On the Strong $L^p$-Hodge decomposition over complete Riemannian manifolds

Xiang-Dong Li $^{a,b,*},^1$

$^a$ School of Mathematical Sciences, Fudan University, 220, Handan Road, Shanghai, 200433, PR China

$^b$ Institut de Mathématiques, Université Paul Sabatier, 118, route de Narbonne, 31062, Toulouse Cedex 9, France

Received 23 February 2009; accepted 4 August 2009

Communicated by Paul Malliavin

Abstract

We establish the Strong $L^p$-Hodge decomposition theorem and the $L^p$-Poincaré inequalities on differential forms over complete non-compact Riemannian manifolds. As applications, we prove some $L^p$-estimates and existence theorems for the de Rham operator as well as some vanishing theorems for the $L^p$-cohomology and the $L^p$-torsion over complete non-compact Riemannian manifolds with suitable geometric conditions.

© 2009 Elsevier Inc. All rights reserved.

Keywords: de Rham operator; $L^p$-cohomology; Poincaré inequalities; Riesz potentials; Riesz transforms; Strong $L^p$-Hodge decomposition

1. Introduction

1.1. Backgrounds

Let $(M, g)$ be a complete Riemannian manifold, $d$ the exterior differential operator, $d^*$ the formal $L^2$-adjoint of $d$ with respect to the Riemannian volume measure $dv(x) := \sqrt{\det g(x)} \, dx$,
and $\Box_k := dd^* + d^*d$ the Hodge Laplacian on $k$-forms. In 1949, Kodaira [13] proved that the $L^2$-space of $k$-forms on $(M, g)$ has the orthogonal decomposition

$$L^2(\Lambda^k T^* M) = H_{k,2}(M) \oplus dC_0^\infty(\Lambda^{k-1} T^* M) \oplus d^*C_0^\infty(\Lambda^{k+1} T^* M),$$

where $H_{k,2}(M) := \text{Ker}(\Box_k) \cap L^2(\Lambda^k T^* M)$, and $C_0^\infty(\Lambda^i T^* M)$, $i = k \pm 1$, denote the space of smooth $i$-forms with compact support. Moreover, we have the following Strong $L^2$-Hodge decomposition theorem on complete Riemannian manifolds, cf. [11].

**Theorem 1.1.** Let $(M, g)$ be a complete Riemannian manifold, $n = \dim M$, and $k = 0, \ldots, n$. Suppose that the Hodge Laplacian $\Box_k$ has a spectral gap in $L^2$, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\|\omega\|_2 \leq \lambda_1^{-1} \langle \langle \Box_k \omega, \omega \rangle \rangle, \quad \forall \omega \in (\text{Ker} \, \Box_k)^\perp \cap L^2(\Lambda^k T^* M).$$

(1)

Then the Strong $L^2$-Hodge decomposition theorem holds:

$$L^2(\Lambda^k T^* M) = H_{k,2}(M) \oplus dW^{1,2}(\Lambda^{k-1} T^* M) \oplus d^*W^{1,2}(\Lambda^{k+1} T^* M),$$

where $W^{1,2}(\Lambda^i T^* M)$, $i = k \pm 1$, denote the $(1, 2)$-Sobolev spaces of $i$-forms on $M$ with respect to the volume measure (for definition see Theorem 1.2 below).

To develop the regularity theory for boundary value problems and for hydrodynamic systems, Spencer and Morrey initiated the study of the $L^p$-Hodge theory on compact Riemannian manifolds. See [28] and the references therein. Inspired by earlier results in [28], Scott [34] proved the following Strong $L^p$-Hodge decomposition theorem on compact Riemannian manifolds.

**Theorem 1.2.** (See Scott [34].) Let $(M, g)$ be a compact Riemannian manifold. Then, for all $1 < p < \infty$, the $L^p$-space of $k$-forms on $(M, g)$ has the Strong $L^p$-Hodge direct sum decomposition

$$L^p(\Lambda^k T^* M) = H_{k,p}(M) \oplus dW^{1,p}(\Lambda^{k-1} T^* M) \oplus d^*W^{1,p}(\Lambda^{k+1} T^* M),$$

(2)

where $H_{k,p}(M) := \text{Ker}(\Box_k) \cap L^p(\Lambda^k T^* M)$ is the space of $L^p$-harmonic $k$-forms on $(M, g)$, and

$$W^{1,p}(\Lambda^i T^* M) := \{ \omega \in L^p(\Lambda^i T^* M) : \|\omega\|_{1,p} < +\infty \}, \quad i = k \pm 1,$

denote the $(1, p)$-Sobolev spaces of $i$-forms on $M$ with respect to the volume measure, on which the Sobolev norm $\| \cdot \|_{1,p}$ is defined by

$$\|\omega\|_{1,p} := \|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p.$$
1.2. Motivation

Indeed, even in the case of Euclidean spaces, the so-called Weak $L^p$-Hodge decomposition theorem has played an indispensable role in the study of boundary valued problems, Navier–Stokes equations and other nonlinear PDEs, cf. [21,28,33,34]. In [36], Strichartz initiated the study of Riesz transforms and Riesz potentials on complete non-compact Riemannian manifolds. Moreover, he implicitly pointed out that there is a deep connection between the Weak $L^p$-Hodge decomposition theory and the $L^p$-boundedness of the Riesz transforms on complete Riemannian manifolds. In our previous papers [17,18], we established the Weak $L^p$-Hodge decomposition theorem on complete Riemannian manifolds with suitable geometric conditions. For related results, see Lohoué [26], Auscher, Coulhon, Duong, Hofmann [2], Carron, Coulhon, Hassell [4] and the author [14]. However, when $p \neq 2$, it seems that one cannot find any known result in the literature which ensures the Strong $L^p$-Hodge decomposition (2) on complete non-compact Riemannian manifolds. The main purpose of this paper is to study this fundamental problem and to establish the Strong $L^p$-Hodge theory on complete non-compact Riemannian manifolds. Inspired by the pioneering works of Strichartz [36] and Lohoué [23,24,26], we are able to obtain a criterion which provides us with an explicit and precise condition for the validity of the Strong $L^p$-Hodge decomposition on complete Riemannian manifolds. See Theorem 2.1. Moreover, we establish the Strong $L^p$-Hodge decomposition theorem on two classes of complete Riemannian manifolds with suitable geometric conditions. See Theorems 2.2 and 2.3.

1.3. Applications

We now describe two problems which are naturally related to the study of the Strong $L^p$-Hodge theory on complete non-compact Riemannian manifolds.

The first related problem is the issue of the $L^p$-cohomology theory on complete non-compact Riemannian manifolds. Let $(M, g)$ be a complete Riemannian manifold, $k = 0, 1, \ldots, n$, and $p \in [1, \infty]$. Let

$$
\Omega^k_p(M) := \{ \omega \in L^p(\Lambda^k T^* M): d\omega \in L^p(\Lambda^{k+1} T^* M) \}.
$$

By definition, the $k$-th unreduced and the $k$-th reduced $L^p$-cohomology on $(M, g)$, denoted by $H^{k,p}(M)$ and $\tilde{H}^{k,p}(M)$ respectively, are defined as follows:

$$
H^{k,p}(M) := \frac{(\text{Ker } d \cap \Omega^k_p(M))}{d \Omega^k_p(M)},
$$

$$
\tilde{H}^{k,p}(M) := \frac{(\text{Ker } d \cap \Omega^k_p(M))}{d \Omega^{k-1}_p(M)}.
$$

As the range of $d_{k-1}$, i.e., $d \Omega^{k-1}_p(M)$, may not be closed in $L^p(\Lambda^k T^* M)$, $H^{k,p}(M)$ need not necessarily coincide with $\tilde{H}^{k,p}(M)$. This leads to the definition of $k$-th $L^p$-torsion on $(M, g)$:

$$
T^{k,p}(M) := \frac{H^{k,p}(M)}{\tilde{H}^{k,p}(M)}.
$$

When $p = 2$, it has been known that, as a consequence of the Weak $L^2$-Hodge decomposition theorem, the $L^2$-cohomology on all complete Riemannian manifolds is reduced, i.e., for all $k = 0, 1, \ldots, n$, $H_{k,2}(M) = \tilde{H}^{k,2}(M)$, or equivalently, $T^{k,2}(M) = 0$. Moreover, as a consequence of
the Strong $L^2$-Hodge decomposition theorem, if the $L^2$-Poincaré inequality (1) holds, then the Hodge–de Rham injection

$$H_{k,2}(M) \rightarrow H^{k,2}(M),$$

$$\omega \mapsto [\omega]$$

is an isomorphism. So far, there have been extensive studies on the $L^2$-cohomology on complete Riemannian or Kähler manifolds, see e.g. [4,11] and the references therein.

In recent years, many people have generalized various known results on $L^2$-cohomology to $L^p$-cohomology on complete non-compact Riemannian manifolds, see [4,5,10,30–32]. However, as pointed out by Pansu [30], the study of $L^p$-cohomology on complete non-compact Riemannian manifolds is much less developed than the one of $L^2$-cohomology. In this paper, we establish some vanishing theorems of the $L^p$-cohomology and the $L^p$-torsion on two classes of complete Riemannian manifolds with suitable geometric conditions. See Theorems 2.8 and 2.9.

The second related problem is the issue of the $L^p$-estimates and existence theorems of the de Rham operator on complete Riemannian manifolds. Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu(x) = e^{-\phi(x)} \, d\nu(x)$ be a weighted volume measure. Given $\alpha \in L^p(\Lambda^{k+1}T^*M, \mu)$ satisfying $d\alpha = 0$, it is natural to ask whether the de Rham equation

$$d\omega = \alpha$$

(3)

has a solution $\omega$ in $L^p(\Lambda^{k+1}T^*M, \mu)$ and whether it satisfies some good $L^p$-estimates. By the famous Poincaré lemma, it is well known that, for any given closed $(k+1)$-form $\alpha \in C^\infty(U, \Lambda^{k+1}T^*U)$ defined in an open subset $U \subset M$ near a given point $x \in M$, the de Rham equation $d\omega = \alpha$ is always locally solvable in $C^\infty(U, \Lambda^{k+1}T^*U)$. However, whether or not the de Rham equation (3) has a global $L^p$ solution with respect to a weighted measure $d\mu = e^{-\phi} \, d\nu$ is a nontrivial problem and depends strongly on the geometry and the topology of manifolds, as well as the choice of the weight function $\phi$. In this paper, we obtain some affirmative answers to this problem on complete Riemannian manifolds with suitable geometric conditions. See Theorems 2.6 and 2.7. Moreover, we prove some $L^p$-Poincaré inequalities on differential forms on complete Riemannian manifolds. See Theorems 2.5 and 6.1.

