \mathbb{C} \setminus \mathbb{T}$, $j = 1, \ldots, n$.

Proof. In view of Theorem 2.13, we only need to prove that u is quasi-definite in $\mathbb{P}_n$ when (24) and (25) hold. Define $\Phi_j = \mathcal{S}_j \cdots \mathcal{S}_1 \binom{1}{1}$. Then, (24), (25) and the recurrence relation for $(\Psi_j)_{j=0}^{n+r}$ yield for $j = 0, \ldots, n$,

$$A\,\Phi_j = A\mathcal{S}_j \cdots \mathcal{S}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathcal{S}_j \cdots \mathcal{S}_1 \mathcal{C}_0 \Psi_r = \mathcal{C}_j \mathcal{T}_{j+r} \cdots \mathcal{T}_{r+1} \Psi_r = \mathcal{C}_j \Psi_{j+r}.$$

Therefore, Theorem 3.1 shows that u is quasi-definite in $\mathbb{P}_n$. □

The above results yield a direct relation between the Schur parameters of $u = vL$ and v, which can be obtained setting $z = 0$ in the equivalent version $\mathcal{C}_j \mathcal{B}_{j+r} = \mathcal{A}_j \tilde{\mathcal{C}}_{j-1}$ of (25) and using $C_j^*(0) = A(0)$.

Corollary 3.4. *If α is the leading coefficient of A, the j-th Schur parameter a_j of $u = vL$ can be obtained from the $j + r$-th Schur parameter b_{j+r} of v by*

$$a_j = \frac{\alpha b_{j+r} - \overline{D_{j-1}^*(0)}}{C_{j-1}(0)}. \tag{26}$$

Theorem 3.3 and Corollary 3.4 provide an algorithm to obtain the Schur parameters (a_j) of $u = vL$ from the Schur parameters (b_j) of v.

Algorithm D. • Determination of $\mathcal{C}_0 \in \mathbb{J}_r$ from initial condition (24) and Ψ_r, A.

- For $j = 1, 2, \ldots$.
  - While $C_{j-1}(0) \neq 0$, calculation of a_j from (26) and $b_{j+r}, \mathcal{C}_{j-1}$.
  - Determination of $\mathcal{C}_j \in \mathbb{J}_r$ from recurrence (25) and $a_j, b_{j+r}, \mathcal{C}_{j-1}$.

The fact that the j-th step of the above algorithm actually gives a matrix $\mathcal{C}_j \in \mathbb{J}_r$ is a consequence of Lemma 2.8(ii) and the equivalence between $C_{j-1}(0) \neq 0$ and $\mathcal{C}_{j-1} \in \mathbb{J}_r^{\text{reg}}$ when $\mathcal{C}_{j-1} \in \mathbb{J}_r$.

In short, the fact that Algorithm D works from $j = 1$ to $j = n$ will be called the *n-consistence* of recurrence (25). We will say that the recurrence is *consistent* if it works for any $j \geq 1$. Of course, this is an abuse of language because the consistence depends, not only on recurrence (25), but also on initial condition (24).

The consistence relies on the fact that $C_j(0) \neq 0$ at each step. Suppose that the recurrence fails at the $(n+1)$-th step, i.e., it is n-consistent and not $(n+1)$-consistent. Then $C_{n-1}(0) \neq 0$ and $C_n(0) = 0$, that is, $\mathcal{C}_{n-1} \in \mathbb{J}_r^{\text{reg}}$ but $\mathcal{C}_n \in \mathbb{J}_r \setminus \mathbb{J}_r^{\text{reg}}$. Recurrence (25) shows that this is equivalent to $|a_{n-1}| \neq 1$ and $|a_n| = 1$. So, the n-consistence condition can be written as $|a_j| \neq 1$ for $j = 1, \ldots, n-1$, which means that $u = vL$ has a finite segment of MOP of length n, i.e., it is quasi-definite in $\mathbb{P}_{n-1}$.

Contrary to Proposition 3.2, Algorithm D only requires the knowledge of the Schur parameters of v and a single MOP ψ_r with the same degree r as the polynomial perturbation L. Furthermore, this algorithm makes the calculation of determinants completely unnecessary. As an example, we will develop explicitly Algorithm D for $r = 1$.

3.1.1. The case $r = 1$

Different kinds of direct modifications of degree 1 have been considered previously. The Christoffel transformation on the unit circle, corresponding to a polynomial perturbation

$L(z) = (z - \zeta)(z^{-1} - \overline{\zeta})$, $\zeta \in \mathbb{C}$, is studied in [26,9]. Other direct perturbations of degree 1 are analyzed in [39,5,14,15]. Our method yields a complementary approach which unifies the discussion of these cases and, at the same, allows us to deal with non-quasi-definite functionals.

Consider a hermitian functional v with MOP (ψ_j) and a hermitian Laurent polynomial L of degree 1. We can write $L = P + P_*$, $P(z) = \alpha z + \beta$, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$, so $A(z) = zL(z) = \alpha z^2 + 2\beta z + \overline{\alpha}$. The MOP (φ_j) of the modified functional $u = vL$, if they exist, are given by

$$A\varphi_j = (\alpha z + c_j)\psi_{n+1} + d_j \psi_{n+1}^*,$$

for some $c_j \in \mathbb{R}$, $d_j \in \mathbb{C}$. This relation and its reversed can be combined in

$$A\Phi_j = \mathcal{C}_j \Psi_{j+1}, \quad \mathcal{C}_j = \begin{pmatrix} \alpha z + c_j & d_j \\ \overline{d}_j z & \overline{\alpha} + c_j z \end{pmatrix}.$$

Also, recurrence (25) becomes

$$\begin{cases} c_{j-1} + \overline{d}_{j-1} a_j = c_j + d_j \overline{b}_{j+1}, \\ \overline{\alpha} a_j = c_j b_{j+1} + d_j, \\ c_{j-1} a_j + d_{j-1} = \alpha b_{j+1}, \end{cases}$$

which can be written as

$$a_j = \frac{\alpha b_{j+1} - d_{j-1}}{c_{j-1}}, \qquad \begin{pmatrix} 1 & \overline{b}_{j+1} \\ b_{j+1} & 1 \end{pmatrix} \begin{pmatrix} c_j \\ d_j \end{pmatrix} = \begin{pmatrix} c_{j-1} + \overline{d}_{j-1} a_j \\ \overline{\alpha} a_j \end{pmatrix}. \tag{27}$$

On the other hand, initial condition (24) is $\alpha z^2 + 2\beta z + \overline{\alpha} = (z + c_1)(z + b_1) + d_0(\overline{b}_1 z + 1)$, i.e.,

$$\begin{pmatrix} 1 & \overline{b}_1 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 2\beta - \alpha b_1 \\ 1 \end{pmatrix}. \tag{28}$$

This provides unique c_0, d_0 for any P and any possible value of $b_1 \in \mathbb{C} \setminus \mathbb{T}$.

Finally, Algorithm D can be explicitly formulated in the following way:

- Calculation of c_0, d_0 from P, b_1 using (28).
- For $j = 1, 2, \ldots,$ while $c_{j-1} \neq 0$, calculation of a_j, c_j, d_j from $b_{j+1}, c_{j-1}, d_{j-1}$ using (27).

This algorithm provides the Schur parameters of $u = vL$ and informs us about its quasi-definiteness: the maximum subspace $\mathbb{P}_n$ where u is quasi-definite is given by the first index n of inconsistency of the algorithm.

We can think in reducing the general problem to the case $r = 1$ by factoring the polynomial A. Suppose that $A = A_1 A_2$, $\deg A_1 = 2r_1$, $\deg A_2 = 2r_2$, with A_i self-reciprocal, and denote by $\mathcal{C}_j^{(1)}$, $\mathcal{C}_j^{(2)}$ the J-self-reciprocal matrices associated with the direct problem $w = vA_1 z^{-r_1}$, $u = wA_2 z^{-r_2}$ respectively. If $\mathcal{U}_j$ are the transfer matrices for the functional w with MOP (ξ_j) and $\Xi_j = \binom{\xi_j}{\xi_j^*}$, then $A_1 \Xi_j = \mathcal{C}_j^{(1)} \Psi_{j+r_1}$ and $A_2 \Phi_j = \mathcal{C}_j^{(2)} \Xi_{j+r_2}$. This implies the equality $A\Phi_j = \mathcal{C}_j^{(2)} \mathcal{C}_{j+r_2}^{(1)} \Psi_{j+r}$, so $\mathcal{C}_j = \mathcal{C}_j^{(2)} \mathcal{C}_{j+r_2}^{(1)}$. However, this does not always reduce a direct problem to simpler ones because the length of the finite segments of MOP for w can be not big enough to get the actual relations between all the MOP of u and v.

3.2. Inverse problem

In this subsection we will study a problem which can be considered as the inverse of that one of the previous section. More precisely, given an hermitian functional u with a finite segment of MOP $(\varphi_j)_{j=0}^n$ and a hermitian Laurent polynomial L of degree r, we will try to obtain information about the hermitian solutions v of $u = vL$ and their finite segments of MOP $(\psi_j)_{j=0}^m$.

First of all we will clarify the structure of the set

$$H(u, L) := \{v \text{ hermitian} : u = vL\}.$$

The equation $u = vL$ is equivalent to $u[z^n] = v[z^n L]$, $n \geq 0$, which, denoting $\mu_n = u[z^n]$, $m_n = v[z^n]$ and $L(z) = \sum_{j=-r}^{r} \alpha_j z^j$, $\alpha_{-j} = \overline{\alpha}_j$, becomes

$$\mu_n = \sum_{j=-r}^{r} \alpha_j m_{n+j}, \quad n \geq 0. \tag{29}$$

The first equation ($n = 0$)

$$\mu_0 = 2\mathrm{Re} \sum_{j=0}^{r} \alpha_j m_j \tag{30}$$

is simply a constraint between the first $r + 1$ moments $m_0, \ldots, m_r$ of v. The rest of the equations determine the moments m_n, $n > r$. Since any hermitian solution v is determined by its moments m_n, $n \geq 0$, the general solution depends on $2r$ real independent parameters obtained establishing in the set $\{m_0, m_1, \ldots, m_r\}$, $m_0 \in \mathbb{R}$, $m_1, \ldots, m_r \in \mathbb{C}$, the constraint (30).

There is another way to describe the set of hermitian solutions v starting from a particular one v_0. Then the hermitian solutions are those functionals with the form $v = v_0 + \varDelta$, where $\varDelta$ is any hermitian functional satisfying $\Delta L = 0$, i.e.,

$$\varDelta = \sum_{i=1}^{p} \sum_{k_i=0}^{q_i-1} M_{k_i}^{(i)} \delta^{(k_i}(z - \zeta_i), \quad M_{k_i}^{(i)} \in \mathbb{C}, \qquad M_{k_j}^{(j)} = \overline{M}_{k_i}^{(i)} \quad \text{if } \zeta_j = 1/\overline{\zeta}_i.$$

ζ_i, $i = 1, \ldots, p$, being the roots of $A = z^r L$ and q_i the multiplicity of ζ_i. Again we see that the hermitian solutions are parametrized by $2r$ real parameters: the independent real and imaginary parts of the coefficients $M_{k_i}^{(i)}$. Furthermore, this approach shows that the inverse problem is related to the study of the influence of Dirac's deltas and their derivatives on the quasi-definiteness and the MOP of a hermitian functional.

For convenience we will denote by $H_r(u)$ the set of hermitian functionals v which are solutions of $u = Lv$ for some hermitian Laurent polynomial L of degree r, i.e.,

$$H_r(u) = \bigcup_{\substack{L=P+P_* \\ \deg P=r}} H(u, L).$$

The main result of this section characterizes the functionals of $H_r(u)$ which are quasi-definite in some subspace $\mathbb{P}_m$.

