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Weighted graphs defining facets: A connection between stable set and linear ordering polytopes[☆]

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ABSTRACT

A graph is α-critical if its stability number increases whenever an edge is removed from its edge set. The class of α-critical graphs has several nice structural properties, most of them related to their defect which is the number of vertices minus two times the stability number. In particular, a remarkable result of Lovász [L. Lovász, Some finite basis theorems on graph theory, in: A. Hajnal, V.T. Sós (Eds.), Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, in: Colloquia Mathematica Societatis János Bolyai, vol. 18, North-Holland, Amsterdam, 1978, pp. 717–729] is the finite basis theorem for α-critical graphs of a fixed defect. The class of α-critical graphs is also of interest for at least two topics of polyhedral studies. First, Chvátal [V. Chvátal, On certain polytopes associated with graphs, Journal of Combinatorial Theory Ser. B 18 (1975) 138–154] shows that each α-critical graph induces a rank inequality which is facet-defining for its stable set polytope. Investigating a weighted generalization, Lipták and Lovász [L. Lipták, L. Lovász, Facets with fixed defect of the stable set polytope, Mathematical Programming 88 (Ser. A) (2000) 33–44; L. Lipták, L. Lovász, Critical facets of the stable set polytope, Combinatorica 21 (2001) 61–88] introduce critical facet-graphs (which again produce facet-defining inequalities for their stable set polytopes) and they establish a finite basis theorem. Second, Koppen [M. Koppen, Random utility representation of binary choice probabilities: Critical graphs yielding critical necessary conditions, Journal of Mathematical Psychology 39 (1995) 21–39] describes a construction that delivers from any α-critical graph a facet-defining inequality for the linear ordering polytope. Doignon et al. [J.-P. Doignon, S. Fiorini, G. Joret, Facets of the linear ordering polytope: A unification for the fence family through weighted graphs, Journal of Mathematical Psychology 50 (3) (2006) 251–262] handle the weighted case and thus define facet-defining graphs. Here we investigate relationships between the two weighted generalizations of α-critical graphs. We show that facet-defining graphs (for the linear ordering polytope) are obtainable from 1-critical facet-graphs (linked with stable set polytopes). We then use this connection to derive various results on facet-defining graphs, the most prominent one being derived from Lipták and Lovász's finite basis theorem for critical facet-graphs. At the end of the paper we offer an alternative proof of Lovász's finite basis theorem for α-critical graphs.

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1. Introduction

A (finite, simple, undirected) graph G is α-critical if its stability number $\alpha(G)$ (defined as the maximum cardinality of a subset of mutually nonadjacent vertices) increases whenever an edge is removed from its edge set. These graphs have several interesting structural properties, most of which being related to their *defect* $\delta = |G| - 2\alpha(G)$. An important result of Lovász [12] shows for instance that for every fixed defect $\delta \geq 1$, there exists a finite collection of graphs from which every connected α-critical graph with defect δ can be derived using a certain edge subdivision operation.

One of the interests of α-critical graphs lies in their connection with facets of some polytopes arising in combinatorial optimization: Chvátal [2] and Koppen [8] showed how to obtain facets of respectively the stable set and linear ordering polytopes from (connected) α-critical graphs. This link was investigated further in the recent years and led to the introduction of two generalizations of α-critical graphs, one called *critical facet-graphs* [10,11,15] and the other *facet-defining graphs* [1,4]. Graphs in both families are vertex-weighted, and give rise to facets of the stable set and linear ordering polytopes, respectively.

Although examples show that the classes of critical facet-graphs and facet-defining graphs are (inclusion-wise) incomparable, some of the known results on their respective structures are intriguingly similar (see e.g. [11,4]). The purpose of this paper is to explain precisely how critical facet-graphs and facet-defining graphs are related to each other.

In a recent contribution, Fiorini [5] already showed that a subclass of the former, which we call *1-critical facet-graphs*, are facet-defining graphs. Here we prove a converse result: Every facet-defining graph can be obtained from some 1-critical facet-graph using a simple contraction operation. This connection conveys a great deal of information on facet-defining graphs. In particular, the main result of Lipták and Lovász [10], an extension of Lovász's finite basis theorem to the class of critical facet-graphs, translates naturally to facet-defining graphs.

The paper is organized as follows. We first give the necessary definitions and preliminaries in Section 2. We then present in Section 3 our main result which relates facet-defining graphs to 1-critical facet-graphs, and use it to derive new results on facet-defining graphs from the theory of critical facet-graphs. Finally, in Section 4, we go back to α-critical graphs and offer an alternative proof for the finite basis theorem of Lovász. The latter theorem is not only at the heart of the theory of α-critical graphs, but also a key ingredient in Lipták and Lovász's proof for the extension of the result to critical facet-graphs.

2. The stable set and linear ordering polytopes

In this section, we define the stable set and linear ordering polytopes, the two classes of weighted graphs under consideration, and the corresponding facets. We also state the main known results on these two classes of weighted graphs.