The results and the methods used in this paper can be viewed as a product of an interplay between differential geometry, functional analysis, harmonic analysis, partial differential equations, topology as well as stochastic analysis in the study of problems in geometric analysis on complete non-compact Riemannian manifolds, which has been initiated and developed by many people in the past years. The reader might ask the question whether there will be some geometric and topological applications of the Strong $L^p$-Hodge theory and the $L^p$-cohomology theory on complete non-compact Riemannian manifolds. To answer this fundamental question, we would like to say that, on the one hand, there is no unique Hodge theory on complete non-compact Riemannian manifolds, and the Strong $L^p$-Hodge theory and the $L^p$-cohomology theory have been proved to be a suitable substitute of the classical Hodge theory and the de Rham cohomology theory when one considers geometry and topology on complete non-compact Riemannian manifolds. On the other hand, the interested reader can find that, at least from the technical point of view, there is a significant difference between the $L^2$-theory and the $L^p$-theory for $p \neq 2$ on complete non-compact manifolds. Moreover, it is worth to say that the Strong $L^p$-Hodge theory
and the $L^p$-cohomology theory provide with us much more rich and fine information of analytical, geometrical and topological information of many important subjects (e.g., harmonic forms, cohomology, estimates and existence theorem of the de Rham operator, etc.) on complete non-compact Riemannian manifolds. Many fundamental problems still remain open, for example, it is very interesting to know under which condition on a complete non-compact Riemannian manifold on which the $L^p$-cohomology is of finite dimension for $p \neq 2$.

Our study on the $L^p$-estimates and existence theorems of the de Rham operator over complete Riemannian manifolds has been strongly motivated by an effort of using a new point of view to understand the Andreotti–Hörmander–Vesentini $L^2$-estimate and existence theorems of the Cauchy–Riemann operator $\bar{\partial}$ on pseudo-convex domains and on complete Kähler manifolds [1,6,12]. Indeed, in the case where $M$ is a complete Kähler manifold and the de Rham operator $d$ is replaced by the Cauchy–Riemann operator $\bar{\partial}$ on $M$, the de Rham equation (3) becomes the famous Cauchy–Riemann equation $\bar{\partial} \omega = \alpha$ on a complete Kähler manifold, where $\alpha$ is an $(k,l)$-form on $M$ such that $\bar{\partial} \alpha = 0$, $k = 0, \ldots, n$, $l = 1, \ldots, n$, $n = \dim_{\mathbb{C}} M$. In [20], we will use the method developed in this paper to establish some $L^p$-estimates and existence theorems of $\bar{\partial}$ and to prove some vanishing theorem of the $L^p$-Dolbeault cohomology on complete Kähler manifolds with suitable geometric conditions.

The rest of this paper is organized as follows. In Section 2, we state the main results of this paper. In Section 3, we give some known and new results on the $L^p$-boundedness of the Riesz transforms and the Riesz potentials over complete non-compact Riemannian manifolds. In Section 4, we prove the Strong $L^p$-Hodge decomposition theorem on complete non-compact Riemannian manifolds. In Sections 5 and 6, we prove some $L^p$-Poincaré inequalities on differential forms as well as some $L^p$-estimates and existence theorems of the de Rham operator on complete Riemannian manifolds. In Section 7, we prove some vanishing theorems of the $L^p$-cohomology and the $L^p$-torsion on complete Riemannian manifolds.

2. Main results and applications

2.1. Notations

We now fix some notations. Throughout this paper, let $(M, g)$ be a complete Riemannian manifold, $\phi \in C^2(M)$, $d\nu(x) := \sqrt{\det g(x)} \, dx$ the Riemannian volume measure, and $d\mu(x) := e^{-\phi(x)} \, d\nu(x)$ a weighted volume measure. Let $d$ be the exterior differential operator, $d^*_{\phi}$ the formal $L^2$-adjoint of $d$ with respect to $\mu$, i.e., for all $\alpha \in C^\infty_0(\Lambda^k T^* M)$ and $\beta \in C^\infty_0(\Lambda^{k+1} T^* M)$, we have

$$\int_M \langle d\alpha, \beta \rangle \, d\mu = \int_M \langle \alpha, d^*_{\phi} \beta \rangle \, d\mu.$$ 

The weighted Hodge Laplacian (also called the Witten Laplacian in the literature) with respect to $\mu$ is defined by

$$\Box_{\phi} := dd^*_\phi + d^*_\phi d.$$
Note that, when $\phi$ is a constant, we have $\Box_{\phi} = \Box$. For all $k = 0, 1, \ldots, n$, let $\Box_k = \Box|C^\infty_0(\Lambda^k T^* M)$ and $\Box_{\phi,k} = \Box_{\phi}|C^\infty_0(\Lambda^k T^* M)$. For all $p > 1$, let $L^p(\Lambda^k T^* M, \mu)$ be the completion of $C^\infty_0(\Lambda^k T^* M)$ with respect to the $L^p$-norm $\| \cdot \|_p$ defined by
\[
\| \omega \|_p^p := \int_M |\omega(x)|^p d\mu(x).
\]
By [36], $\Box_k$ is essentially self-adjoint on $L^2(\Lambda^k T^* M, \nu)$. Similarly, we can prove that $\Box_{\phi,k}$ is essentially self-adjoint on $L^2(\Lambda^k T^* M, \mu)$. Throughout this paper, we use the same notions $\Box_k$ and $\Box_{\phi,k}$ to denote their Friedrich extensions on $L^2(\Lambda^k T^* M, \nu)$ or $L^2(\Lambda^k T^* M, \mu)$. The definitions of the weighted $k$-th $L^p$-cohomology $H^{k,p}(M, \mu)$ and the weighted $k$-th reduced $L^p$-cohomology $\bar{H}^{k,p}(M, \mu)$ will be given in Section 2.3. Let
\[
W^{1,p}(\Lambda^k T^* M, \mu) := \{ \omega \in L^p(\Lambda^k T^* M, \mu) : |d\omega|, |d^*\omega| \in L^p(\mu) \},
\]
\[
L^pH^k(M) := \{ \omega \in L^p(\Lambda^k T^* M) : d\omega = 0, \ u^* \omega = 0 \},
\]
\[
H_{k,p}(M, \mu) := \text{Ker}(\Box_{\phi,k}) \cap L^p(\Lambda^k T^* M, \mu).
\]
Let $\nabla$ be the Levi-Civita connection, $R$ the curvature tensor of $\nabla$, and $\Delta$ the Laplace–Beltrami operator on $(M, g)$. At any fixed $x \in M$, let $e_1, \ldots, e_n$ be a normal orthonormal basis of $T_y M$ for $y \in M$ near $x$ such that $\nabla_{e_i} e_j(x) = 0$ for all $i, j = 1, \ldots, n$, and $e^*_1, \ldots, e^*_n$ be its dual basis. The $k$-th Weitzenböck curvature at $x$ is defined by
\[
W_k(x) := \sum_{i,j} e^*_i \wedge i_{e_j} R(e_i, e_j), \quad \forall x \in M,
\]
where $i_{e_j}$ denotes the interior multiplication induced by the contraction of the vector field $e_j$ on $\Lambda^k T^* M$. Note that $W_k \in \text{End}(\Lambda^k T^* M)$. Recall the well-known Bochner–Lichnerowicz–Weitzenböck formula
\[
\Box_k = -\Delta + W_k,
\]
where $\Delta = \text{Tr} \nabla^2$ is the covariant Laplace–Beltrami operator acting on $k$-forms. Let $\nabla^*_\phi$ be the formal $L^2$-adjoint of $\nabla$ with respect to $\mu$. Let $\Delta_{\phi} = -\nabla^*_\phi \nabla$ be the weighted covariant Laplace–Beltrami operator acting on $k$-forms. By [9], we have
\[
\Delta_{\phi} = \Delta - \nabla_{\phi^*}.
\]
The $k$-th weighted Weitzenböck curvature $W_{\phi,k} \in \text{End}(\Lambda^k T^* M)$ is defined by
\[
W_{\phi,k} := W_k + dA^k \nabla^2 \phi,
\]
where $\nabla^2 \phi$ denotes the Hessian of $\phi$ with respect to the Levi-Civita connection $\nabla$, and $dA^k \nabla^2 \phi : \Lambda^k T^* M \to \Lambda^k T^* M$ is defined by: for all $v_1, \ldots, v_k \in T^*_x M$,
\[
dA^k \nabla^2 \phi(v_1 \wedge \cdots \wedge v_k) := \sum_{i=1}^k v_1 \wedge \cdots \wedge \nabla^2 \phi(v_i) \wedge \cdots \wedge v_k.
\]
By [9], the weighted Bochner–Lichnerowicz–Weitzenböck formula holds

\[ \square_{\phi,k} = -\Delta_{\phi} + W_{\phi,k}. \]

Throughout this paper, see Remark 2.4 below for the reason, we assume that \((M, g)\) is an \(L\)-stochastically complete Riemannian manifold, where \(L = \Delta - \nabla \phi \cdot \nabla\). That is to say, the heat semigroup \(e^{tL}\) is Markovian, i.e., \(e^{tL}1(x) = 1\) for all \(x \in M\), or equivalently, the heat equation \(\partial_t u = Lu\) has a unique solution \(u(t, \cdot) \in L^\infty\) for all \(t > 0\) provided that \(u(0, \cdot) \in L^\infty\). In the probabilistic literature, this is equivalent to say that the lifetime of the \(L\)-diffusion on \(M\) is infinity.