Theorem 3.5. *Let u be quasi-definite in $\mathbb{P}_n$.*

(i) *If $n \geq r$, there is a (unique up to factors) solution $v \in H_r(u)$ quasi-definite in $\mathbb{P}_{n+r}$ for each $b_1, \ldots, b_{2r} \in \mathbb{C} \setminus \mathbb{T}$, $b_{2r+1}, \ldots, b_{n+r} \in \mathbb{C}$, $\mathcal{X}_r, \ldots, \mathcal{X}_n \in \mathbb{J}_r^{\mathrm{reg}}$ such that*

$$\mathcal{X}_r \varPhi_r = \varPsi_{2r}, \quad \varPsi_{2r} = \mathcal{T}_{2r} \cdots \mathcal{T}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{31}$$

$$\mathcal{T}_{j+r} \mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = r + 1, \ldots, n. \tag{32}$$

The relation between v and b_j, $\mathcal{X}_j$ is that $\Psi_{j+r} = \mathcal{X}_j \Phi_j$ provides the $j+r$-th MOP of v for $j = r, \ldots, n$, and $b_j \in \mathbb{C} \setminus \mathbb{T}$, $j = 1, \ldots, n+r$, are the first $n+r$ Schur parameters of v. Besides, $u = vL$ with $\det \mathcal{X}_j \propto A$, *$j = r, \ldots, n$.*

(ii) *There is a (unique up to factors) solution $v \in H_r(u)$ quasi-definite in $\mathbb{P}_{n+r}$ for each $b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$, $b_{r+1}, \ldots, b_{n+r} \in \mathbb{C}$, $\mathcal{X}_0, \ldots, \mathcal{X}_n \in \mathbb{J}_r^{\mathrm{reg}}$ such that*

$$\mathcal{X}_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Psi_r, \quad \Psi_r = \mathcal{T}_r \cdots \mathcal{T}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{33}$$

$$\mathcal{T}_{j+r} \mathcal{X}_{j-1} = \mathcal{X}_j \mathcal{S}_j, \quad j = 1, \ldots, n. \tag{34}$$

The relation between v and b_j, $\mathcal{X}_j$ is that $\Psi_{j+r} = \mathcal{X}_j \Phi_j$ provides the $j+r$-th MOP of v for $j = 0, \ldots, n$, and $b_j \in \mathbb{C} \setminus \mathbb{T}$, $j = 1, \ldots, n+r$, are the first $n+r$ Schur parameters of v. Besides, $u = vL$ with $\det \mathcal{X}_j \propto A$, *$j = 0, \ldots, n$.*

Proof. We will prove only (i), the proof of (ii) is similar. Bearing in mind Theorems 2.11 and 2.13 we only need to show that (31) and (32) imply that $\mathcal{X}_j \Phi_j$ gives for $j = r, \ldots, n$ the $j+r$-th MOP of a unique $v \in H_r(u)$ whose first $n+r$ Schur parameters are b_j, $j = 1, \ldots, n+r$.

Let us define $\Psi_j = \mathcal{T}_j \cdots \mathcal{T}_1 \binom{1}{1}$. Since $\mathcal{X}_j \in \mathbb{J}_r^{\mathrm{reg}}$, $j = r, \ldots, n$, recurrence (32) implies that $|b_j| \neq 1$, not only for $j = 1, \ldots, 2r$, but also for $j = 2r+1, \ldots, n+r$. Therefore, $(\psi_j)_{j=0}^{n+r}$ is a finite segment of MOP with respect to some hermitian functional $\hat{v}$.

From (31), (32) and the recurrence relation for $(\Phi_j)_{j=0}^{n}$ we obtain for $j = r, \ldots, n$,

$$\Psi_{j+r} = \mathcal{T}_{j+r} \cdots \mathcal{T}_{2r+1} \Psi_{2r} = \mathcal{T}_{j+r} \cdots \mathcal{T}_{2r+1} \mathcal{X}_r \Phi_r = \mathcal{X}_j \mathcal{S}_j \cdots \mathcal{S}_{r+1} \Phi_r = \mathcal{X}_j \Phi_j.$$

Hence, Theorem 2.11 proves that $u \equiv \hat{v}\hat{L}$ in $\mathbb{P}_n$ for some hermitian Laurent polynomial $\hat{L}$ of degree r. Multiplying $\hat{L}$ by a real factor we can get a hermitian Laurent polynomial L of degree r such that $u = \hat{v}L$ in $\mathbb{P}_n$.

The equality $u = \hat{v}L$ in $\mathbb{P}_n$, as well as the fact that $(\psi_j)_{j=0}^{n+r}$ is a finite segment of MOP for $\hat{v}$, only depends on the first $n+r+1$ moments $\hat{v}[z^j]$, $j = 0, \ldots, n+r$, of $\hat{v}$. Let us define a new hermitian functional v fixing its moments $m_j = v[z^j]$ by $m_j = \hat{v}[z^j]$ for $j \leq n+r$, and m_j given by (29) for $j \geq n+r+1$. Then v is a solution of $u = vL$, has $(\psi_j)_{j=0}^{n+r}$ as a finite segment of MOP and its first $n+r$ Schur parameters are $\psi_j(0) = b_j$, $j = 1, \ldots, n+r$.

Finally, the first $n+r$ Schur parameters of a functional v determine its finite segment of MOP of length $n+r+1$ and, thus, its first $n+r+1$ moments up to a common factor. Requiring also $u = vL$ for a given hermitian Laurent polynomial of degree r fixes the rest of the moments up to the common factor due to (29). Therefore, the conditions of (i) define a unique hermitian functional v up to factors because L is determined up to real factors by $\det \mathcal{X}_j$. □

We have the following relation between the Schur parameters of u and $v \in H_r(u)$. To prove it simply choose $z = 0$ in the equivalent version $\mathcal{B}_{j+r} \tilde{\mathcal{X}}_{j-1} = \mathcal{X}_j \mathcal{A}_j$ of (34) and use that X_j is monic, i.e., $X_j^*(0) = 1$.

Corollary 3.6. *The $j+r$-th Schur parameter b_{j+r} of $v \in H_r(u)$ can be obtained from the j-th Schur parameter a_j of u by*

$$b_{j+r} = \frac{a_j - \overline{Y_{j-1}^*(0)}}{\overline{X_{j-1}(0)}}. \tag{35}$$

Theorem 3.5 and Corollary 3.6 provide algorithms generating the solutions of the inverse problem which are quasi-definite in some subspace $\mathbb{P}_m$. The algorithms are based on the consistence of recurrence (32) or (34), what can be defined in a similar way to the case of Algorithm D. We have several possibilities depending on the initial data.

If we know that L has degree r but not its explicit form, we can proceed in the following ways.

Algorithm I1. • Choice of Ψ_{2r}, i.e., of $b_1, \ldots, b_{2r} \in \mathbb{C} \setminus \mathbb{T}$.

- Determination of $\mathcal{X}_r \in \mathbb{J}_r$ from initial condition (31) and Φ_r, Ψ_{2r}.
- For $j = r+1, r+2, \ldots$.
  - While $X_{j-1}(0) \neq 0$, calculation of b_{j+r} from (35) and a_j, $\mathcal{X}_{j-1}$.
  - Determination of $\mathcal{X}_j \in \mathbb{J}_r$ from recurrence (32) and a_j, b_{j+r}, $\mathcal{X}_{j-1}$.

Algorithm I2. • Choice of Ψ_r, i.e., of $b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$.

- Choice of a solution $\mathcal{X}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ of initial condition (33) using Ψ_r, i.e., choice of a monic polynomial X_0 of degree r with $X_0(0) \neq 0$ and determination of $Y_0 = \psi_r - X_0$.
- For $j = 1, 2, \ldots$.
  - While $X_{j-1}(0) \neq 0$, calculation of b_{j+r} from (35) and a_j, $\mathcal{X}_{j-1}$.
  - Determination of $\mathcal{X}_j \in \mathbb{J}_r$ from recurrence (34) and a_j, b_{j+r}, $\mathcal{X}_{j-1}$.

Algorithms I1 and I2 generate the elements of $H_r(u)$ which are quasi-definite in $\mathbb{P}_{2r}$ and $\mathbb{P}_r$ respectively. For any of these two algorithms we recover the polynomial perturbation through $A \propto \det \mathcal{X}_j$.

On the contrary, if we know explicitly the hermitian polynomial L of degree r, the following scheme yields the elements of $H(u, L)$ which are quasi-definite in $\mathbb{P}_r$.

Algorithm I3. • Choice of Ψ_r i.e., of $b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$.

- Determination of $\mathcal{X}_0 = \frac{\mathrm{Adj}(\mathcal{C}_0)}{C_0(0)}$ from initial condition (24) and Ψ_r, A.
- For $j = 1, 2, \ldots,$
  - While $X_{j-1}(0) \neq 0$, calculation of b_{j+r} from (35) and a_j, $\mathcal{X}_{j-1}$.
  - Determination of $\mathcal{X}_j \in \mathbb{J}_r$ from recurrence (34) and a_j, b_{j+r}, $\mathcal{X}_{j-1}$.

We can ensure that any step of the above algorithms generates a matrix $\mathcal{X}_j \in \mathbb{J}_r$ due to Lemma 2.8(ii) and the fact that $X_{j-1}(0) \neq 0$ is equivalent to $\mathcal{X}_{j-1} \in \mathbb{J}_r^{\mathrm{reg}}$ when $\mathcal{X}_{j-1} \in \mathbb{J}_r$.

The n-consistence of the above algorithms, which means that they work for $j \leq n$, is equivalent to the existence of a finite segment of MOP of length $n + r$ for the corresponding solution v of $u = vL$. Such n-consistence can be written as $X_j(0) \neq 0$, $j \leq n-1$, which holds iff $|b_j| \neq 1$, $j \leq n + r - 1$.

Comparing the above algorithms we see that the arbitrariness in the parameters $b_{r+1}, \ldots, b_{2r}$ is equivalent to the arbitrariness of the polynomial modification L of degree r. This means that any of the infinitely many solutions $\mathcal{X}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ of $\mathcal{X}_0 \Phi_0 = \Psi_r$ should be determined by $\det \mathcal{X}_0$, a result which is proved in the next proposition.

Proposition 3.7. *Given* $b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$ *and a self-reciprocal polynomial* A *of degree* $2r$, *there exists a unique solution* $\mathcal{X}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ *of* $\mathcal{X}_0 \binom{1}{1} = \Psi_r$, $\Psi_r = \mathcal{T}_r \cdots \mathcal{T}_1 \binom{1}{1}$, *such that* $\det \mathcal{X}_0 \propto A$.

Proof. Given Ψ_r, each solution of $\mathcal{X}_0 \Phi_0 = \Psi_r$ with the form

$$\mathcal{X}_0 = \begin{pmatrix} X_0 & Y_0 \\ zY_0^* & X_0^* \end{pmatrix}, \qquad \deg X_0 = r, \qquad \deg Y_0 \leq r-1,$$

is determined by a monic polynomial X_0 because $Y_0 = \psi_r - X_0$. Then $\mathcal{X}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ iff $X_0(0) \neq 0$. Therefore, $\det \mathcal{X}_0 = X_0^* \psi_r + X_0 \psi_r^* - \psi_r \psi_r^*$. Hence, if A is a self-reciprocal polynomial of degree $2r$ and $\lambda \in \mathbb{R}$,

$$\det \mathcal{X}_0 = \lambda A \Leftrightarrow \lambda A + \psi_r \psi_r^* = X_0^* \psi_r + X_0 \psi_r^*. \tag{36}$$

From Remark 2.3 we know that $\lambda A + \psi_r \psi_r^* = C\psi_r + D\psi_r^*$ for some polynomials $C \in \mathbb{P}_r$, $D \in \mathbb{P}_{r-1}$. Since $\lambda A + \psi_r \psi_r^*$ is self-reciprocal in $\mathbb{P}_{2r}$, $(C - zD^*)\psi_r = (C^* - D)\psi_r^*$, so $C^* - D = c\psi_r$ and $C - zD^* = c\psi_r^*$ for some $c \in \mathbb{R}$. Then, the identity

$$\lambda A + \psi_r \psi_r^* = \left(C - \frac{c}{2}\psi_r^*\right)\psi_r + \left(D + \frac{c}{2}\psi_r\right)\psi_r^* = \frac{1}{2}(C + zD^*)\psi_r + \frac{1}{2}(C^* + D)\psi_r^*$$

proves that (36) holds with $X_0 = \frac{1}{2}(C^* + D)$, thus $\mathcal{X}_0$ satisfies $\det \mathcal{X}_0 = \lambda A$ with such a choice. Furthermore, X_0 is monic of degree r iff $X_0^*(0) = 1$, which (36) shows that corresponds to $\lambda = X_0(0)/A(0)$.