2.1. The stable set polytope and critical facet-graphs

The *stable set polytope* $\mathrm{STAB}(G)$ of a graph G is defined as the convex hull of the incidence vectors of all stable sets of G. In other words, letting V and E respectively denote the vertex and edge sets of G, the stable set polytope of G is the integer hull of the polytope

$$P := \{x \in \mathbb{R}^V \mid x_i + x_j \leq 1 \ \forall ij \in E, 0 \leq x_i \leq 1 \ \forall i \in V\},$$

that is,

$$\mathrm{STAB}(G) = \mathrm{conv}(P \cap \mathbb{Z}^V).$$

A central question in polyhedral combinatorics is to determine the facets of $\mathrm{STAB}(G)$. While this is believed to be impossible in general for complexity theoretic reasons, see, e.g., Papadimitriou and Yannakakis [14], there exist numerous published works focusing on special classes of graphs or special families of facets. A large number of these papers are concerned with facets defined by *rank inequalities*, that is, inequalities of the form

$$\sum_{v \in S} x_v \leq \alpha(G[S])$$

for some $S \subseteq V$. In particular, one might ask when the rank inequality obtained for $S = V$, i.e., $\sum_{v \in V} x_v \leq \alpha(G)$, defines a facet of $\mathrm{STAB}(G)$. In 1975, Chvátal [2] showed that this is the case whenever G is a connected α-critical graph, where G is said to be *α-critical* if $\alpha(G - e) > \alpha(G)$ for every $e \in E(G)$. Thus α-critical graphs are of particular relevance to the polyhedral theory of the stable set polytopes. The literature on these graphs is quite rich, most contributions dating back to the 60's and 70's (see [13] for a survey). Two concepts turn out to be of key importance for the study of α-critical graphs: an invariant called the 'defect' and an operation known as taking 'odd subdivisions'. The *defect* of a graph G is defined as $\delta = |G| - 2\alpha(G)$. This invariant is always nonnegative when G is α-critical. An *odd subdivision* of a graph G is any graph that can be obtained from G by replacing edges with odd-length paths. Any odd subdivision of a connected α-critical graph G with at least three vertices is again α-critical and has the same defect (see, e.g., [13]). A central result, due to Lovász [12], shows essentially that α-critical graphs are naturally classified by their defect. It is known as the *finite basis theorem for α-critical graphs*.

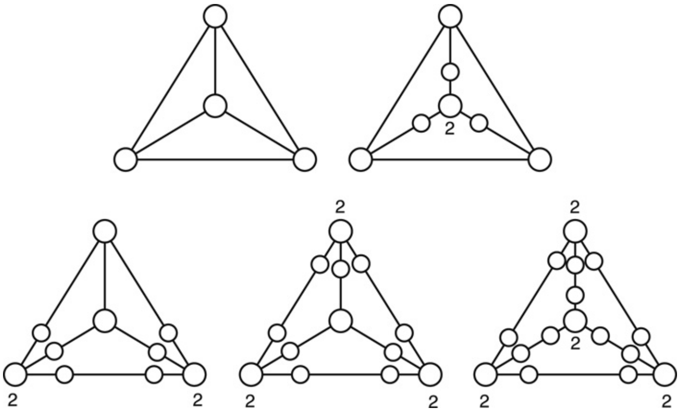

Fig. 1. A basis for critical facet-graphs with defect 2 (only weights different from 1 are indicated).

Theorem 1 (*Lovász [12]*). *For every integer $\delta \geq 1$, there exists a finite collection of graphs such that every connected α-critical graph with defect δ is an odd subdivision of a graph in the collection.*

Let $G \preceq H$ whenever H is an odd subdivision of G. This defines a partial order on graphs. Consider the set of all connected α-critical graphs partially ordered by $\preceq$. The graphs with fixed defect form a partition of this poset into upper monotone sets. Then the finite basis theorem amounts to say that each of these upper monotone sets contains a finite number of minimal elements.

Let G be any graph. Now consider a weight function a on the vertices of G, that is, a function $a : V \to \mathbb{Z}_+$. The pair (G, a) is referred to as a *(vertex)-weighted graph*. From now on, in order to avoid some trivialities, we will always assume that weighted graphs have at least three vertices and $a(v) > 0$ for all vertices v. Letting $\alpha(G, a)$ denote the maximum weight of a stable set in G, the weighted graph (G, a) is said to be *critical* if $\alpha(G - e, a) > \alpha(G, a)$ for all edges e. Moreover, (G, a) is said to be a *facet-graph* if the inequality

$$\sum_{v \in V} a(v)\, x_v \leq \alpha(G, a)$$

defines a facet of STAB(G) and G is connected (recall that we also assume that G contains at least three vertices and the weights are positive). The critical facet-graphs are the natural weighted counterpart of α-critical graphs. Many results from the theory of α-critical graphs were extended to critical facet-graphs, see the works of Sewell [15] and Lipták and Lovász [10,11].

The *defect* of a weighted graph (G, a) is defined as $\delta = a(V(G)) - 2\alpha(G, a)$. As was the case for α-critical graphs, this invariant turned out to be crucial for studying critical facet-graphs. The following result reveals much of the structural information conveyed by the defect of a critical facet-graph.