By Theorem 1.4 in [15], if there exist a constant \(C > 0\) and a fixed point \(o \in M\) such that

\[ \text{Ric} + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n} \geq -C \left( 1 + d^2(o, x) \right), \quad \forall x \in M \setminus \text{cut}(o), \]

where \(n = \dim M\), \(m\) is any constant such that \(m \geq n\) and \(m = n\) if and only if \(\phi\) is a constant, then \((M, g)\) is \(L\)-stochastically complete. Throughout this paper, for all \(p > 1\), let

\[ p^* = \max \left\{ p, \frac{p}{p - 1} \right\}. \]

By the spectral decomposition theorem, the \(L^2\)-Hodge orthogonal projection \(H : L^2(\Lambda^k T^* M, \mu) \to \text{Ker} \square_{\phi,k} \cap L^2(\Lambda^k T^* M, \mu)\) is a bounded operator on all weighted complete Riemannian manifolds \((M, g, \mu)\). Moreover,

\[ H := I - dd^*_{\phi} \square_{\phi,k}^{-1} - d^*_{\phi} d \square_{\phi,k}^{-1}, \]

where \(dd^*_{\phi} \square_{\phi,k}^{-1}\) and \(d^*_{\phi} d \square_{\phi,k}^{-1}\) are singular integral operators which are defined as follows: for all \(\omega \in L^2(\Lambda^k T^* M, \mu) \cap (\text{Ker} \square_{\phi,k})^\perp\),

\[ dd^*_{\phi} \square_{\phi,k}^{-1} \omega := \lim_{N \to \infty} \int_0^N dd^*_{\phi} e^{-t \square_{\phi,k}} \omega \, dt \quad \text{in} \ L^2(\Lambda^2 T^* M, \mu), \]

\[ d^*_{\phi} d \square_{\phi,k}^{-1} \omega := \lim_{N \to \infty} \int_0^N d^*_{\phi} d e^{-t \square_{\phi,k}} \omega \, dt \quad \text{in} \ L^2(\Lambda^2 T^* M, \mu). \]

By [17], if the Riesz transforms \(d \square_{\phi,k}^{-1/2}\) and \(d^* \square_{\phi,k}^{-1/2}\) are bounded in \(L^p\) and \(L^q\), where \(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1\), then \(H : C^\infty_0(\Lambda^k T^* M) \subset L^2(\Lambda^k T^* M, \mu) \to \text{Ker} \square_{\phi,k} \cap L^2(\Lambda^k T^* M, \mu)\) extends as a bounded operator from \(L^p(\Lambda^k T^* M, \mu)\) to \(\text{Ker} \square_{\phi,k} \cap L^p(\Lambda^k T^* M, \mu)\). That is, for some constant \(C_p > 0\) it holds

\[ \|H \omega\|_p \leq C_p \|\omega\|_p, \quad \forall \omega \in C^\infty_0(\Lambda^k T^* M). \]
We denote it again by $H$ and call it the $L^p$-Hodge projection. Throughout this paper, we denote

$$L^p \left( \Lambda^k T^* M, \mu \right) \cap (\text{Ker} \square_\phi)^\perp := (I - H)L^p \left( \Lambda^k T^* M, \mu \right).$$

### 2.2. Main results

Now we state the main results of this paper. Our first result provides us with a criterion for the validity of the Strong $L^p$-Hodge decomposition on complete Riemannian manifolds.

**Theorem 2.1.** Let $p > 1$, $q = \frac{p}{p-1}$. Let $(M, g)$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu(x) = e^{-\phi(x)} d\nu(x)$. Suppose that the Riesz transforms $d\square_\phi^{-1/2}$ and $d^*\phi \square_\phi^{-1/2}$ are bounded in $L^p$ and $L^q$, and the Riesz potential $\square_\phi^{-1/2}$ is bounded in $L^p$ (see Section 3 for definitions). Then the Strong $L^p$-Hodge direct sum decomposition holds on $k$-forms:

$$L^p \left( \Lambda^k T^* M, \mu \right) = H_{k,p}(M, \mu) \oplus dW_{1,p} \left( \Lambda^{k-1} T^* M, \mu \right) \oplus d^*\phi W_{1,p} \left( \Lambda^{k+1} T^* M, \mu \right),$$

and we have the $L^p$-Gaffney identification

$$L^p H^k(M, \mu) = H_{k,p}(M, \mu).$$

Moreover, the Hodge–de Rham injection

$$j_H : H_{k,p}(M, \mu) \rightarrow H^{k,p}(M, \mu), \quad \omega \mapsto [\omega]$$

is an isomorphism. More precisely, we have

$$H_{k,p}(M, \mu) \simeq H^{k,p}(M, \mu) = \bar{H}_{k,p}(M, \mu).$$

Finally, the following $L^p$-Poincaré inequality holds on $k$-forms:

$$\|\omega - H\omega\|_p \leq C_{p,k} \left( \|d\omega\|_p + \|d^*\phi \omega\|_p \right), \quad \forall \omega \in C_0^\infty \left( \Lambda^k T^* M \right),$$

(4)

where $H : L^p \left( \Lambda^k T^* M, \mu \right) \rightarrow H_{k,p}(M, \mu)$ denotes the $L^p$-Hodge projection, and $C_{p,k} > 0$ is a constant depending on the $L^p$ bound of the Riesz potential $\square_\phi^{-1/2}$ as well as the $L^p$ and $L^q$ bounds of the Riesz transforms $d\square_\phi^{-1/2}$ and $d^*\phi \square_\phi^{-1/2}$.

When $p \neq 2$, Theorem 2.1 is new in the literature and has been inspired by the pioneering works of Strichartz [36] and Lohoué [23,24,26]. People might ask the question whether the sufficient condition given in Theorem 2.1 is also the necessary one for the validity of the Strong $L^p$-Hodge decomposition theorem on complete Riemannian manifolds. Here we would like to point out that, on the one hand, using the Banach open mapping theorem, one can show that the $L^p$-boundedness of the Green operator $\square_\phi^{-1/2}$ or $\square_\phi^{-1/2}$ is a necessary condition of the validity of the Strong $L^p$-Hodge decomposition for $k$-forms on complete Riemannian manifolds, while on the other hand, as was shown by Lohoué [25], even the Weak $L^p$-Hodge decomposition theorem
does not hold on the four dimensional simply connected hyperbolic space $\mathbb{H}^4$ of constant sectional curvature $-1$ for $p$ big enough. Indeed, the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ as well as the Riesz potential $\Box_k^{-1/2}$ are not bounded in $L^p$ on $\mathbb{H}^4$ for $p$ big enough. This also shows that any one of the two conditions stated in Theorem 2.1 cannot be eliminated in order to establish the Strong $L^p$-Hodge decomposition theory on complete non-compact Riemannian manifolds.

Based on Theorem 2.1, we establish the Strong $L^p$-Hodge decomposition theorem on two classes of complete Riemannian manifolds with suitable geometric conditions. More precisely, we have the following two results.

**Theorem 2.2.** Let $(M, g)$ be a complete and $L$-stochastically complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu(x) = e^{-\phi(x)}dv(x)$. Suppose that there exist two constants $a \geq 0$ and $\rho > 0$ such that $W_{\phi,k} \geq \rho, \quad W_{\phi,i} \geq -a, \quad i = k \pm 1$.

Then, for all $p > 1$, the Strong $L^p$-Hodge direct sum decomposition holds

$$L^p(\Lambda^k T^* M, \mu) = H_{k,p}(M, \mu) \oplus dW^{1,p}(\Lambda^{k-1} T^* M, \mu) \oplus d^*W^{1,p}(\Lambda^{k+1} T^* M, \mu).$$

Moreover, for all $p \in (1, \infty)$ and for all $\omega \in C_0^\infty(\Lambda^k T^* M)$, we have the following $L^p$-Poincaré inequality on $k$-forms

$$\|\omega - H\omega\|_p \leq C_{k,a}(1 + \frac{\sqrt{a/\rho}}{\rho}(p^* - 1)^{3/2} (\|d\omega\|_p + \|d^*\phi\omega\|_p),$$

where $H: L^p(\Lambda^k T^* M, \mu) \to H_{k,p}(M, \mu)$ denotes the $L^p$-Hodge projection, and $C_{k,a} > 0$ is a constant depending only on $k$ and $a$.

**Theorem 2.3.** Let $(M, g)$ be a complete and $L$-stochastically complete Riemannian manifold. Suppose that there exists a constant $a_k \geq 0$ such that

$$W_{\phi,i} \geq -a_k, \quad i = k - 1, k, k + 1,$$

and

$$\lambda_k := \inf_{\omega \neq 0} \frac{\langle \Box_{\phi,k}\omega, \omega \rangle}{\|\omega\|^2_2} > 0.$$

Let

$$p_1 = 2 - \frac{2\lambda_k}{2\lambda_k + a_k}, \quad p_2 = 2 + \frac{2\lambda_k}{a_k}.$$

Then, for all $p \in (p_1, p_2)$, we have

$$H_{k,p}(M, \mu) = 0,$$
and the Strong \( L^p \)-Hodge direct sum decomposition holds
\[
L^p(\Lambda^k T^* M, \mu) = dW^{1,p}(\Lambda^{k-1} T^* M, \mu) \oplus d^{*\mu}_\phi W^{1,p}(\Lambda^{k+1} T^* M, \mu).
\]

Moreover, for all \( p \in (p_1, p_2) \), we have the following \( L^p \)-Poincaré inequality
\[
\|\omega\|_p \leq C_{k,p,a_k,\lambda_k}(\|d\omega\|_p + \|d^{*\mu}_\phi \omega\|_p), \quad \forall \omega \in C^\infty_0(\Lambda^k T^* M),
\]
where \( C_{k,p,a_k,\lambda_k} > 0 \) is a constant depending only on \( k, p, a_k \) and \( \lambda_k \).