Now, let $\mathcal{X}_0, \hat{\mathcal{X}}_0 \in \mathbb{J}_r^{\mathrm{reg}}$ be such that $\mathcal{X}_0 \Phi_0 = \hat{\mathcal{X}}_0 \Phi_0 = \Psi_r$. Assume that $\det \hat{\mathcal{X}}_0 = \lambda \det \mathcal{X}_0$ for some $\lambda \in \mathbb{R}$. Using an obvious notation, this means that $\hat{X}_0^* \psi_r + (\hat{X}_0 - \psi_r)\psi_r^* = \lambda(X_0^* \psi_r + (X_0 - \psi_r)\psi_r^*)$. The uniqueness of the polynomials C, D in Remark 2.3 ensures that $\hat{X}_0^* = \lambda X_0^*$ and $\hat{X}_0 - \psi_r = \lambda(X_0 - \psi_r)$, which implies that $\hat{\mathcal{X}}_0 = \mathcal{X}_0$. □

The previous results show that the solutions of the inverse problem are parametrized by their first r or $2r$ Schur parameters, depending on whether we fix the polynomial perturbation or only its degree. Of course, such a parametrization works only for the solutions which are quasi-definite (at least) in $\mathbb{P}_r$ and $\mathbb{P}_{2r}$ respectively. Each of these solutions will have a finite segment of MOP of maximum length determined by the consistence level of the corresponding algorithm.

Following Theorem 2.12, we also could parametrize the solutions of the direct and inverse problem using mixed data between the Schur parameters and the polynomial perturbation. This would give algorithms adapted to situations where we could have only partial information about the perturbation. Among them, the algorithms described above would be only the extreme cases.

3.2.1. The case $r = 1$

As an example of the previous discussion we will analyze the particular case of the inverse problem corresponding to a hermitian Laurent polynomial perturbation L of degree 1 of a given hermitian functional u. Bearing in mind the comments at the beginning of Section 3.2, and taking into account the possibilities for the roots of a self-reciprocal polynomial $A = zL$ of degree 2, this is equivalent to the analysis of functionals v with the form

(a) $v_0 + M\delta(z - \zeta) + \overline{M}\delta(z - 1/\overline{\zeta}),\ \zeta \in \mathbb{D} \setminus \{0\},\ M \in \mathbb{C}$,
(b) $v_0 + M_1\delta(z - \zeta) + M_2\delta'(z - \zeta),\ \zeta \in \mathbb{T},\ M_i \in \mathbb{R}$,
(c) $v_0 + M_1\delta(z - \zeta_1) + M_2\delta(z - \zeta_2),\ \zeta_1 \neq \zeta_2,\ \zeta_i \in \mathbb{T},\ M_i \in \mathbb{R}$,

where v_0 is a particular solution of the inverse problem $u = vL$. In case (a), if u is positive definite, we can take v_0 as a multiple of the functional associated with the measure $d\mu(z)/|z-\zeta|^2$, where $d\mu$ is the measure related to u. Then (a) is known as the Geronimus transformation of the measure $d\mu$. The Geronimus transformation of an arbitrary positive measure on the unit circle has been studied in [13,11]. Other inverse transformations of degree 1 have been analyzed in [6]. Our approach permits us to deal with the above three transformations simultaneously, as well as to work with non-quasi-definite functionals.

So, we consider the MOP (φ_j) with respect to a hermitian linear functional u and we define the monic polynomials (ψ_j)

$$\psi_{j+1} = (z + x_j)\varphi_j + y_j\varphi_j^*, \quad j \geq 0, \tag{37}$$

with $\psi_0(z) = 1$ and $x_j, y_j \in \mathbb{C}$. The polynomials (ψ_j) are the only candidates to be MOP of a solution v of $u = vL$.

We can write (37) in a matrix form as

$$\Psi_{j+1} = \mathcal{X}_j \Phi_j, \quad \mathcal{X}_j = \begin{pmatrix} z + x_j & y_j \\ \overline{y}_j z & 1 + \overline{x}_j z \end{pmatrix}, \ j \geq 0, \tag{38}$$

and (34) becomes

$$\begin{cases} x_{j-1} + \overline{y}_{j-1} b_{j+1} = x_j + y_j \overline{a}_j, \\ b_{j+1} = x_j a_j + y_j, \\ \overline{x}_{j-1} b_{j+1} + y_{j-1} = a_j, \end{cases} \tag{39}$$

or equivalently,

$$b_{j+1} = \frac{a_j - y_{j-1}}{\overline{x}_{j-1}}, \qquad \begin{pmatrix} 1 & \overline{a}_j \\ a_j & 1 \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_{j-1} + \overline{y}_{j-1} b_{j+1} \\ b_{j+1} \end{pmatrix}. \tag{40}$$

So, Algorithm I2 reads as follows:

- Choice of $b_1 \in \mathbb{C} \setminus \mathbb{T}$ and $x_0 \in \mathbb{C}^*$ which determines $y_0 = b_1 - x_0$.
- For $j = 1, 2, \ldots,$ while $x_{j-1} \neq 0$, calculation of b_{j+1}, x_j, y_j from a_j, x_{j-1}, y_{j-1} using (40).

For any choice of x_0 we can recover the polynomial perturbation through $A \propto \det \mathcal{X}_0 = \overline{x}_0 z^2 + (1 + |x_0|^2 - |y_0|^2)z + x_0$. According to Proposition 3.7, given b_1, each choice of x_0 in the previous algorithm provides a solution of the inverse problem corresponding to a different polynomial perturbation. These solutions have well defined MOP ψ_0, ψ_1, so the algorithm provides all the solutions of the inverse problem which are quasi-definite at least in $\mathbb{P}_1$. The maximum length of the finite segments of MOP for a particular solution is equal to the consistence level of the algorithm starting with the values b_1 and x_0 defining such solution.

It is remarkable that, when $r = 1$, the consistence of Algorithm I2 is equivalent to the compatibility of (34), i.e., any solution of (39) for $j \leq n$ starting with $x_0 \neq 0$ necessarily satisfies $x_j \neq 0$ for $j \leq n-1$. We can see this by induction: if (39) has a solution for $j \leq n+1$, then $x_{n-1} \neq 0$ due to the induction hypothesis, so $x_n = 0$ would give $b_{n+1} \in \mathbb{T}$ according to (34); on the other hand, setting $x_n = 0$ in (39) for $j = n, n+1$ we get $y_n = b_{n+1}$ and $y_n = a_{n+1}$, which is a contradiction because $a_{n+1} \notin \mathbb{T}$.

3.2.2. *An example of the inverse problem*

As an application, we will solve the inverse problem for an arbitrary hermitian polynomial perturbation L of degree 1, when u is the functional associated with the Lebesgue measure on the unit circle

$$dm(z) = \frac{1}{2\pi i} \frac{dz}{z} = \frac{d\theta}{2\pi}, \quad z = e^{i\theta}.$$

More precisely, we will characterize the quasi-definite solutions $v \in H_1(u)$. Indeed, we will do something more than this because our methods permits us to characterize all the solutions

$v \in H_1(u)$ which are quasi-definite at least in $\mathbb{P}_1$, providing also the maximum subspace $\mathbb{P}_m$ where each of such solutions is quasi-definite.

As we pointed out in the previous section, we are actually studying a modification of the Lebesgue measure which includes the Geronimus transformation $dm(z)/|z-\zeta|^2 + M\delta(z-\zeta) + \overline{M}\delta(z-1/\overline{\zeta})$, $\zeta \in \mathbb{D} \setminus \{0\}$, $M \in \mathbb{C}$, as a particular case. Previous related results can be found in [13], where the authors characterize the quasi-definiteness of such Geronimus transformation in terms of ζ and M. On the contrary, we will characterize the quasi-definiteness of a general inverse transformation of degree 1 in terms of the coefficients of the polynomial perturbation and the initial conditions which define a solution of the inverse problem. This will show that such inverse modifications are structured in "circles" with the same number of MOP. We will go even further, analyzing other aspects of the inverse problem like, for instance, the asymptotics of the parameters related to the different solutions. As a consequence we will see that, among the Geronimus transformations of the Lebesgue measure, the Bernstein–Szegő one $dm(z)/|z-\zeta|^2$ is somewhat singular, while another quasi-definite but non-positive definite Bernstein–Szegő one acts as an attractor for the asymptotics of the remaining Geronimus transformations. Finally, an interpretation as a Newton algorithm will shed light on some peculiarities of the algorithm giving the solutions for this inverse problem.

The Lebesgue functional u is positive definite with MOP $\varphi_n(z) = z^n$, $n \geq 0$, and Schur parameters $a_n = 0$, $n \geq 1$, so that (39) becomes

$$\begin{cases} x_{n-1} + b_{n+1}\overline{y}_{n-1} = x_n, \\ b_{n+1} = y_n, \\ x_{n-1}\overline{b}_{n+1} + \overline{y}_{n-1} = 0. \end{cases} \tag{41}$$

Following Algorithm I2, every choice of $b_1 \in \mathbb{C} \setminus \mathbb{T}$ and $x_0 \in \mathbb{C}^*$ determines $y_0 = b_1 - x_0$ providing initial conditions for the above recurrence. Each of such initial conditions is associated with a different solution of the inverse problem we are considering, and this solution is quasi-definite exactly when the related initial conditions make (41) compatible for every $n \in \mathbb{N}$, i.e., $x_n \neq 0$ for all n. The corresponding orthogonal polynomials (ψ_n) are

$$\psi_{n+1}(z) = (z + x_n)z^n + y_n.$$

The second equation in (41) permits us to eliminate b_n and formulate equivalently the recurrence only in terms of x_n and y_n,

$$\begin{cases} x_n = \dfrac{|x_{n-1}|^2 - |y_{n-1}|^2}{\overline{x}_{n-1}}, \\ y_n = b_{n+1} = -\dfrac{y_{n-1}}{\overline{x}_{n-1}}. \end{cases} \tag{42}$$

The second equation in (42) is solved by

$$y_n = (-1)^n \frac{y_0}{\overline{x}_0 \cdots \overline{x}_{n-1}}, \tag{43}$$

so we only must care about the first equation in (42).

If $L = P + P_*$ with $P(z) = \alpha z + \beta$, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$, we know that

$$\det \mathcal{X}_n(z) = \overline{x}_n z^2 + (1 + |x_n|^2 - |y_n|^2)z + x_n \propto A(z) = \alpha z^2 + 2\beta z + \overline{\alpha}.$$

Therefore,

$$\frac{x_n}{\overline{x}_n} = \frac{\overline{\alpha}}{\alpha}, \qquad \frac{1 + |x_n|^2 - |y_n|^2}{\overline{x}_n} = 2\frac{\beta}{\alpha}. \tag{44}$$

This implies that $x_n = s_n \frac{\overline{\alpha}}{|\alpha|}$, $s_n \in \mathbb{R}$, and the first equation of (42) is equivalent to

$$x_n = 2\tilde{\omega} - \frac{1}{\overline{x}_{n-1}}, \quad \tilde{\omega} = \frac{\beta}{\alpha}. \tag{45}$$

That is, we have reduced the compatibility of (41) to the compatibility of (45) for x_n, which can be rewritten in terms of s_n as

$$s_n = 2\omega - \frac{1}{s_{n-1}}, \quad \omega = \frac{\beta}{|\alpha|}, \tag{46}$$

while the compatibility means simply that $s_n \neq 0$ for all n. If $s_j \neq 0$ for $j < n$ but $s_n = 0$ then the related solution is not quasi-definite but has only the first $n+1$ MOP $\psi_0, \ldots, \psi_n$.

The key idea to calculate s_n is to write (46) as a continued fraction

$$s_n = 2\omega - \frac{1|}{|2\omega} - \frac{1|}{|2\omega} - \cdots - \frac{1|}{|2\omega} - \frac{1|}{|s_0}.$$

According to the general theory of continued fractions (see for instance [42]),

$$s_n = \frac{s_0 Q_{n-1} - Q_{n-2}}{s_0 P_{n-1} - P_{n-2}},$$

where P_n and Q_n satisfy the difference equations

$$\begin{aligned} &Q_k = 2\omega Q_{k-1} - Q_{k-2}, \qquad Q_0 = 2\omega, \qquad Q_{-1} = 1, \\ &P_k = 2\omega P_{k-1} - P_{k-2}, \qquad P_0 = 1, \qquad P_{-1} = 0. \end{aligned}$$

Since $P_1 = 2\omega = Q_0$, we get $Q_k = P_{k+1}$.