Theorem 2 (*Lipták and Lovász [11]*). *If (G, a) is a critical facet-graph with defect δ, then $\deg(v) \leq a(v) + \delta \leq 2\delta$ for every $v \in V(G)$, and $\deg(v) \leq 2\delta - 1$ when $\delta > 1$.*

Let (G, a) be a weighted graph (G, a) and e be one of its edges. The *strength* of the edge e is defined as $\alpha(G-e, a)-\alpha(G, a)$. Notice that if (G, a) is a critical facet-graph then the strength of any of its edges is positive. Consider now the following operation on (G, a): select some of its edges, and replace each with a path of length 3 where the two new vertices have weight equal to the strength of the edge. The resulting weighted graph is referred to as an *elementary odd subdivision* of (G, a). We say that a weighted graph is an *odd subdivision* of (G, a) if it is obtained from (G, a) by applying the operation finitely many times.

Lemma 1 (*Wolsey [17]*). *Every elementary odd subdivision of a critical facet-graph is again a critical facet-graph with the same defect. The three new edges have the same strength as the edge they replace.*

The following result generalizes Lovász's finite basis theorem for α-critical graphs (Theorem 1).

Theorem 3 (*Lipták and Lovász [10]*). *For every integer $\delta \geq 1$, there exists a finite collection of critical facet-graphs such that every critical facet-graph with defect δ is an odd subdivision of a graph in the collection.*

Such a collection of graphs is (explicitly) known for $\delta = 1, 2$ only. Using Theorem 2, it is not difficult to check that critical facet-graphs with defect 1 are the odd cycles with the all-one weighting, that is, the odd subdivisions of $(K_3, \mathbb{1})$. For $\delta = 2$, Sewell [15] proved the following.

Theorem 4 (*Sewell [15]*). *Every critical facet-graph with defect 2 is an odd subdivision of one of the graphs depicted in Fig. 1.*

2.2. *The linear ordering polytope and facet-defining graphs*

Given a complete directed graph with nonnegative weights on its arcs, the *linear ordering problem* asks to layout the vertices of the graph on an oriented line in such a way that the total weight of the arcs going from left to right is maximized. More precisely, solving the linear ordering problem consists in finding a strict linear ordering (that is, a spanning acyclic subtournament) of maximum total weight in a given weighted complete directed graph. The 0/1-polytope naturally associated to this problem is known as the linear ordering polytope. Let N and A respectively denote the node and arc set of the complete directed graph given as input and let $n = |N|$. Then the *linear ordering polytope* P^N_{LO} (we sometimes denote it simply by P^n_{LO}) is the integer hull of the polytope

$$Q = \{x \in \mathbb{R}^A \mid x_{ij} + x_{jk} + x_{ki} \leq 2 \;\forall \{ij, jk, ki\} \subseteq A,\; x_{ij} + x_{ji} = 1 \;\forall ij \in A, x_{ij} \leq 1 \;\forall ij \in A\}.$$

Equivalently, the linear ordering polytope is the convex hull of the incidence vectors of all strict linear orderings contained in $D = (N, A)$. The literature dealing with the polyhedral structure of the linear ordering polytope is quite abundant (with an approximate number of 50 references), although not as abundant as the literature on the stable set polytope. A prominent class of facets for this former polytope are the so-called fence inequalities which where independently discovered by Grötschel, Jünger and Reinelt [6] and Cohen and Falmagne [3]. They were generalized in two different ways by Leung and Lee [9] (also Suck [16]) and Koppen [8]. Then the authors of the present paper proposed a further generalization unifying the two generalizations mentioned above, following an idea of Christophe, Doignon and Fiorini [1]. The resulting class of inequalities is known as the *graphical inequalities*. We give a definition of these inequalities in the next paragraph. To avoid any confusion, let us emphasize that, while arc-weighted directed graphs briefly appeared in the definition of the linear ordering problem, all weighted graphs considered in the sequel will be vertex-weighted undirected graphs (as in Section 2.1).

The *worth* of a subset S of vertices of a weighted graph (G, a) is defined as $a(S) - \|S\|$, where $\|S\| = |E(G[S])|$ denotes the number of edges of G with both ends in the set S. The maximum worth of a set of vertices in (G, a) is denoted by $\beta(G, a)$. In other words, we let

$$\beta(G, a) := \max_{S \subseteq V(G)} \{a(S) - \|S\|\}.$$

Notice $\beta(G, a) \geq \alpha(G, a)$ because $\|S\| = 0$ whenever S is a stable set. As precedingly, let V and E respectively denote the vertex and edge set of G. Suppose that N contains V and, furthermore, a set V' disjoint from V and of the same cardinality. Let $v \mapsto v'$ denote any bijection from V onto V'. The *graphical inequality* defined by (G, a) then reads

$$\sum_{v \in V} a(v)\, x_{vv'} - \sum_{vw \in E} (x_{vw'} + x_{wv'}) \leq \beta(G, a). \tag{1}$$

A weighted graph (G, a) is a *facet-defining graph* if the corresponding graphical inequality defines a facet of the linear ordering polytope (as stated above, we also assume $|V(G)| \geq 3$ and $a(v) > 0$ for all $v \in V(G)$). Suppose for a moment that $a(v) = 1$ for all vertices v. That is, a is the all-one function $\mathbb{1}$. Koppen [8] showed that in this case (G, a) is facet-defining precisely when G is a connected α-critical graph distinct from K_2. This result is reminiscent of the aforementioned result of Chvátal [2] on the stable set polytope. This is not a coincidence, as we now explain.