**Remark 2.4.** In a report on the earlier submitted version of this paper, an anonymous referee asked the author why the stochastic completeness assumption is needed in the statement of Theorems 2.2 and 2.3. Indeed, to prove Theorems 2.2 and 2.3, we need to use the fact that the Riesz transforms \( d\square^{-1/2} \) and \( d^{*\mu}_\phi \square^{-1/2} \) are bounded in \( L^p \) and \( L^q \) with respect to \( \mu \) on complete and \( L \)-stochastically complete Riemannian manifolds with suitable geometric conditions mentioned in Theorems 2.2 and 2.3, where \( \frac{1}{p} + \frac{1}{q} = 1 \). See also Theorems 3.1 and 3.3. The proof of Theorem 3.1 is given in [17]. In [17], we have used the assumption that the \( L \)-diffusion process on \( M \) has infinite lifetime. Moreover, we have used the commutation formulae
\[
d e^{-t\square_{\phi,k}} \omega(x) = e^{-t\square_{\phi,k+1}} d \omega(x), \quad (6)
\]
\[
d^{*\mu}_\phi e^{-t\square_{\phi,k}} \omega(x) = e^{-t\square_{\phi,k-1}} d^{*\mu}_\phi \omega(x), \quad (7)
\]
for all \( t > 0, x \in M \), and \( \omega \in C^\infty_0(\Lambda^k T^* M) \). See also [3]. This is the case if the heat equations on \((k \pm 1)\)-forms
\[
\partial_t u = \square_{\phi,k \pm 1} u
\]
have unique solutions \( u(t, \cdot) = e^{-t\square_{\phi,k \pm 1}} u(0, \cdot) \) in \( L^\infty(\Lambda^{k \pm 1} T^* M) \) for all \( t > 0 \) provided that \( u(0, \cdot) \in L^\infty(\Lambda^{k \pm 1} T^* M) \). By the generalized Weitzenböck formula, the weighted Hodge Laplacians \( \square_{\phi,k \pm 1} \) are Schrödinger operators acting on \((k \pm 1)\)-forms. By the same argument based on the maximum principle as used in Dodziuk [7] or a martingale argument as used in the proof of the Feynman–Kac type representation formulae for the heat semigroups \( e^{-t\square_{\phi,k \pm 1}} \) on \((k \pm 1)\)-forms (cf. [9,27]), if \((M, g)\) is \( L \)-stochastically complete and the weighted Weitzenböck curvature operators \( W_{\phi,k \pm 1} \) are bounded from below, we can show that the heat equations \( \partial_t u = \square_{\phi,k \pm 1} u \) have unique solutions in \( L^\infty(\Lambda^{k \pm 1} T^* M) \) for all \( t > 0 \) provided that \( u(0, \cdot) \in L^\infty(\Lambda^{k \pm 1} T^* M) \). In fact, if \((M, g)\) is \( L \)-stochastically complete and if \( W_{\phi,k} \) is bounded from below by a negative constant, we have the Feynman–Kac type representation formula for the heat semigroup generated by the weighted Hodge Laplacian (cf. [9,27])
\[
e^{-t\square_{\phi,k}} \omega(x) = E_x[M^*_t \omega(X_t)], \quad \forall t > 0, x \in M, \ \omega \in C^\infty_0(\Lambda^k T^* M), \quad (8)
\]
where \( X_t \) denotes the \( L \)-diffusion process starting at \( x \) on \( M \), and \( M^*_t \) denotes the adjoint of \( M_t \in \text{End}(\Lambda^k T^* X_t M, \Lambda^k T^*_t M) \), which is the unique solution to the covariant SDE along the trajectory of the \( L \)-diffusion process \( (X_t) \)
\[
\nabla_{od X_t} M_t = -M_t W_{\phi,k}(X_t), \quad M_0 = \text{Id}_{\Lambda^k X_0^* T^* M}.
\]
Here $\nabla_{odX_t} := U_t \circ \frac{\partial}{\partial t} \circ U_t^{-1}$ denotes the Stratonovich stochastic covariant derivative with respect to the Levi-Civita connection along the trajectory of $\{X_s, s \in [0, t]\}$, and $U_t : \Lambda^k T_x^*M \rightarrow \Lambda^k T_x^*M$ denotes the Itô stochastic parallel transport along the trajectory of $\{X_s, s \in [0, t]\}$. See [9].

Recently, Anton Thalmaier has kindly pointed out to the author that, if one considers the Friedrich extension (denoted here by $\Box_{\phi, k}$) of the weighted Hodge Laplacian $\Box_{\phi, k}$ in $L^2(\Lambda^k T^*M, \mu)$ and defines $\omega(x, t) := e^{-t\Box_{\phi, k}} \omega(x)$ to be the solution to the heat equation $\partial_t \omega = -\Box_{\phi, k} \omega$, then (6) and (7) remain true without requiring that $(M, g)$ is $L$-stochastically complete. More precisely, for $x \in M$, $t > 0$ and all $\omega \in C^\infty_0(\Lambda^k T^*M)$,

\begin{align*}
de^{-t\Box_{\phi, k}} \omega(x) &= e^{-t\Box_{\phi, k+1}} d\omega(x), \\
d^*_\phi e^{-t\Box_{\phi, k}} \omega(x) &= e^{-t\Box_{\phi, k-1}} d^*_\phi \omega(x),
\end{align*}

See Theorem B.7 in [8]. Indeed, when we use the Friedrich extension of the weighted Hodge Laplacian in $L^2(\Lambda^k T^*M, \mu)$, we need to add the Dirichlet boundary condition (on the “boundary” where the $L$-diffusion process $(X_t)$ is killed) to solve the heat equation $\partial_t u = -\Box u$. See e.g. [7]. However, (6) and (7) cannot be true for all $\omega \in C^\infty(\Lambda^k T^*M) \cap L^\infty(\Lambda^k T^*M)$ without the $L$-stochastic completeness assumption. It might be possible that Theorems 2.2 and 2.3 remain true without the $L$-stochastic completeness assumption. To avoid the technique problem involving the “boundary condition” and the domain of definition in (9) and (10), we make the $L$-stochastic completeness assumption in Theorems 2.2, 2.3 and throughout all of this paper.

2.3. Applications

2.3.1. $L^p$-estimates and existence theorems of Hodge system

As the first application of our main results, we prove some $L^p$-estimates and existence theorems of the Hodge system on complete Riemannian manifolds.

**Theorem 2.5.** Under the same condition and notation as in Theorem 2.2 (respectively, Theorem 2.3), for all $p > 1$ (respectively, $p \in (p_1, p_2)$), and for all $\alpha \in L^p(\Lambda^{k+1} T^*M, \mu)$, $\beta \in L^p(\Lambda^{k-1} T^*M, \mu)$ such that

\begin{align*}
d\alpha &= 0 \quad \text{and} \quad d^*_\phi \beta = 0,
\end{align*}

there exists a unique $\omega \in L^p(\Lambda^k T^*M, \mu) \cap (\ker \Box_{\phi})^\perp$ solving the Hodge system

\begin{align*}
d\omega &= \alpha \quad \text{and} \quad d^*_\phi \omega = \beta
\end{align*}

and satisfying the following estimate

\begin{align*}
\|\omega\|_p \leq C_{k, a}(1 + \sqrt{a/\rho})(p^*-1)^{3/2} \sqrt{\rho} (\|\alpha\|_p + \|\beta\|_p),
\end{align*}

where $C_{k, a} > 0$ is a constant depending only on $k$ and $a$ (respectively,
\[ \|\omega\|_p \leq C_{k,p,a_k,\lambda_k} (\|\alpha\|_p + \|\beta\|_p), \quad (12) \]

where \( C_{k,p,a_k,\lambda_k} > 0 \) is a constant depending on \( p, k \) and \( a_k \) and \( \lambda_k \).

2.3.2. \( L^p \)-estimates and existence theorem of de Rham operator

As the second application of our main results, we prove some \( L^p \)-estimates and existence theorems of the de Rham operator on complete Riemannian manifolds.

**Theorem 2.6.** Suppose that there exist two constants \( a \geq 0 \) and \( \rho > 0 \) such that

\[ W_{\phi,k} \geq \rho, \quad W_{\phi,k-1} \geq -a. \]

Then, for all \( p > 1 \), and for all \( \alpha \in L^p(\Lambda^k T^* M, \mu) \) such that \( d\alpha = 0 \), there exists \( \omega \in L^p(\Lambda^{k-1} T^* M, \mu) \) such that

\[ d\omega = \alpha, \]

and there exists a constant \( C_{k,a} > 0 \) depending only on \( k \) and \( a \) such that

\[ \|\omega\|_p \leq \frac{C_{k,a} (1 + \sqrt{a/\rho}) (p^*-1)^{3/2}}{\sqrt{\rho}} \|\alpha\|_p. \quad (13) \]

**Theorem 2.7.** Suppose that there exists a constant \( a_k \geq 0 \) such that

\[ W_{\phi,k} \geq -a_k, \quad W_{\phi,k-1} \geq -a_k, \]

and the \( L^2 \)-bottom of spectrum of \( \Box_{\phi,k} \) is strictly positive, i.e.,

\[ \lambda_k := \inf_{\omega \neq 0} \frac{\langle \Box_{\phi,k} \omega, \omega \rangle}{\|\omega\|_2^2} > 0. \]

Then, for all \( p \in (p_1, p_2) \), where \( p_1 \) and \( p_2 \) are given as in Theorem 2.3, and for all \( \alpha \in L^p(\Lambda^k T^* M, \mu) \) such that \( d\alpha = 0 \), there exists \( \omega \in L^p(\Lambda^{k-1} T^* M, \mu) \) such that

\[ d\omega = \alpha, \]

and satisfying

\[ \|\omega\|_p \leq C_{k,p,a_k,\lambda_k} \|\alpha\|_p, \]

where \( C_{k,p,a_k,\lambda_k} > 0 \) is a constant depending only on \( k, p, a_k \) and \( \lambda_k \).
2.3.3. Vanishing theorems of \( L^p \)-cohomology and \( L^p \)-torsion

As the third application of our main results, we prove some vanishing theorems of the \( L^p \)-cohomology and the \( L^p \)-torsion on complete Riemannian manifolds. To state our results, for all \( p > 1 \), we introduce

\[
\Omega^k_p(M, \mu) := \{ \omega \in L^p(\Lambda^k T^* M, \mu) : d\omega \in L^p(\Lambda^{k+1} T^* M, \mu) \},
\]

\[
L^p H^k(M, \mu) := \{ \omega \in W^{1,p}(\Lambda^k T^* M, \mu) : d\omega = 0, \ d^* \phi \omega = 0 \},
\]

\[
H^k_p(M, \mu) := \{ \omega \in L^p(\Lambda^k T^* M) : \Box_{\phi^k} \omega = 0 \}.
\]