On the other hand, the recurrence and initial conditions for P_k show that $P_k = U_k(\omega)$, where U_k is the second kind Chebyshev polynomial of degree k,

$$U_k(\omega) = \frac{\lambda^{k+1} - \lambda^{-(k+1)}}{\lambda - \lambda^{-1}}, \quad \lambda = \omega + \sqrt{\omega^2 - 1}.$$

The parameter λ is one of the roots of the characteristic polynomial

$$B(z) = A(-z\overline{\alpha}/|\alpha|) = z^2 - 2\omega z + 1, \tag{47}$$

no matter which one because both of them are inverse of each other.

Hence,

$$s_n = \frac{s_0 U_n(\omega) - U_{n-1}(\omega)}{s_0 U_{n-1}(\omega) - U_{n-2}(\omega)}, \quad n \geq 1.$$

As a consequence, the solution of the inverse problem is quasi-definite if and only if

$$s_0 U_n(\omega) \neq U_{n-1}(\omega), \quad n \geq 0. \tag{48}$$

In case $s_0 U_j(\omega) \neq U_{j-1}(\omega)$ for $j < n$ but $s_0 U_n(\omega) = U_{n-1}(\omega)$, the related solution of the inverse problem is quasi-definite in $\mathbb{P}_n$ but not in $\mathbb{P}_{n+1}$.

Besides, from the solution for s_n we can obtain the rest of the variables of interest for the inverse problem. In particular, for $n \geq 1$,

$$x_n = \frac{\overline{\alpha}}{|\alpha|} \frac{s_0 U_n(\omega) - U_{n-1}(\omega)}{s_0 U_{n-1}(\omega) - U_{n-2}(\omega)},$$
$$b_{n+1} = y_n = \left(-\frac{\overline{\alpha}}{|\alpha|}\right)^n \frac{y_0}{s_0 U_{n-1}(\omega) - U_{n-2}(\omega)}.$$

We can express these variables, as well as the quasi-definiteness condition (48), in terms of other parameters. For instance, following Algorithm I2, we can use as free parameters b_1 and x_0. Then, using (44) and the relation $y_0 = b_1 - x_0$, we get for some $\kappa \in \mathbb{R}^*$,

$$\alpha = \kappa \overline{x}_0, \qquad \beta = \frac{\kappa}{2}(1 - |b_1|^2 + 2\mathrm{Re}(\overline{x}_0 b_1)). \tag{49}$$

If we chose the approach of Algorithm I3, then the free parameters must be b_1 and α, β, so we should express s_0, x_0 and y_0 in terms of them. From (49), bearing in mind that $\kappa = |\alpha|/s_0$, we obtain

$$s_0 = \frac{|\alpha|}{2} \frac{1 - |b_1|^2}{\beta - \mathrm{Re}(\alpha b_1)}, \qquad x_0 = \frac{\overline{\alpha}}{2} \frac{1 - |b_1|^2}{\beta - \mathrm{Re}(\alpha b_1)}, \qquad y_0 = -\frac{1}{2} \frac{A(-b_1)}{\beta - \mathrm{Re}(\alpha b_1)}. \tag{50}$$

Finally, we can use the point of view of Algorithm I1. This implies that we restrict our attention to the solutions of the inverse problem which are quasi-definite at least in $\mathbb{P}_2$, and not only in $\mathbb{P}_1$, which was the case till now. Then, according to Algorithm I1, b_1 and b_2 could be used as free parameters too. This can be done using (49) and the relation

$$b_2 = y_1 = -\frac{y_0}{\overline{x}_0} = \frac{x_0 - b_1}{\overline{x}_0},$$

which determines x_0 as the following function of b_1 and b_2,

$$x_0 = \frac{1}{1 - |b_2|^2}(b_1 + \overline{b}_1 b_2).$$

Also, b_2 can be expressed in terms of α, β and b_1 using (50), which gives

$$b_2 = \frac{A(-b_1)}{\alpha(1 - |b_1|^2)}.$$

The fact that the iterations (46) generating the solutions of the inverse problem and the quasi-definiteness condition (48) are given in terms of α, β and s_0 uniquely suggests the possibility of using these variables to parametrize such solutions. However, this is not possible because an arbitrary value of α, β and s_0 can be associated with no value of b_1 or with infinitely many values of b_1. Indeed, the first identity of (50) can be written as

$$|b_1 - x_0|^2 = B(s_0),$$

which shows that we have the following possibilities:

- If $B(s_0) < 0$ there is no solution associated with α, β and s_0.
- If $B(s_0) = 0$ there is exactly one solution associated with α, β and s_0: that one determined by α, β and $b_1 = x_0 = s_0\overline{\alpha}/|\alpha|$.

- If $B(s_0) > 0$ there are infinitely many solutions associated with α, β and s_0: those ones determined by α, β and any value of b_1 in the circle with center x_0 and radius $\sqrt{B(s_0)}$. Therefore such solutions are parametrized by a phase.

In consequence, given $P(z) = \alpha z + \beta$, the inequality $B(s_0) \geq 0$ determines the permitted values of s_0. The set of solutions associated with P and a permitted value s_0 will be called the circle of solutions for P and s_0, and will be denoted $\mathfrak{C}(P, s_0)$. Eventually $B(s_0) = 0$ and $\mathfrak{C}(P, s_0)$ degenerates into a single solution. From (50) we see that, once P is fixed, the circles $\mathfrak{C}(P, s_0)$, $s_0 \in \mathbb{R}^*$, $B(s_0) \geq 0$, do not intersect between themselves, and $|b_1| \neq 1$ for any functional of such circles. Hence, these circles generate a partition in the set of functionals of $H(u, P + P_*)$ which are quasi-definite in $\mathbb{P}_1$. The fact that the quasi-definiteness condition depends only on $\omega = \beta/|\alpha|$ and s_0 means that all the functionals of a circle $\mathfrak{C}(P, s_0)$ have the same number of MOP.

It seems that the presence of the circles of solutions with similar properties should have to do with some symmetry of the problem. The most obvious one is the rotation symmetry. If $u = vL$, then $u_\theta = v_\theta L_\theta$ for any angle θ, where the rotation of a Laurent polynomial f and a functional v are defined by $f_\theta(z) = f(\mathrm{e}^{-\mathrm{i}\theta} z)$ and $v_\theta[f] = v[f_{-\theta}]$. When $u_\theta = u$ we find that $v \in H_1(u)$ implies $v_\theta \in H_1(u)$. The only functional u which is invariant under any rotation is that one defined by the Lebesgue measure, so only in this case we can ensure that $H_1(u)$ is partitioned in "circles of rotated solutions" obtained by the rotation of one of them.

Bearing in mind that we are identifying equivalent functionals and that the rotation of a functional preserves its quasi-definiteness properties, the rotation symmetry permits us to reduce the analysis of the set $H_1(u)$ for the Lebesgue functional u to the case $\alpha = 1$ because each "circle of rotated solutions" has a representative with a monic polynomial P. However, the reduction of the analysis to such canonical cases is not possible for any other hermitian functional u.

Nevertheless, the rotation symmetry of the Lebesgue measure is not responsible of the circles of solutions $\mathfrak{C}(P, s_0)$ that we have found: the solutions of any circle $\mathfrak{C}(P, s_0)$ have a common polynomial P, while the solutions of a "circle of rotated solutions" are related to different polynomials P obtained by a rotation of one of them; furthermore, the rotation of a functional also rotates its Schur parameters around the origin, but the parameters b_1 of the solutions of a circle $\mathfrak{C}(P, s_0)$ are obtained rotating one of them around $x_0 \neq 0$. The search for the "symmetry transformations" relating the functionals of a circle $\mathfrak{C}(P, s_0)$ remains as an open problem.

Some particular quasi-definite solutions deserve a special mention, i.e., the solutions with constant coefficients x_n, y_n, which are characterized by any of the statements of the following equivalence, which follow easily from the previous results:

$$s_n = s_0, \quad n \geq 0 \Leftrightarrow x_n = x_0, \quad n \geq 0 \Leftrightarrow y_n = 0, \quad n \geq 0 \Leftrightarrow b_n = 0, \quad n \geq 2 \Leftrightarrow$$
$$\Leftrightarrow b_2 = 0 \Leftrightarrow y_0 = 0 \Leftrightarrow b_1 = x_0 \Leftrightarrow A(-b_1) = 0 \Leftrightarrow A(-x_0) = 0 \Leftrightarrow B(s_0) = 0.$$

Therefore, these constant solutions correspond exactly to the case where a circle of solutions degenerates into a single solution. The corresponding functionals are those ones associated with the Bernstein–Szegő polynomials $\psi_{n+1}(z) = (z + b_1)z^n$. Since $-b_1$ must be a root of A, such solutions can appear only when A has roots outside the unit circle, which corresponds to the Geronimus transformation of the Lebesgue measure.

It is advisable to discuss the three possibilities (a)–(c) pointed out at the beginning of Section 3.2.1 according to the location of the roots of the polynomial A. The reason is that the qualitative behavior of the solutions of the inverse problem depend strongly on the case at hand. Before doing this we must remark that, since $B(z) = A(-z\overline{\alpha}/|\alpha|)$, the roots ζ_1, ζ_2 of A

are related to the roots λ, λ^{-1} of B through $\zeta_1 = -\lambda\overline{\alpha}/|\alpha|$, $\zeta_2 = -\lambda^{-1}\overline{\alpha}/|\alpha|$, and the three cases we want to discuss can be characterized in terms of ω. Concerning this discussion, notice that, once b_1 is fixed, any restriction on ω becomes a restriction on the initial value x_0 by (49).

We will comment the asymptotics in each of the cases (a)–(c) using the notation $p_n \sim q_n$ to mean that $\lim(p_n/q_n) = 1$.

(a) $A(z) = \alpha(z-\zeta)(z-1/\overline{\zeta}),\ \zeta \in \mathbb{D}\setminus\{0\} \Leftrightarrow |\omega| > 1$.

This case corresponds to B having two different roots $\lambda, \lambda^{-1} \in \mathbb{R}$, thus we can suppose $|\lambda| < 1$ so that $\zeta = -\lambda\overline{\alpha}/|\alpha|$. Then, the quasi-definiteness condition (48) becomes

$$s_0 \neq \lambda\frac{1-\lambda^{2n}}{1-\lambda^{2n+2}}, \quad n \geq 0, \tag{51}$$

or equivalently

$$x_0 \neq -\zeta\frac{1-|\zeta|^{2n}}{1-|\zeta|^{2n+2}}, \quad n \geq 0,$$

which can be also understood as a restriction on b_1 because, together with A, it determines x_0 through (50).

Given only α and β, not any value of s_0 is permitted because $B(s_0)$ can be negative. This happens when $\lambda_1 < s_0 < \lambda_2$, where λ_1, λ_2 are the roots λ, λ^{-1} of B but ordered so that $\lambda_1 < \lambda_2$. Therefore, the values of s_0 associated with a solution of the inverse problem are those lying on $(-\infty, \lambda_1] \cup [\lambda_2, \infty)$. Then, the corresponding sequence of MOP is infinite or finite depending on whether the quasi-definiteness condition (51) is satisfied for every n or not.