Theorem 5 (*Fiorini [5], Corollary 16*). *Let (G, a) be a critical facet-graph with $G = (V, E)$. As above, assume that V is contained in N and $v \mapsto v'$ is a bijection between V and a subset V' of N which is disjoint from V. Finally, for an edge $e \in E$, let $s(e)$ denote its strength. Then there exists a unique integer γ such that the inequality*

$$\sum_{v \in V} a(v)\, x_{vv'} - \sum_{vw \in E} s(vw)\, (x_{vw'} + x_{wv'}) \leq \gamma \tag{2}$$

is facet-defining for the linear ordering polytope.

A critical facet-graph (G, a) is said to be *k-critical* if the strength of any of its edges is at most k. Suppose that (G, a) is a 1-critical facet-graph and consider inequality (2). Because the strength of every edge of (G, a) equals 1, the left-hand side of Eq. (2) equals the left-hand side of Eq. (1), that is, the graphical inequality associated to (G, a). It follows that $\gamma = \beta(G, a)$ and thus Eq. (2) is a facet-defining graphical inequality and (G, a) is a facet-defining graph. This shows that 1-critical facet-graphs are always facet-defining graphs. In the next section we prove that, conversely, any facet-defining graph (G, a) has a 'unit odd subdivision' which is a 1-critical facet-graph.

3. The connection and some of its consequences

In this section we state and prove our main result which relates facet-defining graphs to 1-critical facet-graphs. We then derive new results on facet-defining graphs from Theorems 2 and 3. Thus, in particular, we derive a finite basis theorem for facet-defining graphs. At the end of the section, we provide the basis for subdefects 1 and 2.

Let (G, a) be an arbitrary weighted graph. The *subdefect* (G, a) is defined as $\lambda = a(V(G)) - 2\beta(G, a)$. Notice that the subdefect of a weighted graph never exceeds its defect (hence the name). A *unit odd subdivision* of (G, a) is any graph obtained from (G, a) by replacing edges with odd-length paths, where the new vertices have weight 1. Conversely, a graph (G', a') is said to be a *shrinking* of (G, a) if (G, a) is a unit odd subdivision of (G', a').

The following properties of facet-defining graphs were proved in [1] (see also [4]).

Lemma 2 (*Christophe, Doignon and Fiorini [1]*). *Let (G, a) be a facet-defining graph. Then*

(A) *the only solution to the system*

$$\left\{ \sum_{v \in T} y_v + \sum_{e \in E(T)} y_e = \beta(G, a) \mid T \subseteq V(G),\ T \text{ maximum worth set} \right\}$$

is the trivial solution: $y_v = a(v)$ for all $v \in V(G)$, $y_e = -1$ for all $e \in E(G)$;
(B) *for every $uv \in E(G)$ and $X \subseteq \{u, v\}$, there exists a maximum worth set $T \subseteq V(G)$ with $T \cap \{u, v\} = X$;*
(C) $\deg(v) \geq 2$ *for every $v \in V(G)$;*
(D) *any unit odd subdivision of (G, a) is also facet-defining with the same subdefect, and*
(E) *any shrinking of (G, a) is also facet-defining with the same subdefect.*

By Lemma 1, the notions of odd subdivision and unit odd subdivision coincide for 1-critical facet-graphs. The next lemma shows that this is also the case for the defect and subdefect.

Lemma 3. *If (G, a) is a 1-critical facet-graph then its defect equals its subdefect, and hence $\alpha(G, a) = \beta(G, a)$.*

Proof. Consider any (unit) odd subdivision (G', a') of (G, a) where no edge of G remains. Because every edge in (G', a') is incident to at least one new vertex, which all have weight 1, we have $\alpha(G', a') = \beta(G', a')$. Indeed, any set $S \subseteq V(G')$ can be turned in a stable set whose worth is at least that of S by iteratively removing any vertex of weight 1 adjacent to some other vertex in S. By Lemma 1, (G', a') is a critical facet-graph with the same defect as (G, a). Now Theorem 5 implies that (G', a') is a facet-defining graph. Then, by Lemma 2(E), (G, a) is also a facet-defining graph and has the same subdefect as (G', a'). Since the defect of (G', a') equals its subdefect, we deduce that the same holds for (G, a). The lemma follows. □

We now turn to the main contribution of this paper: a precise connection between facet-defining graphs and critical facet-graphs.

Proposition 1. *A weighted graph is facet-defining if and only if it is a shrinking of a 1-critical facet-graph. Moreover, the subdefect of the former equals the defect of the latter.*

We remark that there are facet-defining graphs which are not facet-graphs, for instance the last two graphs in Fig. 2.

Proof of Proposition 1. Assume first that a graph (G', a') is a shrinking of a 1-critical facet-graph (G, a). Then (G, a) is a facet-defining graph (Theorem 5), and so is (G', a') (Lemma 2(E)). Moreover, (G, a) has equal subdefect and defect (Lemma 3). Also, (G', a') and (G, a) have same subdefect (Lemma 2(E)). Hence, the subdefect of (G', a') equals the defect of (G, a).