The \( k \)-th unreduced and the \( k \)-th reduced weighted \( L^p \)-cohomology on \((M, g, \mu)\), denoted by \( H^{k,p}(M, \mu) \) and \( \tilde{H}^{k,p}(M, \mu) \) respectively, are defined by

\[
H^{k,p}(M, \mu) := (\ker d \cap \Omega^k_p(M, \mu)) / d\Omega^{k-1}_p(M, \mu),
\]

\[
\tilde{H}^{k,p}(M, \mu) := (\ker d \cap \Omega^k_p(M, \mu)) / d\Omega^{k-1}_p(M, \mu),
\]

where \( d\Omega^{k-1}_p(M, \mu) \) denotes the closure of \( d\Omega^{k-1}_p(M, \mu) \) in \( L^p(\Lambda^k T^* M, \mu) \). As the range of \( d_{k-1} \), i.e., \( d\Omega^{k-1}_p(M, \mu) \), may not be closed in \( L^p(\Lambda^k T^* M, \mu) \), \( H^{k,p}(M, \mu) \) need not necessarily coincide with \( \tilde{H}^{k,p}(M, \mu) \). This leads to the definition of the \( k \)-th weighted \( L^p \)-torsion

\[
T^{k,p}(M, \mu) := \frac{H^{k,p}(M, \mu)}{\tilde{H}^{k,p}(M, \mu)}.
\]

**Theorem 2.8.** Under the same condition as in Theorem 2.6, we have

\[
\tilde{H}^{k,p}(M, \mu) = H^{k,p}(M, \mu) = 0.
\]

**Theorem 2.9.** Under the same condition and notation as in Theorem 2.3, for all \( p \in (p_1, p_2) \), the \( k \)-th weighted \( L^p \)-cohomology is reduced and vanishes, i.e.,

\[
\tilde{H}^{k,p}(M, \mu) = H^{k,p}(M, \mu) = 0.
\]

Under the same condition and notation as in Theorem 2.7, for all \( p \in (p_1, p_2) \), the \( k \)-th weighted \( L^p \)-cohomology is reduced

\[
\tilde{H}^{k,p}(M, \mu) = H^{k,p}(M, \mu).
\]

2.4. Examples

**Example 2.10.** Typical example satisfying the conditions in Theorems 2.2, 2.5, 2.6 and 2.8 is the Gaussian space \((\mathbb{R}^n, g_0, \mu)\), where \( g_0 \) is the Euclidean metric, \( \mu \) is the Gaussian measure

\[
d\mu(x) = \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{n/2}} \, dx.
\]
In this case, \( W_{\phi,k} = k \text{Id}, \) \( k = 1, \ldots, n, \) and \( \Delta_{\phi} = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} - x_j \frac{\partial}{\partial x_j} . \) Hence

\[
\square_{\phi,k} = \Delta_{\phi} + k \text{Id}.
\]

Therefore, the Strong \( L^p \)-Hodge decomposition theorem and the \( L^p \)-Poincaré inequality on forms hold on the Gaussian space. As a consequence, for all \( p > 1, k = 1, \ldots, n, \) and \( \alpha \in L^p(\Lambda^k \mathbb{R}^n, \mu) \) satisfying \( d\alpha = 0, \) there exists \( \omega \in L^p(\Lambda^{k-1} \mathbb{R}^n, \mu) \) such that \( d\omega = \alpha \) and

\[
\int_{\mathbb{R}^n} |\omega|^p \, d\mu \leq C_{k,p} \int_{\mathbb{R}^n} |\alpha|^p \, d\mu,
\]

where \( C_{k,p} > 0 \) is a constant depending only on \( k \) and \( p. \) Moreover, using the fact that the \( L^p \)-norm of the Riesz transforms and the Riesz potentials on Gaussian spaces are independent of dimension (cf. Theorem 3.1), we can prove that the Strong \( L^p \)-Hodge decomposition theorem and the \( L^p \)-Poincaré inequality on forms hold on infinite dimensional Wiener spaces.

**Example 2.11.** Theorems 2.3, 2.5, 2.7 and 2.9 extend some earlier results due to N. Lohoué [23, 26] and Chayet and Lohoué [5]. In [23,26], Lohoué proved that, if \( M \) is a complete Riemannian manifold with bounded geometry condition (i.e., \( M \) is a complete Riemannian manifold on which the Riemannian curvature tensor and its first and second order covariant derivatives are bounded, and the injectivity radius is uniformly bounded below by a positive constant) and with positive \( L^2 \)-bottom of spectrum of the Hodge Laplacian \( \square_k, \) i.e.,

\[
\lambda_k := \inf_{\omega \neq 0} \frac{\langle \square_k \omega, \omega \rangle_{L^2(\nu)}}{\|\omega\|^2_2} > 0,
\]

then there exist some \( p_1 \in (1, 2) \) and \( p_2 = \frac{p_1}{p_1-1} \in (2, \infty) \) such that, for all \( p \in (p_1, p_2), \) the Riesz potential \( \square_k^{-1/2} \) is bounded in \( L^p, \) and the Riesz transforms \( d\square_k^{-1/2} \) and \( d^*\square_k^{-1/2} \) are bounded in \( L^p. \) Combining this with Theorem 2.2, the Strong \( L^p \)-Hodge decomposition holds for all \( p \in (p_1, p_2). \) Under the same condition, Chayet and Lohoué [5] proved that for all \( p \in (p_1, p_2), \) the \( k \)-th \( L^p \)-cohomology on \( M \) is reduced and vanishes, and the \((k+1)\)-th \( L^p \)-cohomology on \( M \) is reduced.

**2.5. Remarks**

**Remark 2.12.** By standard cut-off argument and integration by parts formula on complete Riemannian manifolds, we can prove that, for all \( k = 0, 1, \ldots, n, \) and for all \( \omega \in C_0^\infty(\Lambda^k T^* M), \)

\[
\langle \square_{\phi,k} \omega, \omega \rangle = \|d\omega\|^2_2 + \|d^*_\phi \omega\|^2_2.
\]

Equivalently, for all \( \omega \in C_0^\infty(\Lambda^k T^* M), \) we have

\[
\|d\square_{\phi,k}^{-1/2} \omega\|_2^2 + \|d^*_\phi \square_{\phi,k}^{-1/2} \omega\|_2^2 = \|\omega - \text{H} \omega\|_2^2.
\]
where $H : \mathcal{L}^2(\Lambda^k T^* M, \mu) \to \mathcal{L}^2(\Lambda^k T^* M, \mu) \cap \text{Ker}(\Box_{\phi,k})$ is the $\mathcal{L}^2$-Hodge orthogonal projection. Therefore the Riesz transforms $d_{\Box_{\phi,k}}^{-1/2}$ and $d_{\phi,k}^{* -1/2}$ are always bounded in $\mathcal{L}^2$ with respect to $d\mu = e^{-\phi} d\nu$ on any complete Riemannian manifold $(M, g)$ and for any $\phi \in C^2(M)$. In view of this and according to Theorem 2.1, for the validity of the Strong $\mathcal{L}^2$-Hodge decomposition on complete Riemannian manifolds, one needs only to assume that the Riesz potentials associated with the Hodge Laplacian or the weighted Hodge Laplacian is bounded in $\mathcal{L}^2$, which is equivalent to the requirement that the $\mathcal{L}^2$-Poincaré inequality (1) holds on $k$-forms, cf. Theorem 1.1.

**Remark 2.13.** To people who might ask the question why we work on complete non-compact Riemannian manifolds with weighted measure, let us mention that: On Euclidean space one can only get the Weak $\mathcal{L}^p$-Hodge decomposition theorem for all $p \in (1, \infty)$, while on the Gaussian space $(\mathbb{R}^n, \gamma_n(dx))$ or even on the infinite dimensional Wiener space equipped with the Wiener measure, the Strong $\mathcal{L}^p$-Hodge decomposition theorem holds for all $p \in (1, \infty)$, where $\gamma_n(dx) = \frac{e^{-\|x\|^2}}{(2\pi)^{n/2}} dx$ is the Gaussian measure on $\mathbb{R}^n$. Indeed, the weighted $\mathcal{L}^2$-Hodge theory has been a very important tool in the study of many problems on complete non-compact Riemannian manifolds, in particular in the case of complete non-compact Kähler manifolds (cf. [1,6,12]).

### 3. Riesz transforms and Riesz potentials

We now recall the definitions of the Riesz transforms and the Riesz potentials associated with the weighted Hodge Laplacian on complete Riemannian manifolds. Let $p > 1$. We say that $\omega \in \mathcal{L}^p(\Lambda^k T^* M, \mu)$ belongs to the domain of the Riesz transform $d_{\Box_{\phi,k}}^{-1/2}$, respectively, $d_{\phi,k}^{* -1/2}$, in $\mathcal{L}^p$, if the following limit

$$d_{\Box_{\phi,k}}^{-1/2} \omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N d e^{-t\Box_{\phi,k}} \omega \frac{dt}{\sqrt{t}}$$

exists in $\mathcal{L}^p(\Lambda^{k+1} T^* M, \mu)$, respectively,

$$d_{\phi,k}^{* -1/2} \omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N d_{\phi}^{*} e^{-t\Box_{\phi,k}} \omega \frac{dt}{\sqrt{t}}$$

exists in $\mathcal{L}^p(\Lambda^{k-1} T^* M, \mu)$. Moreover, we say that $\omega \in \mathcal{L}^p(\Lambda^k T^* M, \mu)$ belongs to the domain of the Riesz potential $\Box_{\phi,k}^{-1/2}$ in $\mathcal{L}^p$ if the following limit

$$\Box_{\phi,k}^{-1/2} \omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N e^{-t\Box_{\phi,k}} \omega \frac{dt}{\sqrt{t}}$$

exists in $\mathcal{L}^p(\Lambda^k T^* M, \mu)$. Similarly, one can define the Riesz transforms $d(a + \Box_{\phi,k})^{-1/2}$, $d_{\phi,k}^{*} (a + \Box_{\phi,k})^{-1/2}$ as well as the Riesz potential $(a + \Box_{\phi,k})^{-1/2}$, where $a \geq 0$ is a constant.
Let $p > 1$. Recall that the Riesz potential $\square^{-1/2} \phi,k$ is bounded $L^p$, if there exists a constant $A_p > 0$ such that for all $\omega \in L^p(\Lambda^k T^* M, \mu)$ with $\square_{\phi,k} \omega \neq 0$,

$$\|\square^{-1/2} \phi,k \|_p \leq A_p \|\omega\|_p,$$

where $\|\cdot\|_p$ denotes the $L^p$-norm with respect to the weighted volume measure $d\mu = e^{-\phi} dv$. Equivalently, for all $\omega \in L^p(\Lambda^k T^* M, \mu)$, it holds that

$$\|\omega\|_p \leq A_p \|\square^{-1/2} \phi,k \omega\|_p.$$

Similarly, we say that the Riesz transforms $d\square^{-1/2} \phi,k$ and $d^* \square^{-1/2} \phi,k$ are bounded in $L^p$, if there exist some constants $B_p > 0$ and $C_p > 0$ such that for all $\omega \in L^p(\Lambda^k T^* M, \mu)$ with $\square_{\phi,k} \omega \neq 0$,

$$\|d\square^{-1/2} \phi,k \|_p \leq B_p \|\omega\|_p,$$

$$\|d^* \square^{-1/2} \phi,k \|_p \leq C_p \|\omega\|_p.$$

Equivalently, for all $\omega \in L^p(\Lambda^k T^* M, \mu)$, it holds that

$$\|d\omega\|_p \leq B_p \|\square^{-1/2} \phi,k \omega\|_p,$$

$$\|d^* \omega\|_p \leq C_p \|\square^{-1/2} \phi,k \omega\|_p.$$

### 3.1. The $L^p$-boundedness of Riesz transforms

By Remark 2.12, the Riesz transforms $d\square^{-1/2} \phi,k$ and $d^* \square^{-1/2} \phi,k$ are always bounded in $L^2$ on all complete Riemannian manifolds $(M, g)$ and for all $\phi \in C^2(M)$. Now it is natural to raise the following fundamental