There are two quasi-definite constant solutions: $s_n = \lambda$, $x_n = -\zeta = b_1$, $y_n = 0$ and $s_n = \lambda^{-1}$, $x_n = -1/\overline{\zeta} = b_1$, $y_n = 0$. Both of them give rise to a Bernstein–Szegő solution with $b_n = 0$, $n \geq 2$, but the first one is positive definite with measure $\mathrm{d}m(z)/|z-\zeta|^2$, while the second one is indefinite. As we will see, the solution $\mathrm{d}m(z)/|z-\zeta|^2$ is somewhat singular among the solutions of the inverse problem, so in what follows we will consider only $s_0 \neq \lambda$, i.e., $x_0 \neq -\zeta$. Then,

$$\begin{aligned}
&s_0U_n(\omega) - U_{n-1}(\omega) \sim \frac{s_0-\lambda}{1-\lambda^2}\lambda^{-n}, \qquad b_2 = \frac{(b_1+\zeta)(b_1+1/\overline{\zeta})}{1-|b_1|^2},\\
&b_{n+1} = y_n \sim b_2\frac{x_0(1-|\zeta|^2)}{x_0+\zeta}\zeta^{n-1} = -\overline{\alpha}\frac{b_1+1/\overline{\zeta}}{\overline{\alpha}\overline{b}_1+\alpha\zeta}(1-|\zeta|^2)\zeta^{n-1},\\
&\lim b_n = \lim y_n = 0, \qquad \lim s_n = \lambda^{-1}, \qquad \lim x_n = -1/\overline{\zeta}.
\end{aligned}$$

Furthermore, the related orthogonal polynomials obey the asymptotics

$$\begin{aligned}
&\psi_{n+1}(z) \sim -\overline{\alpha}\frac{b_1+1/\overline{\zeta}}{\overline{\alpha}\overline{b}_1+\alpha\zeta}(1-|\zeta|^2)\zeta^{n-1}, \quad |z| < |\zeta|,\\
&\psi_{n+1}(z) \sim (z-1/\overline{\zeta})z^n, \quad |z| > |\zeta|.
\end{aligned}$$

We observe that the parameters of the indefinite Bernstein–Szegő solution provide the asymptotics of the parameters for all the solutions except for $\mathrm{d}m(z)/|z-\zeta|^2$. Also, the indefinite Bernstein–Szegő polynomials $(z-1/\overline{\zeta})z^n$ yield the large z asymptotics of the rest of MOP which solve the inverse problem, with the exception again of the positive definite Bernstein–Szegő polynomials $(z-\zeta)z^n$.

(b) $A(z) = \alpha(z-\zeta)^2,\ \zeta \in \mathbb{T} \Leftrightarrow |\omega| = 1.$

This is equivalent to state that B has a double root $\lambda = \omega \in \{-1, 1\}$, which is related to ζ by $\zeta = -\lambda\overline{\alpha}/|\alpha|$. No quasi-definite solution with constant x_n can appear now, thus $s_0 \neq \lambda$ and $x_0 \neq -\zeta$ for any quasi-definite solution. The confluent form of the Chebyshev polynomials $U_n(\omega) = (n+1)\lambda^n$ yields the quasi-definiteness condition

$$s_0 \neq \lambda \frac{n}{n+1}, \quad n \geq 0, \tag{52}$$

i.e.,

$$x_0 \neq -\zeta \frac{n}{n+1}, \quad n \geq 0,$$

where, once b_1 is chosen, x_0 is fixed by (50) with $\beta = \lambda|\alpha|$.

If we fix only α and β, then s_0 can take any real value because now B is non-negative on $\mathbb{R}$.

We have the relations

$$\begin{aligned}
&s_0 U_n(\omega) - U_{n-1}(\omega) \sim (s_0-\lambda)n\lambda^n, \qquad b_2 = \frac{(b_1+\zeta)^2}{1-|b_1|^2},\\
&b_{n+1} = y_n \sim b_2 \frac{x_0}{x_0+\zeta}\frac{\zeta^{n-1}}{n} = -\overline{\alpha}\frac{b_1+\zeta}{\overline{\alpha}\overline{b}_1+\alpha\zeta}\frac{\zeta^{n-1}}{n},\\
&\lim b_n = \lim y_n = 0, \qquad \lim s_n = \lambda, \qquad \lim x_n = -\zeta,
\end{aligned}$$

and the asymptotics of the corresponding orthogonal polynomials is

$$\begin{aligned}
&\psi_{n+1}(z) \sim -\overline{\alpha}\frac{b_1+\zeta}{\overline{\alpha}\overline{b}_1+\alpha\zeta}\frac{\zeta^{n-1}}{n}, \quad |z| < 1,\\
&\psi_{n+1}(z) \sim (z-\zeta)z^n, \quad |z| > 1.
\end{aligned}$$

We see that in this case there is a so well defined asymptotics for any solution as in (a). However, contrary to $|\omega| > 1$, the asymptotics of the frontier case $|\omega| = 1$ defines no quasi-definite solution of the inverse problem.

(c) $A(z) = \alpha(z-\zeta_1)(z-\zeta_2),\ \zeta_1 \neq \zeta_2,\ \zeta_k \in \mathbb{T} \Leftrightarrow |\omega| < 1.$

Now B has two different roots $\lambda, \overline{\lambda} \in \mathbb{T}$ so that $\zeta_1 = -\lambda\overline{\alpha}/|\alpha|$ and $\zeta_2 = -\overline{\lambda}\overline{\alpha}/|\alpha|$. The quasi-definiteness condition (48) reads as

$$s_0 \mathrm{Im}(\lambda^{n+1}) \neq \mathrm{Im}(\lambda^n), \quad n \geq 0,$$

that is,

$$\overline{x}_0(\zeta_1^{n+1} - \zeta_2^{n+1}) \neq \zeta_2^n - \zeta_1^n, \quad n \geq 0,$$

which again can be considered as a constraint on b_1 due to (50).

Concerning the possible choices of s_0 when fixing only α and β, any real value of s_0 is possible since B is now positive on $\mathbb{R}$.

Analogously to case (b), $s_0 \neq \lambda, \overline{\lambda}$ and $x_0 \neq -\zeta_1, -\zeta_2$ for any quasi-definite solution. Writing $\lambda = e^{i\theta}$, $\theta \notin \mathbb{Z}\pi$, and $s_0 - \lambda = |s_0-\lambda|e^{i\gamma}$,

$$s_0 U_n(\omega) - U_{n-1}(\omega) = |s_0-\lambda|\frac{\sin((n+1)\theta+\gamma)}{\sin\theta},$$

thus the quasi-definiteness condition can be stated as

$$n\theta + \gamma \notin \mathbb{Z}\pi, \quad n \geq 1,$$

and we find the identities

$$s_n = \frac{\sin((n+1)\theta + \gamma)}{\sin(n\theta + \gamma)} = \cos\theta + \frac{\sin\theta}{\tan(n\theta + \gamma)},$$

$$b_{n+1} = y_n = \left(-\frac{\overline{\alpha}}{|\alpha|}\right)^n \frac{y_0}{|s_0 - \lambda|} \frac{\sin\theta}{\sin(n\theta + \gamma)},$$

which show that in this case s_n and $|b_n|$ do not converge for any quasi-definite solution.

The algorithm (46) giving the solutions of the inverse problem for the Lebesgue measure can be interpreted as a Newton algorithm to find the zeros of a function. It is instructive to discuss the different behavior of the associated Newton algorithm depending on the values of ω and s_0. This approach sheds light on the different asymptotics found in cases (a)–(c). Since we will discuss the behavior depending on the values of ω and s_0, we remember that, given P, there is a set of permitted values s_0 and each choice of s_0 determines a circle of solutions $\mathfrak{C}(P, s_0)$ which degenerates into a single solution when s_0 is a root of B. Remember also that the solutions of such a circle have the same number of MOP.

The Newton algorithm for a real function $f(s)$ of a real variable s is given by the iteration

$$s_n = s_{n-1} - \frac{f(s_{n-1})}{f'(s_{n-1})}.$$

Comparing this with (46) we see that the algorithm providing the parameters s_n of the inverse problem for the Lebesgue measure can be understood as the Newton algorithm for a function $f(s)$ satisfying

$$s - \frac{f(s)}{f'(s)} = 2\omega - \frac{1}{s}.$$

For each value of ω the above differential equation has a unique solution up to a multiplicative constant factor. This essentially unique solution has three qualitatively different expressions depending on the localization of ω in $\mathbb{C}$. Solving the above equation we find three cases (λ_1, λ_2 are the roots of B):

(a) $|\omega| > 1 \Rightarrow f(s) = (\frac{|s-\lambda_2|^{\lambda_2}}{|s-\lambda_1|^{\lambda_1}})^{\frac{1}{\lambda_2-\lambda_1}}$.
(b) $|\omega| = 1 \Rightarrow f(s) = |s - \omega| \exp(\frac{\omega}{\omega - s})$.
(c) $|\omega| < 1 \Rightarrow f(s) = \sqrt{B(s)} \exp(\frac{\omega}{\sqrt{1-\omega^2}} \arctan(\frac{s-\omega}{\sqrt{1-\omega^2}}))$.

The typical behavior of the iterations in these three cases is shown in Figs. 1–8, which represent the function $f(s)$ as well as some of these iterations for different choices of ω and s_0. In any case the function $f(s)$ is analytic in $\mathbb{R} \setminus \{\lambda_1, \lambda_2\}$ and has a minimum at $s = 0$ which can stop the iterations, giving rise to a circle $\mathfrak{C}(P, s_0)$ of non-quasi-definite solutions but with a finite segment of MOP with the same length for all the circle.

When $|\omega| > 1$ the function $f(s)$ diverges to ∞ at $s = \lambda$ and vanishes at $s = \lambda^{-1}$, where λ is the root with smallest module among λ_1, λ_2. Indeed,

$$f(s) = \frac{|s - \lambda^{-1}|^{\frac{1}{1-\lambda^2}}}{|s - \lambda|^{\frac{\lambda^2}{1-\lambda^2}}}, \quad |\lambda| < 1,$$

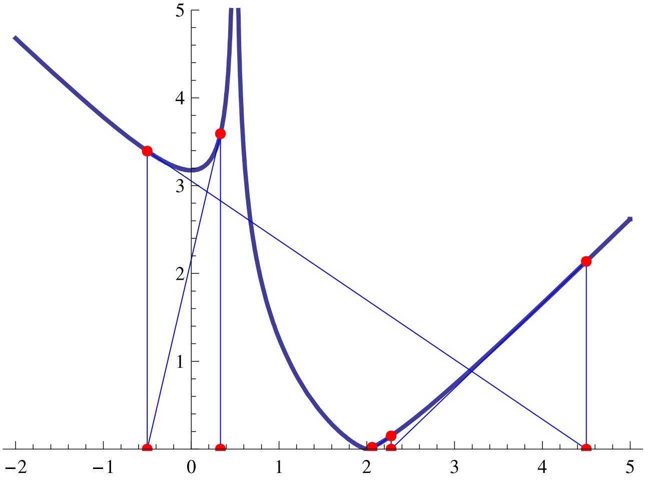

Fig. 1. (Case (a) — quasi-definite circle of solutions) First values of s_n for $\omega = \frac{5}{4}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$, $\sigma_n = 2\frac{4^n-1}{4^{n+1}-1}$, $s_0 = \frac{1}{3} \notin \{\sigma_n\}$. This value of s_0 generates an infinite sequence (s_n) such that $s_n \to \lambda_2^+$ monotonically for $n \geq 2$. Hence, the solutions of the associated circle $\mathfrak{C}(P, s_0)$ are quasi-definite.

thus $f(s)$ attains its absolute minimum at λ^{-1} and $f \in C^{(1)}(\mathbb{R} \setminus \{\lambda\})$ with $f'(\lambda^{-1}) = 0$. Excluding the case $s_0 = \lambda$, the iterations, which must start at a point of $(-\infty, \lambda_1] \cup [\lambda_2, \infty)$, always converge to λ^{-1} (corresponding to a circle of quasi-definite solutions) or they stop at the origin after a finite number of steps (corresponding to a circle of solutions with only a finite segment of MOP).

If $\omega = \pm 1$, then $\lim_{s \to \lambda^{\mp}} f(s) = \infty$ and $\lim_{s \to \lambda^{\pm}} f(s) = \lim_{s \to \lambda^{\pm}} f'(s) = 0$, where $\lambda = \lambda_1 = \lambda_2 = \pm 1$, which plays again the role of an attractor where the iterations converge (circle of quasi-definite solutions) while they do not stop at the origin (circle of solutions with a finite segment of MOP).

On the contrary, $f(s)$ has no divergence neither zero when $|\omega| < 1$, and the origin is then the absolute minimum. In this case, as far as the iterations do not reach the origin (circle of solutions with a finite segment of MOP), they oscillate indefinitely around such a minimum (circle of quasi-definite solutions).