Assume now that (G', a') is a facet-defining graph and let (G, a) be the unit odd subdivision of (G', a') obtained by replacing each edge with a path of length 3 and giving a weight of 1 to the new vertices. Following Lemma 2(D), (G, a) is also facet-defining. Moreover, as in the proof of Lemma 3 (the roles of (G, a) and (G', a') are now interchanged), we have $\beta(G, a) = \alpha(G, a)$. Observe, in passing, that the same holds for all spanning subgraphs of (G, a). Now consider an edge e of (G, a). Then, by Lemma 2(B), we have $\beta(G - e, a) = \beta(G, a) + 1$. On the other hand, we also have $\alpha(G - e, a) = \beta(G - e, a)$ as $G - e$ is a spanning subgraph of (G, a). Hence the strength of every edge of (G, a) equals 1. We now show that (G, a) is also a facet-graph.

Arguing by contradiction, assume that (G, a) is not a facet-graph. This means that (G, a) does not contain $|G|$ linearly independent maximum weight stable sets (since the stable set polytope is full-dimensional). It follows then that the system

$$\left\{ \sum_{v \in S} y_v = \alpha(G, a) \mid S \subseteq V(G),\ S \text{ maximum weight stable set} \right\}$$

has a solution $\tilde{y}$ distinct from the solution $y_v = a(v)$ for all $v \in V(G)$.

For each edge e of G, pick a vertex t_e of weight 1 incident to e. Extend now $\tilde{y}$ to a vector in $\mathbb{R}^{V(G) \cup E(G)}$ by letting $\tilde{y}_e = -\tilde{y}_{t_e}$ for every edge e. Consider any maximum worth set T of (G, a) and let $S := T \setminus \{t_{e_1}, t_{e_2}, \ldots, t_{e_k}\}$, where $E(T) = \{e_1, e_2, \ldots, e_k\}$. Since S is a maximum weight stable set, we obtain

$$\sum_{v \in T} \tilde{y}_v + \sum_{e \in E(T)} \tilde{y}_e = \sum_{v \in T} \tilde{y}_v - \sum_{e \in E(T)} \tilde{y}_{t_e} = \sum_{v \in S} \tilde{y}_v = \alpha(G, a) = \beta(G, a).$$

Hence, this extended vector $\tilde{y}$ is a nontrivial solution of the system defined in Lemma 2(A), contradicting the fact that (G, a) is facet-defining. Therefore, (G, a) is a 1-critical facet-graph. This concludes the proof. □

Several structural properties of facet-defining graphs derive from Proposition 1 combined with known results on critical facet-graphs, as we know illustrate. We first note a direct corollary of Theorem 2:

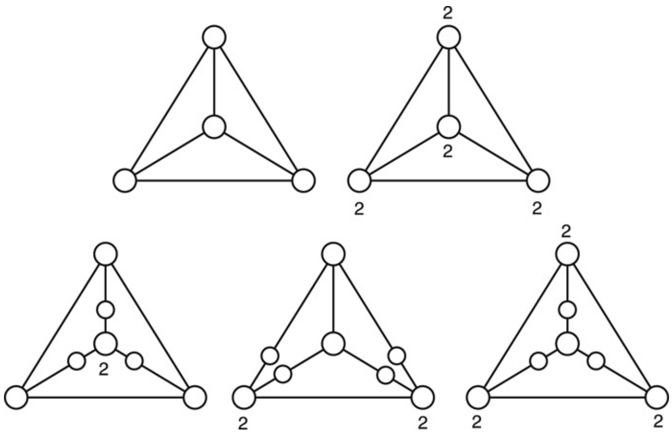

Fig. 2. A basis for facet-defining graphs with subdefect 2 (only weights different from 1 are indicated).

Corollary 1. *If (G, a) is a facet-defining graph with subdefect λ, then $\deg(v) \leq a(v) + \lambda \leq 2\lambda$ for every $v \in V(G)$, and $\deg(v) \leq 2\lambda - 1$ when $\lambda > 1$.*

One of the main interests of Proposition 1 is that the finite basis theorem for critical facet-graphs (Theorem 3) extends naturally to facet-defining graphs.

Corollary 2. *For every integer $\lambda \geq 1$, there exists a finite collection of facet-defining graphs such that every facet-defining graph (G, a) with subdefect λ is a unit odd subdivision of a graph in the collection.*

Before turning to the proof of Corollary 2, we need the following result:

Lemma 4. *In a facet-defining graph a cutset cannot induce K_2.*

Proof. Let (G, a) be a facet-defining graph. Arguing by contradiction, assume that $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v, w\}$ and $vw \in E(G)$. Let $\beta := \beta(G, a)$ and $V_i := V(G_i)$, $E_i := E(G_i)$ for $i = 1, 2$.

The maximum worth sets of (G, a) can be classified in 4 categories, according to their intersection with $\{v, w\}$ (which can be $\varnothing$, $\{v\}$, $\{w\}$ or $\{v, w\}$). It follows from Lemma 2(B) that (G, a) has at least one maximum worth set in each category. For $X \subseteq \{u, v\}$ and $i = 1, 2$, we define c^i_X as

$$c^i_X := (a(T \cap V_i) - \|T \cap V_i\|) - (a(X) - \|X\|)\,,$$

where T is any maximum worth set of (G, a) with $T \cap \{u, w\} = X$. Notice that, since $\{v, w\}$ is a cutset of G, the value of c^i_X is independent of the particular choice of T.