**Stein–Strichartz problem.** (Cf. [35,36].) Let $(M, g)$ be a complete non-compact Riemannian manifold, $k = 0, 1, \ldots, n$, $n = \text{dim } M$, and $\phi \in C^2(M)$. Under which condition on $(M, g)$ and $\phi \in C^2(M)$, the Riesz transforms $d\square^{-1/2} \phi,k$ and $d^* \square^{-1/2} \phi,k$ are bounded in $L^p$ with respect to $d\mu = e^{-\phi} dv$ for some or all $1 < p < \infty$?

Since the beginning of 1980s, many people have studied the Stein–Strichartz problem on complete non-compact Riemannian manifolds with different conditions and using different methods. See [2–4,14,22,26] and the references therein. Here we mention the following result which was originally proved by Bakry [3] and was improved by the author [17].

**Theorem 3.1.** (See Bakry [3], Li [16,17].) Let $M$ be a complete and $L$-stochastically complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu = e^{-\phi} dv$. Suppose that

$$W_{\phi,i} \geq -a_k, \quad i = k - 1, k, k + 1,$$
where $a_k$ is a non-negative constant. Then, for any $p > 1$, there exists a constant $C_{p,k} > 0$ depending only on $p$ and $k$ such that, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$,

$$
\| d(a_k + \Box_{\phi,k})^{-1/2} \omega \|_p \leq C_{p,k} \| \omega \|_p.
$$

$$
\| d^*\Box_{\phi,k}(a_k + \Box_{\phi,k})^{-1/2} \omega \|_p \leq C_{p,k} \| \omega \|_p.
$$

In particular, if $W_{\phi,i} \geq 0$, $i = k$, $k \pm 1$, then the Riesz transforms $d\Box_{\phi,k}^{-1/2}$ and $d^*\Box_{\phi,k}^{-1/2}$ are bounded in $L^p$ for all $p > 1$. Moreover, for all $p > 1$, there exists a constant $C_{p,k} > 0$ such that, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$,

$$
\| d\Box_{\phi,k}^{-1/2} \omega \|_{p,p} \leq C_{p,k} \| \omega \|_p,
$$

$$
\| d^*\Box_{\phi,k}^{-1/2} \omega \|_{p,p} \leq C_{p,k} \| \omega \|_p.
$$

**Remark 3.2.** By [16], if $W_{\phi,1} = \operatorname{Ric} + \nabla^2 \phi \geq 0$, then $\| \nabla(-L)^{-1/2} \|_{p,p} \leq 2(p^* - 1)$ for all $p > \infty$. In [17], the author proved that, under the condition of Theorem 3.1, there exists a constant $C_k$ depending only on $k \geq 1$ such that

$$
C_{p,k} \leq C_k (p^* - 1)^{3/2}, \quad \forall p > 1.
$$

The following result is inspired by Lohoué [22]. When $k = 0$, see also [14].

**Theorem 3.3.** Let $p > 1$. Suppose that the Riesz potential $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p$, and the Riesz transform $d(a + \Box_{\phi,k})^{-1/2}$ (respectively, $d^*(a + \Box_{\phi,k})^{-1/2}$) is bounded in $L^p$ for some $a > 0$. Then the Riesz transform $d\Box_{\phi,k}^{-1/2}$ (respectively, $d^*\Box_{\phi,k}^{-1/2}$) is bounded in $L^p$. More precisely, there exists a constant $C_a > 0$ depending only on $a$ such that, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$,

$$
\| d\Box_{\phi,k}^{-1/2} \omega \|_p \leq C_a (1 + \sqrt{a} \| \Box_{\phi,k}^{-1/2} \|_{p,p}) \| d(a + \Box_{\phi,k})^{-1/2} \|_{p,p} \| \omega \|_p,
$$

(respectively,

$$
\| d^*\Box_{\phi,k}^{-1/2} \omega \|_p \leq C_a (1 + \sqrt{a} \| \Box_{\phi,k}^{-1/2} \|_{p,p}) \| d^*(a + \Box_{\phi,k})^{-1/2} \|_{p,p} \| \omega \|_p.
$$

**Proof.** By assumption, for all $\omega \in L^p(\Lambda^k T^* M, \mu)$, we have

$$
\| d\omega \|_p \leq \| d(a + \Box_{\phi,k})^{-1/2} \|_{p,p} \| \sqrt{a + \Box_{\phi,k}} \omega \|_p.
$$

By Lemma 2.3 in [3], the function $f_a(x) = \sqrt{\frac{a+x}{a+\sqrt{x}}}$ defined on $[0, \infty)$ is the Laplace transform of a bounded signed measure $\nu_a$ with finite total variation $C_a := \| \nu_a \|_{\text{var}} = \int_0^\infty |\nu_a| (dt)$, which depends only on $a$. Hence

$$
\| \sqrt{a + \Box_{\phi,k}} (\sqrt{a + \Box_{\phi,k}})^{-1} \|_{p,p} \leq C_a = \| \nu \|_{\text{var}},
$$
and
\[ \left\| \sqrt{a + \Box_{\phi,k} \omega} \right\|_p \leq C_a \left( \sqrt{a} \left\| \omega \right\|_p + \sqrt{\Box_{\phi,k} \omega} \right). \]

Therefore
\[ \left\| d\omega \right\|_p \leq C_a \left\| d(a + \Box_{\phi,k})^{-1/2} \right\|_{p,p} \left( \sqrt{a} \left\| \omega \right\|_p + \sqrt{\Box_{\phi,k} \omega} \right). \]

Now the Riesz potential $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p(\Lambda^k T^* M, \mu)$
\[ \left\| \omega \right\|_p \leq \left\| \Box_{\phi,k}^{-1/2} \right\|_{p,p} \left\| \Box_{\phi,k}^{1/2} \omega \right\|_p, \]
whence
\[ \left\| d\omega \right\|_p \leq C_a \left\| d(a + \Box_{\phi,k})^{-1/2} \right\|_{p,p} \left( 1 + \sqrt{a} \left\| \Box_{\phi,k}^{-1/2} \right\|_{p,p} \right) \left\| \Box_{\phi,k}^{1/2} \omega \right\|_p. \]

Equivalently, the Riesz transform $d\Box_{\phi,k}^{-1/2}$ is bounded in $L^p$ and satisfies (15). Similarly, we prove the corresponding result for the Riesz transform $d^*_\phi \Box_{\phi,k}^{-1/2}$. 

3.2. The $L^p$-boundedness of Riesz potentials

In this subsection we prove some criteria for the $L^p$-boundedness of the Riesz potentials on complete non-compact Riemannian manifolds.

**Theorem 3.4.** Suppose that there exists a constant $\rho > 0$ such that
\[ W_{\phi,k} \geq \rho. \]

Then, for all $1 < p < \infty$, the Riesz potential $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p(\Lambda^k T^* M, \mu)$. More precisely, for all $\omega \in C_0^\infty(\Lambda^k T^* M) \cap (\text{Ker } \Box_{\phi,k})^\perp$, we have
\[ \left\| \Box_{\phi,k}^{-1/2} \omega \right\|_p \leq \frac{\left\| \omega \right\|_p}{\sqrt{\rho}}. \tag{17} \]

**Proof.** Under the curvature condition $W_{\phi,k} \geq \rho > 0$, we have (cf. [9])
\[ \left| e^{-t \Box_{\phi}} \omega \right|(x) \leq e^{-\rho t} e^{t \lambda} \left| \omega \right|(x), \quad \forall x \in M, \ t > 0. \]

By subordination, and using the Minkowski inequality as well as the $L^p$-contractivity of $e^{t \lambda}$, for all $p > 1$, we have
\[ \|\Box^{-1/2}_\phi\omega\|_p = \left\| \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-t\Box \phi} \omega \frac{dt}{\sqrt{t}} \right\|_p \]

\[ \leq \frac{1}{\Gamma(1/2)} \int_0^\infty \|e^{-t\Box \phi} \omega\|_p \frac{dt}{\sqrt{t}} \]

\[ \leq \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-\rho t} \|e^{|L|\omega}\|_p \frac{dt}{\sqrt{t}} \]

\[ \leq \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-\rho t} \|\omega\|_p \frac{dt}{\sqrt{t}} \]

\[ = \frac{\|\omega\|_p}{\sqrt{\rho}}. \]

The proof of Theorem 3.4 is completed. " 

In [22,23,26], Lohoué studied the \(L^p\)-boundedness of the Riesz transforms and the Riesz potentials on complete non-compact Riemannian manifolds with bounded geometry conditions and with positive \(L^2\)-spectrum. The following result is an extension of Lohoué’s result in [26].