In any case, for each value of ω, the values of s_0 associated with non-quasi-definite solutions can be obtained by the inverse Newton algorithm starting at the origin, so they form a sequence (σ_n) given by

$$\sigma_n = \frac{1}{2\omega - \sigma_{n-1}}, \qquad \sigma_0 = 0. \tag{53}$$

If $s_0 = \sigma_n$, then $s_j = \sigma_{n-j} \neq 0$ for $j < n$ and $s_n = 0$, hence the solutions of the related circle $\mathfrak{C}(P, s_0)$ have only $n + 1$ MOP. When $|\omega| \geq 1$, (σ_n) is a monotone sequence with limit λ, but if $|\omega| < 1$ then (σ_n) is non-convergent and oscillates around the origin. Eventually, $\sigma_{n-1} = 2\omega$ and the iterations (53) stop. To understand this fact notice that (48) shows that

$$\sigma_n = \frac{U_{n-1}(\omega)}{U_n(\omega)}$$

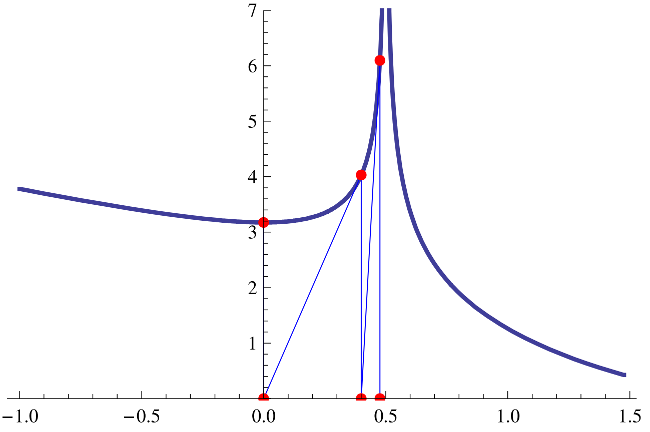

Fig. 2. (Case (a) — non-quasi-definite circle of solutions) Values of s_n for $\omega = \frac{5}{4}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$, $s_0 = \sigma_2 = \frac{10}{21}$. The iterations stop at $n = 2$, thus the solutions of the circle $\mathfrak{C}(P, s_0)$ have only the MOP ψ_0, ψ_1, ψ_2. Since the set $\{\sigma_n\}$ is infinite, there exist non-quasi-definite solutions with an arbitrary number of MOP.

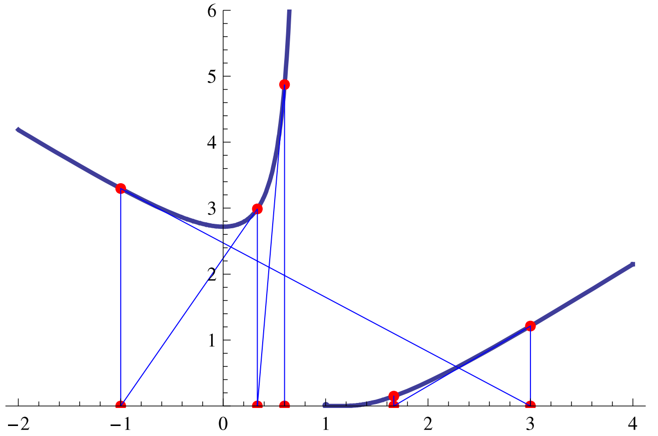

Fig. 3. (Case (b) — quasi-definite circle of solutions) First values of s_n for $\omega = \lambda_1 = \lambda_2 = 1$, $\sigma_n = \frac{n}{n+1}$, $s_0 = \frac{3}{5} \notin \{\sigma_n\}$. The situation is similar to Fig. 1, but now $\lambda_1 = \lambda_2$.

if $U_n(\omega) \neq 0$, otherwise σ_n has no meaning because no value of s_0 can satisfy $s_0 U_n(\omega) = U_{n-1}(\omega)$ when $U_n(\omega) = 0$. The recurrence for U_n implies that $\sigma_{n-1} = 2\omega$ iff $U_n(\omega) = 0$, so this is exactly the case where σ_n does not exist and, besides, $\sigma_{n+1} = 0 = \sigma_0$, hence the values of σ_j, $j \geq n+1$, are simply a reiteration of the values for $j = -1, 0, \ldots, n-1$ if we define $\sigma_{-1} = \infty$. Therefore, (53) always works for $n \geq -1$ if we assume that $\sigma_{n-1} = 2\omega$ gives $\sigma_n = \infty$, which leads to $\sigma_{n+1} = 0$ and yields a periodic sequence (σ_j) in $\mathbb{R} \cup \{\infty\}$ with period $n + 1$.

Summarizing, if ω is a zero of U_n, which can hold only when $|\omega| < 1$, there is a finite number of non-quasi-definite circles of solutions $\mathfrak{C}(P, s_0)$, those ones related to the initial values $s_0 \in \{\sigma_j\}_{j=1}^{n-1}$. Furthermore, if n is the smallest index such that $U_n(\omega) = 0$, the quantities σ_j, $j = 0, \ldots, n-1$, are different from each other, hence there are exactly $n - 1$ non-quasi-definite

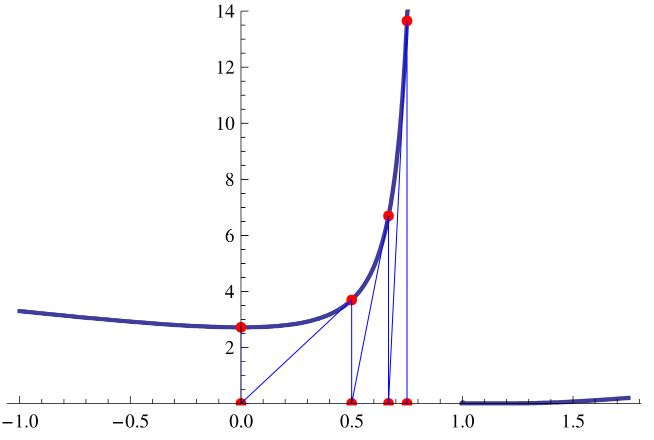

Fig. 4. (Case (b) — non-quasi-definite circle of solutions) Values of s_n for $\omega = \lambda_1 = \lambda_2 = 1$, $s_0 = \sigma_3 = \frac{3}{4}$. The situation is similar to Fig. 2 but now $\lambda_1 = \lambda_2$ and we have chosen s_0 so that the solutions of the circle $\mathfrak{C}(P, s_0)$ have four MOP.

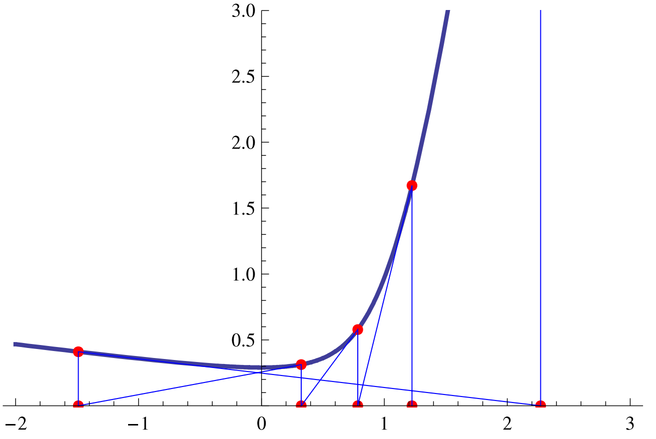

Fig. 5. (Case (c) — quasi-definite circle of solutions) First values of s_n for $\omega = \frac{4}{5}$, $\lambda_{1,2} = \frac{4}{5} \pm \frac{3}{5}\mathrm{i}$, $\sigma_n = 5\frac{\mathrm{Im}((4+3i)^n)}{\mathrm{Im}((4+3i)^{n+1})}$, $s_0 = \sqrt{\frac{3}{2}} \notin \{\sigma_n\} \subset \mathbb{Q}$. The solutions of the associated circle $\mathfrak{C}(P, s_0)$ are quasi-definite because s_0 generates an infinite sequence (s_n) which oscillates indefinitely around the origin.

circles $\mathfrak{C}(P, s_0)$, and the length of the corresponding finite segments of MOP runs from 2 to n when $s_0 = \sigma_1, \ldots, \sigma_{n-1}$. Therefore, there are no non-quasi-definite solutions with more than n MOP.

On the contrary, if $U_n(\omega) \neq 0$ for all n, then $\sigma_j \neq \sigma_k$ for $j \neq k$, thus an infinite denumerable set of non-quasi-definite circles $\mathfrak{C}(P, s_0)$ appear, which correspond to $s_0 \in \{\sigma_j\}_{j=0}^{\infty}$. In this case, given any $n \in \mathbb{N}$, there is exactly one non-quasi-definite circle of solutions with only $n+1$ MOP, which corresponds to $s_0 = \sigma_n$.

As a final remark notice that $U_n(\omega) = 0$ means $\lambda^{2n+2} = 1$, $\lambda \neq \pm 1$. Therefore, not only the sequence (σ_j), but also $(U_j(\omega))$ is in this case periodic with period $n+1$, so (s_j) shows such a periodic behavior too, no matter the choice of s_0.

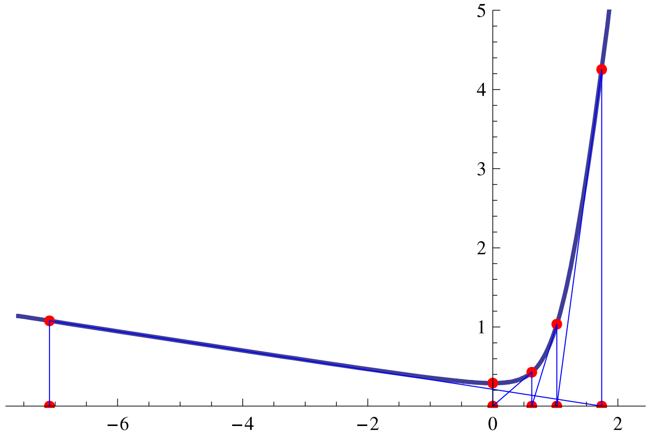

Fig. 6. (Case (c) — non-quasi-definite circle of solutions) Values of s_n for $\omega = \frac{4}{5}$, $\lambda_{1,2} = \frac{4}{5} \pm \frac{3}{5}\mathrm{i}$, $s_0 = \sigma_4 = -\frac{560}{79}$. The iterations stop at $n = 4$, thus the solutions of the related circle $\mathfrak{C}(P, s_0)$ have only five MOP. Like in Figs. 2 and 4, the set $\{\sigma_n\}$ is infinite (but, on the contrary, (σ_n) is not monotone neither convergent) because $\lambda_{1,2}^2$ are not roots of the unity, so there exist non-quasi-definite solutions with an arbitrary number of MOP.

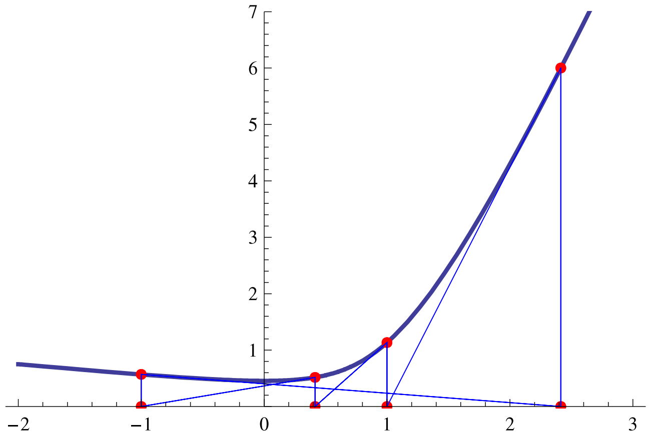

Fig. 7. (Case (c) – quasi-definite circle of solutions – periodic case) Values of s_n for $\omega = \frac{1}{\sqrt{2}}$, $\lambda_{1,2} = e^{\pm \mathrm{i}\frac{\pi}{4}}$, $\sigma_n = \frac{\mathrm{Im}(e^{\mathrm{i}\frac{\pi}{4}n})}{\mathrm{Im}(e^{\mathrm{i}\frac{\pi}{4}(n+1)})}$, $s_0 = 1 \notin \{\sigma_n\} = \{0, \sqrt{2}, 1/\sqrt{2}, \infty\}$. Like in Fig. 5, the solutions of the associated circle $\mathfrak{C}(P, s_0)$ are quasi-definite but, on the contrary, the sequences (s_n) and (σ_n) are periodic with period 4 because $U_3(\omega) = 0$.