Pick any $\gamma^1 \in \mathbb{R}$ distinct from 1 and let, using the fact that $c^2_\varnothing \neq 0$,

$$\gamma^2 := \frac{\beta - \gamma^1 c^1_\varnothing}{c^2_\varnothing}.$$

Define a vector $y \in \mathbb{R}^{V(G) \cup E(G)}$ as follows:

$$y_u := \gamma^i \cdot a(u) \quad \text{for } i = 1, 2 \text{ and } u \in V_i \setminus \{v, w\};$$
$$y_e := \gamma^i \cdot (-1) \quad \text{for } i = 1, 2 \text{ and } e \in E_i \setminus \{vw\};$$
$$y_u := \beta - \gamma^1 c^1_{\{u\}} - \gamma^2 c^2_{\{u\}} \quad \text{for } u \in \{v, w\};$$
$$y_{vw} := \beta - \gamma^1 c^1_{\{v,w\}} - \gamma^2 c^2_{\{v,w\}} - y_v - y_w.$$

This vector y is a nontrivial solution to the system of Lemma 2(A), a contradiction. □

Proof of Corollary 2. In virtue of Lemma 2(C), every vertex of a facet-defining graph has degree at least 2. This is in particular true for 1-critical facet-graphs. Now consider some (sub)defect $\lambda \geq 1$. By Theorem 3, the number of vertices with degree at least 3 in a 1-critical facet-graph with defect λ is bounded from above by some constant c_λ that depends only on λ.

We call an edge *remote* if both of its ends have degree 2. Denote by $\mathcal{B}_\lambda$ the set of facet-defining graphs with subdefect λ having no remote edge. Every facet-defining graph (G, a) with subdefect λ is a unit odd subdivision of some graph in $\mathcal{B}_\lambda$, as easily proved by induction on $|G|$: either $(G, a) \in \mathcal{B}_\lambda$ or (G, a) has a remote edge uv. In the latter case, we find an induced path $u'uvv'$ in G, as otherwise there would be a cutset inducing K_2, which Lemma 4 forbids. Now, by 'shrinking' this path (i.e. removing u, v and adding the edge $u'v'$) and using Lemma 2(E), we are done by induction. Hence, $\mathcal{B}_\lambda$ is a basis for facet-defining graphs with subdefect λ.

We know from Proposition 1 that any graph $(G, a) \in \mathcal{B}_\lambda$ is a shrinking of a 1-critical facet-graph, and thus that the number of vertices with degree at least 3 in (G, a) is bounded by c_λ. Since (G, a) has no remote edge, we deduce $|G| \le c_\lambda + 2\lambda \binom{c_\lambda}{2}$ (cf. Corollary 1), and that $\mathcal{B}_\lambda$ is finite. □

Similarly as for critical facet-graphs, Corollary 1 implies that facet-defining graphs with subdefect 1 are the odd cycles with unit weights. We note that Theorem 4 shows in particular that every critical facet-graph with defect 2 is 1-critical. Hence, we obtain the following corollary.

Corollary 3. *Every facet-defining graph with subdefect* 2 *is a unit odd subdivision of a graph depicted in Fig. 2.*

4. Finite basis for α-critical graphs

As we have seen, the finite basis result for facet-defining graphs (Corollary 2) is a consequence of the corresponding theorem for critical facet-graphs, Theorem 3, which was proved by Lipták and Lovász [10]. The main step of their proof is a lemma which says roughly that every critical facet-graph is the image of an α-critical graph with the same defect under a particular well-behaved homomorphism. The result is then derived from Lovász's finite basis theorem for α-critical graphs (Theorem 1). Hence, the latter theorem is not only important for α-critical graphs, it is also a key result for critical facet-graphs and facet-defining graphs. The purpose of this section is to present an alternative proof for this theorem, restated as follows:

Theorem 6 (*Lovász [12]*). *For every $\delta \ge 1$, there exists a constant c_δ such that every connected α-critical graph with defect δ has at most c_δ vertices with degree at least* 3.

This version implies the one given in Theorem 1. Indeed, every connected α-critical graph which is minimal for the partial order $\preceq$ associated to the odd subdivision operation does not have two adjacent vertices with degree 2. Moreover, it is easily seen that, in a graph G, if two vertices v and w are not adjacent and have exactly the same neighbors, then any edge e incident to v or w is such that $\alpha(G-e) = \alpha(G)$. Hence, there are at most $\binom{r}{2}$ vertices with degree 2 in a minimal connected α-critical graph, where r is the number of vertices with degree at least 3.

The outline of our proof is as follows. We first relate the defect of an α-critical graph G to the maximum order of an acyclic tournament in a collection of directed graphs associated to G. We then use this relationship to transform the problem into a Ramsey-type problem on digraphs, which in turn follows from standard results in Ramsey theory. Let us emphasize that, while this gives a shorter and perhaps simpler proof of the existence of c_δ, the value for c_δ that is implied by our proof is much larger than the one proved in [12].

In this section, by a maximum set of a graph G we mean a maximum worth set of $(G, \mathbb{1})$. A main ingredient in our proof of Theorem 6 is the following simple lemma on sequences of maximum worth sets. Interestingly, this lemma was originally introduced in a more general form in [4, Lemma 16], as a tool to study the subdefect of facet-defining graphs.