**Theorem 3.5.** Let \(M\) be a complete Riemannian manifold, \(\phi \in C^2(M)\), and \(d\mu = e^{-\phi} \, dv\). Suppose that there exists a constant \(a_k \geq 0\) such that

\[ W_{\phi,i} \geq -a_k, \quad i = k, k \pm 1, \]

and

\[ \lambda_k := \inf_{\omega \neq 0} \frac{\|\Box^{-1/2}_\phi k \omega, \omega\|_{L^2(\mu)}}{\|\omega\|^2_2} > 0. \]

Then, for all \(p\) in the following range

\[ 2 - \frac{2\lambda_k}{2\lambda_k + a_k} < p < 2 + \frac{2\lambda_k}{a_k}, \]

the Riesz potential \(\Box^{-1/2}_\phi\) as well as the Riesz transforms \(d\Box^{-1/2}_\phi, d\Box^{-1/2}_\phi\) are bounded in \(L^p\).

**Proof.** The proof is similar to the one in [5,23]. " 

4. The Strong \(L^p\)-Hodge decomposition theorems

In this section we prove the main results of this paper, that is, the Strong \(L^p\)-Hodge decomposition theorems on complete Riemannian manifolds.
4.1. Proof of Theorem 2.1

Without loss of generality, we may assume $\phi = 0$. By [17,36], as the Riesz transforms $d\Box^{-1/2}_k$ and $d^*\Box^{-1/2}_k$ are bounded in $L^p$ and $L^q$, the Weak $L^p$-Hodge direct sum decomposition holds

$$L^p(A^k T^* M) = H_{k,p}(M) \oplus dd^*\Box^{-1}_k L^p(A^k T^* M) \oplus d^*d\Box^{-1}_k L^p(A^k T^* M).$$

By standard cut-off argument, cf. [4], we can prove that, under the assumption of Theorem 2.2,

$$dC^\infty_0(A^{k-1}T^* M) = dd^*\Box^{-1}_k L^p(A^k T^* M),$$

and

$$d^*C^\infty_0(A^{k+1}T^* M) = d^*d\Box^{-1}_k L^p(A^k T^* M),$$

where the closures are taken in the sense of $L^p(A^k T^* M)$. Therefore

$$L^p(A^k T^* M) = H_{k,p}(M) \oplus dC^\infty_0(A^{k-1}T^* M) \oplus d^*C^\infty_0(A^{k+1}T^* M).$$  \(18\)

We first prove the $L^p$-Gaffney identification $L^p H^k(M) = H_{k,p}(M)$. On the one hand, if $\omega \in L^p H^k(M)$, $d\omega = 0$ and $d^*\omega = 0$, then $\Box_k \omega = dd^*\omega + d^*d\omega = 0$, so $\omega \in H_{k,p}(M)$. On the other hand, as the Riesz transforms $d\Box^{-1/2}_k$ and $d^*\Box^{-1/2}_k$ are bounded in $L^p$, for all $\omega \in L^p(A^k T^* M)$, we have

$$\|d\omega\|_p + \|d^*\omega\|_p \leq (\|d\Box^{-1/2}_k\|_{p,p} + \|d^*\Box^{-1/2}_k\|_{p,p}) \|\Box^{-1/2}_k \omega\|_p.$$

Moreover, the Riesz potential $\Box^{-1/2}_k$ is bounded in $L^p$

$$\|\omega\|_p \leq \|\Box^{-1/2}_k\|_{p,p} \|\Box^{-1/2}_k \omega\|_p, \quad \forall \omega \in L^p(A^k T^* M),$$

whence

$$\|\Box^{-1/2}_k \omega\|_p \leq \|\Box^{-1/2}_k\|_{p,p} \|\Box_k \omega\|_p.$$

From the above two inequalities, for all $\omega \in L^p(A^k T^* M)$, we have

$$\|d\omega\|_p + \|d^*\omega\|_p \leq (\|d\Box^{-1/2}_k\|_{p,p} + \|d^*\Box^{-1/2}_k\|_{p,p}) \|\Box^{-1/2}_k\|_{p,p} \|\Box_k \omega\|_p.$$

In particular, for all $\omega \in H_{k,p}(M)$, we have $d\omega = 0$ and $d^*\omega = 0$. This proves the $L^p$-Gaffney identification $L^p H^k(M) = H_{k,p}(M)$.

Next we prove the Strong $L^p$-Hodge decomposition. By assumption, the Green operator $\Box^{-1}_k = \Box^{-1/2}_k \Box^{-1/2}_k$ exists on $L^p$ and $L^q$, and hence by Riesz–Thorin interpolation exists on $L^2$. Moreover, we have

$$\|d\Box^{-1}_k \omega\|_p = \|d\Box^{-1/2}_k \Box^{-1/2}_k \omega\|_p \leq \|d\Box^{-1/2}_k\|_{p,p} \|\Box^{-1/2}_k\|_{p,p} \|\omega\|_p.$$
and
\[ \left\| d^* \square_k^{-1} \omega \right\|_p = \left\| d^* \square_k^{-1/2} \square_k^{-1/2} \omega \right\|_p \leq \left\| d^* \square_k^{-1/2} \right\|_{p,p} \left\| \square_k^{-1/2} \right\|_{p,p} \left\| \omega \right\|_p. \]

Therefore
\[ dd^* \square_k^{-1} L^p(\Lambda^k T^* M) \subset dW^{1,p}(\Lambda^{k-1} T^* M), \]
\[ d^* d\square_k^{-1} L^p(\Lambda^k T^* M) \subset dW^{1,p}(\Lambda^{k+1} T^* M). \]

We now prove that the above inclusions “⊂” are indeed “=”. By density argument and using the Weak $L^p$-Hodge decomposition (18), we have
\[ L^p(\Lambda^k T^* M) = H_{k,p}(M) \oplus dW^{1,p}(\Lambda^{k-1} T^* M) \oplus d^* W^{1,p}(\Lambda^{k+1} T^* M). \]

Thus, to finish the proof of the Strong $L^p$-Hodge decomposition, we need only to prove that $dW^{1,p}(\Lambda^{k-1} T^* M)$ and $d^* W^{1,p}(\Lambda^{k+1} T^* M)$ are closed subspaces in $L^p(\Lambda^k T^* M)$. Let $\eta$ be an element in $dW^{1,p}(\Lambda^{k-1} T^* M)$. By density argument, there exists a sequence $\omega_j \in C^\infty_0(\Lambda^{k-1} T^* M)$ such that $\|d\omega_j - \eta\|_p \to 0$ as $j \to \infty$. By the Weak $L^2$-Hodge decomposition theorem, we have
\[ \omega_j = H\omega_j + dd^* \square_{k-1}^{-1} \omega_j + d^* d\square_{k-1}^{-1} \omega_j. \]

Let $\alpha_j = d^* d\square_{k-1}^{-1} \omega_j$ and $\beta_j = H\omega_j + dd^* \square_{k-1}^{-1} \omega_j$. Then $\omega_j = \alpha_j + \beta_j$. By Weyl’s elliptic regularity lemma, $\alpha_j \in d^* C^\infty(\Lambda^{k-1} T^* M)$. Moreover, $d\beta_j = dH\omega_j + d^2 d^* \square_{k-1}^{-1} \omega_j = 0$. Since the Riesz potential $\square_{k-1}^{-1/2}$ is bounded in $L^p(\Lambda^k T^* M)$ and the Riesz transforms $d^* \square_{k-1}^{-1/2}$ is bounded in $L^p$, we have
\[ \|\alpha_j\|_p = \left\| d^* \square_k^{-1/2} \square_k^{-1/2} d\omega_j \right\|_p \]
\[ \leq \left\| d^* \square_k^{-1/2} \right\|_{p,p} \left\| \square_k^{-1/2} \right\|_{p,p} \left\| d\omega_j \right\|_p. \]

By the fact that $\alpha_j = \omega_j - \beta_j$ and $d\beta_j = 0$, we have $d\alpha_j = d\omega_j$ and
\[ \|d\alpha_j\|_p = \|d\omega_j\|_p. \]

Furthermore, since $d^* \alpha_j = (d^*)^2 d\square_{k-1}^{-1} \omega_j = 0$, we have $\|d^* \alpha_j\|_p = 0$. Define
\[ \|\alpha\|_{1,p} := \|\alpha\|_p + \|d\alpha\|_p + \|d^* \alpha\|_p, \quad \forall \alpha \in W^{1,p}(\Lambda^{k-1} T^* M). \]

Then $(W^{1,p}(\Lambda^{k-1} T^* M), \|\cdot\|_{1,p})$ is a reflexive Banach space for all $1 < p < \infty$. Note that
\[ \|\alpha_j\|_{1,p} = \|\alpha_j\|_p + \|d\alpha_j\|_p + \|d^* \alpha_j\|_p \]
\[ \leq \left( 1 + \left\| d^* \square_k^{-1/2} \right\|_{p,p} \left\| \square_k^{-1/2} \right\|_{p,p} \right) \|d\omega_j\|_p. \]
Thus, as \( \|d\omega_j - \eta\|_p \to 0 \), \( \{\alpha_j\} \) is a bounded sequence in \( W^{1,p}(\Lambda^{k-1}T^*M) \) which is a reflexive Banach space for \( 1 < p < \infty \). Therefore, there exists a subsequence of \( \alpha_j \), denoted again by \( \alpha_j \), and there exists some \( \alpha \in W^{1,p}(\Lambda^{k-1}T^*M) \), such that \( \alpha_j \to \alpha \) weakly in \( W^{1,p}(\Lambda^{k-1}T^*M) \). Since \( \|\cdot\|_p \) is \( L^p \)-weakly lower semi-continuous and using the fact that \( d \) is continuous from \( (W^{1,p}(\Lambda^{k-1}T^*M), \text{weak}) \) to \( (L^p(\Lambda^kT^*M), \text{weak}) \), we have
\[
\|d\alpha - \eta\|_p = \lim_{j \to \infty} \|d\alpha_j - \eta\|_p \\
\leq \liminf_{j \to \infty} \|d\alpha_j - \eta\|_p \\
= \liminf_{j \to \infty} \|d\omega_j - \eta\|_p \\
= 0.
\]
This proves that \( d\alpha = \eta \) and hence \( dW^{1,p}(\Lambda^{k-1}T^*M) \) is closed in \( L^p(\Lambda^kT^*M) \). Similarly, we can prove that \( d^*W^{1,p}(\Lambda^{k+1}T^*M) \) is closed in \( L^p(\Lambda^kT^*M) \). This proves the part of Strong \( L^p \)-Hodge decomposition on \( k \)-forms, which implies immediately the part of Hodge–de Rham isomorphism of Theorem 2.1. The proof of (4) will be given in Section 6.