4. Applications of these techniques

The characterization we have obtained for hermitian functionals related by polynomial perturbations is not only interesting by itself, but provides an efficient tool to answer different questions concerning orthogonal polynomials on the unit circle. In this section we will show two examples of this. The first one exploits the fact that a polynomial perturbation is equivalent to a linear relation with polynomial coefficients between two sequences of orthogonal polynomials and their reversed ones. The second one deals with a problem concerning associated polynomials,

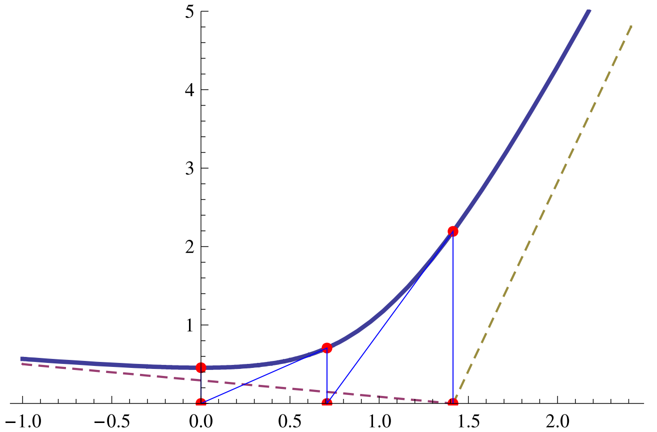

Fig. 8. (Case (c) – non-quasi-definite circle of solutions – periodic case) Values of s_n for $\omega = \frac{1}{\sqrt{2}}$, $\lambda_{1,2} = e^{\pm i\frac{\pi}{4}}$, $s_0 = \sigma_2 = \frac{1}{\sqrt{2}}$. Like in Fig. 6, the solutions of the circle $\mathfrak{C}(P, s_0)$ are non-quasi-definite, although in this case there exist only three MOP. Indeed, contrary to Fig. 6, there is no non-quasi-definite solution with more than three MOP because σ_n takes only three finite values: $\sigma_0 = 0$, $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 1/\sqrt{2}$. The picture, which can be understood also as the inverse Newton algorithm starting at the origin which yields (σ_n), shows clearly that $\sigma_3 = \infty$ because the corresponding tangent line becomes any of the two asymptotes.

which can be solved due to the formulation of a polynomial perturbation in terms of a difference equation for two sequences of Schur parameters.

4.1. Orthogonal polynomials and linear combinations with constant polynomial coefficients

There are in the literature different results on the orthogonality properties of linear combinations of orthogonal polynomials. In particular, it is known that, if (φ_n) and (ψ_n) are MOP on the unit circle, a relation like

$$\psi_{n+r} = \sum_{j=0}^{r} (\lambda_{j,n}\varphi_{n+j} + \kappa_{j,n}\varphi^*_{n+j}), \quad \lambda_{j,n}, \kappa_{j,n} \in \mathbb{C},\ \lambda_{0,n} \neq 0,\ n \geq 0, \tag{54}$$

forces (ψ_n) to be Bernstein–Szegő polynomials when $r > 1$ (see [28]). The result is so strong that it holds assuming (54) only when $n \geq n_0$ for some n_0, and even if we suppose that the sum in (54) is up to and index $r(n)$ depending on n, with the simple restriction $1 < r(n) \leq n/2$ for $n \geq n_0$ (see [29]).

A way to escape from this triviality is to consider a more general relation than (54). Identity (54) implies that $\psi_{n+r} \in (z\mathbb{P}_{n-2})^{\perp_{n+r}} \subset (z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$ for $r \geq 1$, where the orthogonality is understood with respect to the functional associated with (φ_n). Thus, Lemma 2.1 shows that (54) is a particular case of

$$\psi_{n+r} = X_n\varphi_n + Y_n\varphi^*_n, \quad X_n \in \mathbb{P}_r,\ Y_n \in \mathbb{P}_{r-1},\ n \geq 0. \tag{55}$$

However, contrary to (54), a relation like (55) can hold for non-trivial MOP (φ_n) and (ψ_n), since it is always equivalent to a polynomial perturbation relation between the corresponding orthogonality functionals due to Theorem 2.11 and the subsequent comments, together with Proposition 2.14: the hermitian functionals u and v associated with (φ_n) and (ψ_n) must be related by $u = vL$ where $L = P + P_*$ is given by a polynomial P with $\deg P \leq r$; the condition $X_n(0) \neq 0$, which holds for no n or simultaneously for all n, characterizes the case $\deg P = r$.

In this section we will show that the freedom enclosed in (55) is large enough to yield non-trivial solutions even when imposing very strong conditions on X_n and Y_n. More precisely, we will find all the pairs of sequences of MOP (ψ_n) and (φ_n) related by (55) with constant polynomials coefficients, i.e.,

$$\psi_{n+r} = X\varphi_n + Y\varphi_n^*, \quad X \in \mathbb{P}_r,\ Y \in \mathbb{P}_{r-1},\ n \geq 0. \tag{56}$$

We will see that, for $r > 1$, the solutions are the MOP (φ_n) with constant Schur parameters $a_n = a$, together with any MOP sequence (ψ_n) with arbitrary Schur parameters $b_1, \ldots, b_r$, but a fixed value $b_n = b = a\psi_r(\zeta)/\psi_r^*(\zeta)$, $\zeta = (1-a)/(1-\overline{a})$, for $n > r$. This also holds for $r = 1$ as far as $X(z) \neq z - 1$, so that b_1 and a parametrize the solutions in this case. However, if $X(z) = z - 1$ the solutions are parametrized by b_1 and a sequence (a_n) arbitrarily chosen in the perpendicular bisector of $[0, 1 + b_1]$. Then, b_{n+1} must be the symmetric point of a_n with respect to $[0, 1 + b_1]$.

This is not only an academic problem, but its importance relies on the fact that the constant solutions should play the role of fixed points with respect to the asymptotics of the polynomials X_n, Y_n related to the quasi-definite solutions of $H_r(u)$. Therefore, some of these fixed points should act as attractors whose study could give information about the asymptotics for the quasi-definite solutions of $H_r(u)$, similarly to what happens in the example given in Section 3.2.2.

Relation (56) can be rewritten, together with its reversed, as

$$\Psi_{n+r} = \mathcal{X}\Phi_n, \quad \mathcal{X} = \begin{pmatrix} X & Y \\ zY^* & X^* \end{pmatrix}, \ n \geq 0,$$

and the polynomial perturbation is recovered by $A = \det \mathcal{X}$.

As follows from Theorem 2.13 and Proposition 2.14, the problem we want to solve is equivalent to the recurrence $\mathcal{T}_{n+r}\mathcal{X} = \mathcal{X}\mathcal{S}_n$, $n \geq 1$, and the initial condition $\mathcal{X}\Phi_0 = \Psi_r$, i.e.,

$$\begin{cases} \overline{a}_n Y = b_{n+r} Y^*, \\ a_n X - b_{n+r} X^* = (z-1)Y, \end{cases} \quad n \geq 1, \\ \psi_r = X + Y, \quad X \in \mathbb{P}_r,\ Y \in \mathbb{P}_{r-1}. \tag{57}$$

If $Y = 0$, Eqs. (57) yield $b_{n+r}X^* = a_n X$ and $\psi_r = X$. Since ψ_r and ψ_r^* have no common roots, we find that $a_n = b_{n+r} = 0$ for $n \geq 1$. This situation corresponds to u being the functional associated with the Lebesgue measure and MOP $\varphi_n(z) = z^n$, and v a Bernstein–Szegő type functional with the first $r + 1$ MOP generated by arbitrary Schur parameters $b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$, while $\psi_{n+r}(z) = z^n \psi_r(z)$ for $n \geq 1$.

Let us find now the solutions with $Y \neq 0$. Denote for convenience $a = a_n$ and $b = b_{n+r}$. The first equation of (57) simply says that Y is proportional to a self-reciprocal polynomial in $\mathbb{P}_{r-1}$ and $|b| = |a|$. Using such equation and bearing in mind that $\psi_r = X + Y$ and $\psi_r^* = X^* + zY^*$, we can eliminate X and X^* in the second equation of (57), which becomes

$$a\psi_r - b\psi_r^* = [z(1-\overline{a}) - (1-a)]Y. \tag{58}$$

Therefore,

$$b = a\frac{\psi_r(\zeta)}{\psi_r^*(\zeta)}, \quad \zeta = \frac{1-a}{1-\overline{a}},$$

$$
\begin{aligned}
Y(z) &= \frac{a}{1-\overline{a}} \frac{1}{\psi_r^*(\zeta)} \frac{\psi_r^*(\zeta)\psi_r(z) - \psi_r(\zeta)\psi_r^*(z)}{z-\zeta} \\
&= \frac{a}{1-\overline{a}} \varepsilon_r \overline{\left(\frac{\zeta}{\psi_r(\zeta)}\right)} K_{r-1}(z,\zeta),
\end{aligned}
\tag{59}
$$

where we have used the Christoffel–Darboux formula for the n-th kernel $K_n(z,\zeta) = \sum_{j=0}^{n} \varepsilon_j^{-1} \psi_j(z)\overline{\psi_j(\zeta)}$ associated with the MOP (ψ_j).

As a consequence, given ψ_r, the solutions of (57) are determined by an arbitrary choice of $a \in \mathbb{C} \setminus \mathbb{T}$: (59) provides b and Y self-reciprocal in $\mathbb{P}_{r-1}$ up to a factor, solving the first equation of (57), and finally $X = \psi_r - Y$ solves the second equation of (57).

On the other hand, given X, Y, let us see how many solutions a, b of (57) we can expect. If we suppose two different solutions a, b and a', b', (57) gives

$$
\begin{cases}
(\overline{a} - \overline{a}')Y = (b-b')Y^*, \\
(a-a')X = (b-b')X^*.
\end{cases}
\tag{60}
$$

Then, $Y^* \propto Y$, $X^* \propto X$ and, using again (57), we find that $Y = 0$ or $X \propto (z-1)Y$. In the first case $\psi_r = X$, which is not possible because $X^* \propto X$. In the second case Y divides $\psi_r = X + Y$, which implies that Y is a constant because $Y^* \propto Y$. Hence, $X(z) = z - 1$ and the polynomial modification must be of degree $r = 1$.

As a conclusion, given X, Y, the Eqs. (57) have at most one solution a, b when the degree r of the modification is greater than 1, or when it is equal to 1 but $X(z) \neq z - 1$. Thus, concerning the MOP related by (56) we have to distinguish two cases depending on the degree r of the modification.

- $r > 1$.

  In this case, given X, Y, the Schur parameters a_n, b_{n+r} must be constants of equal modulus for $n \geq 1$: the unique solution a, b of Eq. (57). Furthermore, for any choice of $a, b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$ the system (57) has a unique solution in X, Y, b obtained through (59) and the relation $X = \psi_r - Y$. In other words, the MOP related by (56) are those (φ_n) corresponding to a sequence of constant Schur parameters $(a, a, \ldots)$ and those (ψ_n) related to a sequence $(b_1, \ldots, b_r, b, b, \ldots)$ of Schur parameters, where $a, b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$ are arbitrary and b is given by (59). The MOP related by (56) are thus parametrized by $a, b_1, \ldots, b_r \in \mathbb{C} \setminus \mathbb{T}$.
- $r = 1$.

  If $X(z) \neq z - 1$ the conclusions are similar to those corresponding to $r > 1$. However, when $X(z) = z - 1$ the system (57) has infinitely many solutions no matter the choice of $Y = y \in \mathbb{C}$. To see this, let us write (57) explicitly,

$$
\begin{cases}
\overline{a}y = b\overline{y}, \\
a + b = y, \\
b_1 = y - 1.
\end{cases}
$$

  Since $b_1 \notin \mathbb{T}$ forces $y \neq 0$, the solutions a, b are all the symmetric points of the perpendicular bisector $\Pi(y)$ of the segment $[0, y]$. Therefore, the solutions corresponding to $X(z) = z - 1$ can be constructed in the following way: choose $b_1 \in \mathbb{C} \setminus \mathbb{T}$, which determines $y = b_1 + 1$; for each $n \geq 1$ choose $a_n \in \Pi(y) \setminus \mathbb{T}$ and $b_{n+1} \in \Pi(y)$ as its symmetric point with respect to the segment $[0, y]$. This procedure generates all the sequences of Schur parameters (a_n), (b_n) whose MOP (φ_n), (ψ_n) are related by

$$
\psi_{n+1}(z) = (z-1)\varphi_n(z) + y\varphi_n^*(z), \qquad y \in \mathbb{C}.
$$

Hence, the solutions with $X(z) = z - 1$ are parametrized by $b_1 \in \mathbb{C} \setminus \mathbb{T}$ and an infinite sequence $(a_1, a_2, \ldots)$ lying on $\Pi(1 + b_1) \setminus \mathbb{T}$.