Lemma 5 (*Doignon et al. [4]*). *Let G be an α-critical graph with defect δ and $T_1, \ldots, T_k$ be a sequence of maximum worth sets (repetitions are allowed) such that for every vertex u of G there exist indices $i, j \in \{1, \ldots, k\}$ with $u \in T_i$ and $u \notin T_j$. Then*

$$\delta \ge \sum_{i=1}^{k} \|T_i\| - \sum_{j=3}^{k} \|X_j\|,$$

where $X_j := \big((T_1 \cup \cdots \cup T_{j-1}) \cap T_j\big) \cup \big((T_1 \cap \cdots \cap T_{j-1}) \setminus T_j\big)$.

Proof of Theorem 6. Let G be a connected α-critical graph with defect δ. We want to show that the number of vertices with degree at least 3 is bounded from above by some constant c_δ depending only on δ. To this aim, we may assume without loss of generality that G has maximum degree exactly 3. Indeed, nothing has to be proved if G has no vertex with degree at least 3, and if $v \in V(G)$ has degree at least 4, then we can simultaneously decrease the number of vertices of degree more than 3 and increase the number of vertices with degree at least 3 by *splitting* v: partition the neighbors of v into two sets N_1, N_2, each of cardinality at least 2, remove v, add three new vertices v_1, v_2, v', and link v_i to v' and the vertices of N_i, for $i = 1, 2$. It is easily seen that this operation keeps a graph α-critical and does not change the defect (see, e.g., [13] for a proof).

Denote by $v_1, \ldots, v_p$ the vertices of G with degree 3. We assume that no two of them are adjacent, this can always be achieved by taking an appropriate odd subdivision of G. Denote also by $e_{i,1}, e_{i,2}, e_{i,3}$ the three edges incident to v_i, and let $T_{i,j}$ denote any maximum stable set of $G - e_{i,j}$. Notice that $T_{i,j}$ is a maximum worth set of G with $E(T_{i,j}) = \{e_{i,j}\}$.

We define a digraph D_G based on G and the $T_{i,j}$'s. Its vertex set is the set of edges of G which are incident to some degree-3 vertex, i.e.,

$$V(D_G) = \{e_{i,j} \mid 1 \le i \le p \text{ and } 1 \le j \le 3\},$$

and for every distinct $i, k \in \{1, \ldots, p\}$ and $j \in \{1, 2, 3\}$, we put an arc from $e_{i,j}$ to $e_{k,\ell}$ for all $\ell \in \{1, 2, 3\}$ whenever

$$\text{either} \quad (v_k \in T_{i,j+1} \text{ and } v_k \in T_{i,j+2}) \quad \text{or} \quad (v_k \notin T_{i,j+1} \text{ and } v_k \notin T_{i,j+2}),$$

where indices are taken mod 3. Moreover, we color the arc $(e_{i,j}, e_{k,\ell})$ red in the first case, blue in the second.

An acyclic tournament J in D_G is *admissible* if J contains at most one of the three vertices $e_{i,1}, e_{i,2}, e_{i,3}$, for $1 \le i \le p$. In addition, J is said to be *red* (resp. *blue*) if all its arcs are colored red (resp. blue). Our main tool is the following observation:

Claim 1. *If J is a red or blue admissible acyclic tournament in D_G, then $|J| \le \delta$.*

Proof. By renaming the indices if necessary, we may assume $V(J) = \{e_{i,1} \mid 1 \le i \le t\}$ and $A(J) = \{(e_{i,1}, e_{k,1}) \mid 1 \le i < k \le t\}$. Let $\{w_1, \dots, w_\ell\} := V(G) \setminus \{v_1, \dots, v_t\}$ and, for $1 \le i \le \ell$, let S_i (resp. S'_i) be any maximum stable set of G with $w_i \in S_i$ (resp. $w_i \notin S'_i$). Now consider the following sequence of maximum worth sets of G:

$$T_{1,2}, T_{1,3} - v_1, T_{2,2}, T_{2,3} - v_2, \dots, T_{t,2}, T_{t,3} - v_t, S_1, S'_1, \dots, S_\ell, S'_\ell.$$

For the sake of clarity, we will commit a slight abuse of notation and denote by T_i the i-th set of the above sequence of $k := 2t + 2\ell$ sets. Notice that, by construction, our sequence of maximum worth sets satisfies the assumption of Lemma 5. Also, if $uv \in E(T_i)$ then $u, v \notin T_{i+1}$, for $1 \le i < k$. Defining X_j as in Lemma 5, this implies that, for $j \ge 3$, if we have $uv \in E(X_j)$, then we also have $uv \in E(T_j)$. Hence, $E(X_j) \subseteq E(T_j)$.