On the other hand, the solutions with $X(z) \neq z - 1$ are parametrized by $b_1, a \in \mathbb{C} \setminus \mathbb{T}$ with $a \notin \Pi(1 + b_1)$, and the corresponding pair of sequences of Schur parameters is given by $(a, a, \ldots)$ and $(b_1, b, b, \ldots)$ with $b = a(\zeta + b_1)/(1 + \overline{b}_1\zeta)$. This yields all the MOP related by

$$\psi_{n+1}(z) = (z + x)\varphi_n(z) + y\varphi_n^*(z), \quad x, y \in \mathbb{C},\ x \neq -1.$$

Moreover, from this equality for $n = 0$ and (59) we find that the parameters x, y related to a choice of b_1 and a are

$$x = b_1 - y, \quad y = \frac{a(1 - |b_1|^2)}{(1 - \overline{a}) + \overline{b}_1(1 - a)}. \tag{61}$$

Concerning the possible values of the polynomials X and Y, we have to point out that Y must be proportional to a self-reciprocal polynomial in $\mathbb{P}_{r-1}$, as follows from (57). Indeed, (59) shows that $Y(z)$ is proportional to a kernel $K_{r-1}(z, \zeta)$ for some $\zeta \in \mathbb{T}$, thus it has exact degree $r - 1$ unless $Y = 0$. On the other hand, X is a monic polynomial of degree r which cannot be proportional to a self-reciprocal one unless $r = 1$ and $X(z) = z-1$, as follows from the reasoning in the paragraph after (60). This, together with the fact that $\psi_r = X + Y$ must be an orthogonal polynomial, are necessary conditions which must be fulfilled by the polynomial coefficients X, Y. Nevertheless, they are not sufficient conditions for the existence of MOP satisfying (56). To see this consider the case $r = 1$, where these conditions become

$$X(z) = z + x, \qquad Y(z) = y, \quad x \in \mathbb{C} \setminus \mathbb{T} \cup \{-1\}, \qquad x + y \in \mathbb{C} \setminus \mathbb{T}. \tag{62}$$

However, solving (61) for b_1 and a we get

$$b_1 = x + y, \qquad a = y\frac{1 + \overline{x}}{1 - |x|^2},$$

which shows that to get the alluded necessary and sufficient conditions for $r = 1$ we must add to (62) the following one

$$|y| \neq \left|\frac{1 - |x|^2}{1 + x}\right| \quad \text{if } |x| \neq 1.$$

Concerning the polynomial perturbation $L = P + P_*$ such that $u = vL$, we know that $A \propto XX^* - zYY^*$. Hence, when $X(z) = z - 1$ we find that $P(z) \propto z + (|y|^2/2 - 1)$. As for the rest of solutions, related to Schur parameters $(a, a, \ldots)$, $(b_1, \ldots, b_r, b, b, \ldots)$ with b given in (59), we only know that $\deg P \leq r$. The inequality $\deg P < r$ is characterized by any of the statements of the following equivalence, which are consequences of the previous results and the recurrence for (ψ_n):

$$\deg P < r \Leftrightarrow X(0) = 0 \Leftrightarrow Y(0) = b_r \Leftrightarrow b = b_r \Leftrightarrow$$
$$\Leftrightarrow b_r = a\frac{\psi_r(\zeta)}{\psi_r^*(\zeta)} \Leftrightarrow b_r = a\frac{\psi_{r-1}(\zeta)}{\psi_{r-1}^*(\zeta)}.$$

That is, among the values of $a, b_1, \ldots, b_r$ which parametrize the solutions with $X(z) \neq z - 1$, the inequality $\deg P < r$ holds for those ones with b_r determined by $a, b_1, \ldots, b_{r-1}$ through $b_r = a\psi_{r-1}(\zeta)/\psi_{r-1}^*(\zeta)$. The solutions with $\deg P < r$ correspond to $b_n = b$ for $n \geq r$, while the solutions with $\deg P = r$ are those ones with (b_n) given by $(b_1, \dots, b_r, b, b, \dots)$, $b_r \neq b$. Notice that each solution with $\deg P < r$ has a sequence (b_n) with the form $(b_1, \dots, b_s, b, b, \dots)$, $b_s \neq b$, for some $s < r$, and then $\deg P = s$ and one can find new polynomial coefficients $\hat{X} \in \mathbb{P}_s$, $\hat{Y} \in \mathbb{P}_{s-1}$ such that $\psi_{n+s} = \hat{X}\varphi_n + \hat{Y}\varphi_n^*$, $n \geq 0$. In any case, $b = a\psi_j(\zeta)/\psi_j^*(\zeta)$ for $j \geq \deg P$.

4.2. *Associated polynomials and polynomial modifications*

Given a sequence (ψ_n) of MOP with Schur parameters (b_n), the associated polynomials are those MOP (φ_n) with Schur parameters (a_n), $a_n = b_{n+1}$. Despite the similarity of their Schur parameters, the corresponding orthogonality functionals can be quite different. We will consider the following question concerning such functionals: when is the functional u of the associated polynomials (φ_n) a polynomial modification of the functional v related to the original MOP (ψ_n)? We will answer explicitly this question for a polynomial modification of degree 1.

We will find that the solutions are parametrized by the first Schur parameters a_1 and b_1 of (φ_n) and (ψ_n). The associated polynomials (φ_n) are obtained by a rotation of the MOP with constant Schur parameters $(a_1, a_1, \dots)$, where the rotation is determined by a_1 and b_1.

According to Theorem 2.13, this problem is equivalent to the existence of matrices $\mathcal{C}_n \in \mathbb{J}_1$ such that $\mathcal{C}_n\mathcal{B}_{n+1} = \mathcal{A}_n\widetilde{\mathcal{C}}_{n-1}$, $\mathcal{B}_{n+1} = \mathcal{A}_n$, $n \geq 1$, with $\mathcal{C}_0 \in \mathbb{J}_1^{\text{reg}}$ satisfying the initial condition $\mathcal{C}_0\Psi_1 = A\Phi_0$. Let us denote $P(z) = \alpha z + \beta$, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$. The recurrence for $\mathcal{C}_n$ can be written as

$$\begin{pmatrix} \alpha z + c_n & d_n \\ z\overline{d}_n & c_n z + \overline{\alpha} \end{pmatrix} \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} \alpha z + c_{n-1} & zd_{n-1} \\ \overline{d}_{n-1} & c_{n-1}z + \overline{\alpha} \end{pmatrix},$$

for some coefficients $c_n \in \mathbb{R}^*$, $d_n \in \mathbb{C}$. Splitting this matrix recurrence gives the equivalent system of equations

$$\begin{cases} c_n + \overline{a}_n d_n = c_{n-1} + a_n\overline{d}_{n-1}, \\ a_n c_n + d_n = \overline{\alpha} a_n, \\ \alpha a_n = a_n c_{n-1} + d_{n-1}. \end{cases} \tag{63}$$

Taking determinants in the matrix recurrence and setting $z = 0$, we find that $c_n = c_{n-1}$ for $n \geq 1$, so $c_n = c_0$ for $n \geq 0$. Therefore, (63) reads as

$$\begin{cases} \overline{a}_n d_n = a_n\overline{d}_{n-1}, \\ a_n(\overline{\alpha} - c_0) = d_n, \\ a_n(\alpha - c_0) = d_{n-1}, \end{cases} \tag{64}$$

although the first equation is a consequence of the others.

Assume that $\alpha = c_0$. Then, $d_n = 0$ for all n and the initial condition is $A = \alpha(z+1)(z+b_1)$, which is not possible because A is self-reciprocal while $|b_1| \neq 1$. Hence, $\alpha \neq c_0$ and the solution of (64) is

$$a_{n+1} = \lambda^n a_1, \qquad d_n = \lambda^n(\alpha - c_0)a_1, \quad \lambda = \frac{\overline{\alpha} - c_0}{\alpha - c_0}, \ n \geq 0.$$

Besides, the initial condition

$$\alpha z^2 + 2\beta z + \bar{\alpha} = (\alpha z + c_0)(z + b_1) + d_0(\overline{b}_1 z + 1)$$

yields the parameters of the polynomial perturbation,

$$\alpha = \overline{b}_1 c_0 + \overline{d}_0, \qquad \beta = \frac{1}{2}(\alpha b_1 + \overline{b}_1 d_0 + c_0) = \frac{c_0}{2}(1 - |b_1|^2) + \mathrm{Re}(\alpha b_1).$$

Taking into account that $d_0 = (\alpha - c_0)a_1$, we can express $\alpha, \beta, \lambda, d_0$, in terms of a_1, b_1, c_0,

$$\alpha = c_0 \frac{\overline{a}_1(b_1 - a_1) + (\overline{b}_1 - \overline{a}_1)}{1 - |a_1|^2},$$
$$\beta = c_0 \left\{ \frac{1}{2}(1 - |b_1|^2) + \frac{\mathrm{Re}[(\overline{a}_1(b_1 - a_1) + (\overline{b}_1 - \overline{a}_1))b_1]}{1 - |a_1|^2} \right\},$$
$$\lambda = \frac{(b_1 - 1) + a_1(\overline{b}_1 - 1)}{(\overline{b}_1 - 1) + \overline{a}_1(b_1 - 1)},$$
$$d_0 = c_0 a_1 \frac{(\overline{b}_1 - 1) + \overline{a}_1(b_1 - 1)}{1 - |a_1|^2}.$$

The fact that $\deg P = 1$ means that $\alpha \neq 0$. This only excludes the possibility $a_1 = b_1$, which gives $\lambda = 1$ and thus corresponds to the trivial case $a_n = b_n$ for all n, i.e., $u = v$.

Therefore, the arbitrariness in $c_0 \in \mathbb{R}^*$ is simply the freedom of the polynomial perturbation in a multiplicative real factor, and the solutions of the problem are parametrized by $a_1, b_1 \in \mathbb{C} \setminus \mathbb{T}$ with $a_1 \neq b_1$: the MOP (ψ_n) whose associated ones (φ_n) come from a polynomial perturbation of degree 1 of the orthogonality functional of (ψ_n) are those ones with Schur parameters $(b_1, a_1, a_1\lambda, a_1\lambda^2, \ldots)$, where $\lambda \in \mathbb{T}$ is the square of the phase of $(b_1 - 1) + a_1(\overline{b}_1 - 1)$. The associated polynomials (φ_n) have Schur parameters $(a_1, a_1\lambda, a_1\lambda^2, \ldots)$, so they are obtained by a rotation $\varphi_n(z) = \lambda^n \phi_n(\overline{\lambda} z)$ of the MOP (ϕ_n) with constant Schur parameters $(a_1, a_1, a_1, \ldots)$.

We can use α, β and b_1 as free parameters too. The initial condition can be expressed as

$$\begin{pmatrix} 1 & \overline{b}_1 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} \beta - \alpha b_1 \\ \overline{\alpha} \end{pmatrix},$$

with solutions

$$c_0 = 2\frac{\beta - \mathrm{Re}(\alpha b_1)}{1 - |b_1|^2}, \qquad d_0 = \frac{A(-b_1)}{1 - |b_1|^2}.$$

This gives

$$a_1 = \frac{d_0}{\alpha - c_0} = \frac{A(-b_1)}{\alpha(1 - |b_1|^2) - 2(\beta - \mathrm{Re}(\alpha b_1))},$$
$$\lambda = \frac{\overline{\alpha}(1 - |b_1|^2) - 2(\beta - \mathrm{Re}(\alpha b_1))}{\alpha(1 - |b_1|^2) - 2(\beta - \mathrm{Re}(\alpha b_1))},$$

providing a solution whenever $c_0 \neq 0, \alpha$ and $|a_1| \neq 1$, i.e.,

$$\beta \neq \mathrm{Re}(\alpha b_1), \qquad \frac{\alpha}{2}(1 - |b_1|^2) + \mathrm{Re}(\alpha b_1), \qquad \left| \frac{A(-b_1)}{\alpha(1 - |b_1|^2) - 2(\beta - \mathrm{Re}(\alpha b_1))} \right| \neq 1.$$

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