Using Lemma 5, we obtain:

$$\delta \ge \|T_1\| + \|T_2\| + \sum_{j=3}^{k} (\|T_j\| - \|X_j\|) = 1 + \sum_{j=3}^{k} |E(T_j) \setminus E(X_j)|.$$

Each term in the last sum is nonnegative. We now prove that at least $t - 1$ of them are positive, which clearly implies the claim. Pick some $i \in \{2, \dots, t\}$ and denote by x the end of the edge $e_{i,2}$ that is distinct from v_i. If J is red, then by the definition of D_G we have $v_i \in T_j$ for $1 \le j \le 2i-2$. Since T_{2i-1} (which equals $T_{i,2}$) is the only set in our sequence of maximum worth sets that contains both ends of $e_{i,2}$, we deduce $x \notin T_j$ for $1 \le j \le 2i-2$. This shows $x \notin X_{2i-1}$, and hence $e_{i,2} \in E(T_{2i-1}) \setminus E(X_{2i-1})$. Similarly, if J is blue then it follows from the definition of D_G that $v_i \notin T_j$ for $1 \le j \le 2i - 2$, which implies $v_i \notin X_{2i-1}$, and again $e_{i,2} \in E(T_{2i-1}) \setminus E(X_{2i-1})$. □

By the above claim, to prove Theorem 6 it is sufficient to show that if G has many degree-3 vertices, then there is a large monochromatic admissible acyclic tournament in D_G. As D_G is "almost" a complete digraph, this sounds like a Ramsey-type property, and indeed we will reduce it to Ramsey's theorem. To this aim, we proceed with three claims. The first one is an easy consequence of Ramsey's theorem (see e.g. [7] for a proof).

Claim 2. *For every $k \ge 1$, there exists a constant d_k such that for every digraph D with at least d_k vertices, D or its complement $\bar{D}$ contains an acyclic tournament of order k.*

We say that D' is a *blow-up* of a digraph D if it can be obtained as follows: first create three vertices v_1, v_2, v_3 per vertex v of D, then for each arc $(v, w) \in A(D)$, choose some subset $I_{(v,w)} \subseteq \{1, 2, 3\}, I_{(v,w)} \neq \varnothing$, and add the arcs $(v_i, w_1), (v_i, w_2), (v_i, w_3)$ for every $i \in I_{(v,w)}$. Similarly as before, we say that an acyclic tournament in D' is *D-admissible* if for every vertex v of D, it contains at most one of the three corresponding vertices in D'.

Let us give some intuition on acyclic tournaments in blow-ups of digraphs. If $I_{(v,w)} = \{1, 2, 3\}$ for each arc $(v, w) \in A(D)$ in the definition of the blow-up operation, then D' is simply the lexicographic product $D \otimes \bar{K}_3$ of D with the complement of K_3. In particular, in this case a D-admissible acyclic tournament of order k in D' is readily obtained from an acyclic tournament of order k in D. The same holds more generally if, for every $v \in V(D)$, we have $\bigcap_{(v,w) \in A(D)} I_{(v,w)} \neq \varnothing$, because then D' has a subgraph isomorphic to D which contains exactly one the three vertices v_1, v_2, v_3 for each $v \in V(D)$. It turns out that this observation can essentially be extended to the case where the sets $I_{(v,w)}$ are arbitrary nonempty subsets of $\{1, 2, 3\}$: the digraph D' will contain a D-admissible acyclic tournament of order k, provided D contains a large enough acyclic tournament. This is a consequence of the following claim.

Claim 3. *For every $k \ge 1$, there exists a constant a_k such that for every acyclic tournament D on at least a_k vertices, any blow-up D' of D contains a D-admissible acyclic tournament of order k.*

Proof. We prove the claim by induction on k, the case $k = 1$ being trivial. For the inductive step, set $a_k := 3a_{k-1} + 1$. Let $v \in V(D)$ be the unique vertex of D with out-degree $|D| - 1$, and let v_1, v_2, v_3 be the corresponding three vertices in D'. There is at least one of the latter three vertices, say v_1, for which the set $S \subseteq V(D)$ of vertices of D which correspond to the out-neighbors of v_1 in D' has cardinality at least $(a_k - 1)/3 = a_{k-1}$. Let also $S' \subseteq V(D')$ be the set of out-neighbors of v_1 in D'. The digraph $D'[S']$ is clearly a blow-up of $D[S]$, and by the induction hypothesis, $D'[S']$ contains a $D[S]$-admissible acyclic tournament of order $k - 1$. Using v_1 and the latter subgraph we obtain a D-admissible acyclic tournament of order k in D'. □

Claim 4. *For every $k \ge 1$, there exists a constant b_k such that for every digraph D on at least b_k vertices, all blow-ups of either D or $\bar{D}$ contain a D-admissible acyclic tournament of order k.*

Proof. We claim that $b_k := d_{a_k}$ will do. Indeed, by Claim 2, D or $\bar{D}$ contains then an acyclic tournament on a set T of a_k vertices, say without loss of generality D. Then, following Claim 3, every blow-up of $D[T]$ contains a $D[T]$-admissible acyclic tournament of order k, and the same clearly holds if we replace $D[T]$ with D. □

We now have everything we need to conclude. Let R (resp. B) be the digraph on vertex set $\{v_1, \ldots, v_p\}$ where there is an arc from v_i to v_k ($i \neq k$) if v_k is in at least two (resp. at most one) of the three sets $T_{i,1}, T_{i,2}, T_{i,3}$. By the definition of D_G, the red and blue parts of D_G are blow-ups of respectively R and B.

Since $R = \bar{B}$, if $p \geq b_{\delta+1}$ holds, then using Claim 4 we deduce that there exists a monochromatic admissible acyclic tournament of order $\delta + 1$ in D_G. But then, G has defect at least $\delta + 1$ by Claim 1, a contradiction. Hence, $p < b_{\delta+1}$, and $c_\delta := b_{\delta+1} - 1$ will do in the statement of Theorem 6. $\square$

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