4.2. Proof of Theorem 2.2

By Theorem 3.1, the Riesz transforms \( d(a + \Box_{\phi,k})^{-1/2} \) and \( d^*(a + \Box_{\phi,k})^{-1/2} \) are bounded in \( L^p \) for all \( 1 < p < \infty \). On the other hand, using Theorem 3.4, the Riesz potential \( \Box_{\phi,k}^{-1/2} \) is bounded in \( L^p \) for all \( 1 < p < \infty \). By Theorem 3.3, the Riesz transforms \( d\Box_{\phi,k}^{-1/2} \) and \( d^*\Box_{\phi,k}^{-1/2} \) are bounded in \( L^p \) for all \( 1 < p < \infty \). Therefore, we derive the first and the second parts of Theorem 2.2 from Theorem 2.1. The \( L^p \)-Poincaré inequality (5) can be derived from Theorem 6.1 as well as (14), (15), (16) and (17).

4.3. Proof of Theorem 2.3

From the proof of Theorem 3.5, for all \( 1 < p < \infty \) and for all \( t > 0 \), we have
\[
\|e^{-t\Box_{\phi,k}}\|_{p,p} \leq e^{\alpha_k t - 2(\alpha_k + \lambda_k)t/p^*}.
\]
Therefore, for all \( \omega \in H_{k,p}(M, \mu) = \text{Ker} \Box_{\phi,k} \cap L^p(\Lambda^kT^*M, \mu) \) and \( t > 0 \),
\[
\|\omega\|_p = \|e^{-t\Box_{\phi,k}}\omega\|_p \leq e^{\alpha_k t - 2(\alpha_k + \lambda_k)t/p^*}\|\omega\|_p.
\]
Note that, for all \( p \in (p_1, p_2) \), the right hand side of the above inequality tends to zero when taking \( t \to \infty \). Thus, \( \omega = 0 \). This proves that \( H_{k,p}(M, \mu) = 0 \).

By Theorem 3.1, the Riesz transforms \( d(a + \Box_{\phi,k})^{-1/2} \) and \( d^*(a + \Box_{\phi,k})^{-1/2} \) are bounded in \( L^p \) for all \( 1 < p < \infty \). On the other hand, using Theorem 3.5, the Riesz potential \( \Box_{\phi,k}^{-1/2} \) is bounded in \( L^p \) for \( p \in (p_1, p_2) \). By Theorem 3.3 or Theorem 3.5, the Riesz transforms \( d\Box_{\phi}^{-1/2} \) and \( d^*\Box_{\phi}^{-1/2} \) are bounded in \( L^p \) for \( p \in (p_1, p_2) \). Therefore, Theorem 2.3 follows from Theorem 2.1.
5. The $L^p$-Poincaré inequalities on forms

In this section we establish some relationships between the $L^p$-boundedness of Riesz transforms, the $L^p$-boundedness of Riesz potentials and the $L^p$-Poincaré inequalities on differential forms.

**Theorem 5.1.** Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu = e^{-\phi} \, dv$. Suppose that the Riesz transforms $d\Box_{\phi,k}^{-1/2}$ and $d^*\Box_{\phi,k}^{-1/2}$ are bounded in $L^p$ and in $L^q$. Then the following two conditions are equivalent:

(1) The Riesz potential $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p(\Lambda^kT^*M,\mu)$, i.e., there exists a constant $C_{1,p} > 0$ such that for all $\omega \in (\text{Ker} \, \Box_{\phi,k})^\perp \cap L^p(\Lambda^kT^*M,\mu)$, we have

$$\|\Box_{\phi,k}^{-1/2}\omega\|_p \leq C_{1,p} \|\omega\|_p.$$  \hspace{1cm} (19)

(2) The $L^p$-Poincaré inequality holds on $k$-forms, i.e., there exists a constant $C_{2,p} > 0$ such that for all $\omega \in (\text{Ker} \, \Box_{\phi,k})^\perp \cap L^p(\Lambda^kT^*M,\mu)$, we have

$$\|\omega\|_p \leq C_{2,p} (\|d\omega\|_p + \|d^*\phi \omega\|_p).$$  \hspace{1cm} (20)

**Proof.** Without loss of generality, we may assume $\phi = 0$. Suppose that the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in $L^q$ and that the Riesz potential $\Box_k^{-1/2}$ is bounded in $L^p(\Lambda^kT^*M)$. By the Weak $L^2$-Hodge decomposition on $L^2(\Lambda^kT^*M)$: for all $\omega \in C_0^\infty(\Lambda^kT^*M)$, we have

$$\omega = H\omega + \Box_k^{-1} dd^* \omega + \Box_k^{-1} d^* d \omega.$$  

It follows that

$$\|\omega - H\omega\|_p \leq \|\Box_k^{-1/2}\|_{q,q} \|d\Box_k^{-1/2}\|_{p,p} \|d\omega\|_p + \|\Box_k^{-1/2} d\|_{p,p} \|d^* \omega\|_p.$$  

By duality argument, for all $p > 1$ and $q = \frac{p}{p-1}$, we can prove that

$$\|\Box_k^{-1/2} d^*\|_{p,p} = \|d\Box_k^{-1/2}\|_{q,q}^*, \quad \|\Box_k^{-1/2} d\|_{p,p} = \|d^* \Box_k^{-1/2}\|_{q,q}^*,$$

whence

$$\|\omega - H\omega\|_p \leq \|\Box_k^{-1/2}\|_{p,p} (\|d\Box_k^{-1/2}\|_{q,q} \|d\omega\|_p + \|d^* \Box_k^{-1/2}\|_{q,q} \|d^* \omega\|_p).$$  

This yields the $L^p$-Poincaré inequality

$$\|\omega - H\omega\|_p \leq \|\Box_k^{-1/2}\|_{p,p} (\|d\Box_k^{-1/2}\|_{q,q} + \|d^* \Box_k^{-1/2}\|_{q,q}) (\|d\omega\|_p + \|d^* \omega\|_p).$$  \hspace{1cm} (21)

Thus we have proved (20) with $C_{2,p}$ precisely given in (21).
Conversely, assuming that the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in $L^p$ and that the $L^p$-Poincaré inequality on $k$-forms holds, we prove that the Riesz potential $\Box_k^{-1/2}$ is bounded in $L^p(A^kT^*M)$. Indeed, for all $\omega \in L^p(A^kT^*M)$, we have

$$
\|d\omega\|_p \leq \|d\Box_k^{-1/2}\|_{p,p} \|\Box_k^{1/2}\omega\|_p,
$$

$$
\|d^*\omega\|_p \leq \|d^*\Box_k^{-1/2}\|_{p,p} \|\Box_k^{1/2}\omega\|_p.
$$

Substituting these into the $L^p$-Poincaré inequality (20), we have

$$
\|\omega\|_p \leq C_{2,p}(\|d\Box_k^{-1/2}\|_{p,p} + \|d^*\Box_k^{-1/2}\|_{p,p}) \|\Box_k^{1/2}\omega\|_p.
$$

Equivalently, for all $\omega \in (\text{Ker } \Box_k)^\perp \cap L^p(A^kT^*M, \mu)$, it holds that

$$
\|\Box_k^{-1/2}\omega\|_p \leq C_{2,p}(\|d\Box_k^{-1/2}\|_{p,p} + \|d^*\Box_k^{-1/2}\|_{p,p}) \|\omega\|_p. \tag{22}
$$

This proves (19) with $C_{1,p}$ precisely given in (22). □

**Theorem 5.2.** Let $p > 1$, and $q = \frac{p}{p-1}$. Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d\mu = e^{-\phi} dv$. Suppose that the Riesz transforms $d(a + \Box_k)\phi^{-1/2}$ and $d^*(a + \Box_k)\phi^{-1/2}$ are bounded in $L^p$ and in $L^q$, and the Riesz potential $\Box_k^{-1/2}$ is bounded in $L^p$, i.e., there exists a constant $C_{1,p} > 0$ such that for all $\omega \in (\text{Ker } \Box_k)^\perp \cap L^p(A^kT^*M)$,

$$
\|\Box_k^{-1/2}\omega\|_p \leq C_{1,p} \|\omega\|_p.
$$

Then the $L^p$-Poincaré inequality holds on $k$-forms, i.e., there exists a constant $C_{2,p} > 0$ such that for all $\omega \in (\text{Ker } \Box_k)^\perp \cap L^p(A^kT^*M, \mu)$,

$$
\|\omega\|_p \leq C_{2,p}(\|d\omega\|_p + \|d^*\omega\|_p).
$$

**Proof.** Since $\Box_k^{-1/2}$ is self-adjoint on $L^2(A^kT^*M, \mu)$, the duality argument shows that it is automatically bounded in $L^q$. By Theorem 3.3, as the Riesz transforms $d(a + \Box_k)\phi^{-1/2}$ and $d^*(a + \Box_k)\phi^{-1/2}$ are bounded in $L^p$ and in $L^q$, then the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in $L^p$ and in $L^q$. The $L^p$-Poincaré inequality (23) on $k$-forms follows from Theorem 5.1. □

6. $L^p$-estimates and existence theorems of de Rham operator

In this section we prove some $L^p$-estimates and existence theorems of the de Rham operator $d$ over complete non-compact Riemannian manifolds. We first prove a result concerning the $L^p$-estimates and existence theorem of Hodge system on complete Riemannian manifolds.
Theorem 6.1. Under the same condition as in Theorem 2.1, for all \( \alpha \in W^{1,p}(\Lambda^{k+1}T^*M, \mu) \cap (\ker \Box_{\phi,k+1})^\perp \) and \( \beta \in W^{1,p}(\Lambda^{k-1}T^*M, \mu) \cap (\ker \Box_{\phi,k-1})^\perp \) such that

\[
d\alpha = 0 \quad \text{and} \quad d^*_\phi \beta = 0,
\]

there exists a unique \( \omega \in L^p(\Lambda^k T^*M, \mu) \cap (\ker \Box_{\phi,k})^\perp \) solving the weighted Hodge system

\[
d\omega = \alpha \quad \text{and} \quad d^*_\phi \omega = \beta.
\] (23)

Moreover, there exists a constant \( C_{p,k} > 0 \) such that

\[
\| \omega \|_p \leq C_{p,k} (\| \alpha \|_p + \| \beta \|_p).
\]

Indeed, the above solution \( \omega \in L^p(\Lambda^k T^*M, \mu) \cap (\ker \Box_{\phi,k})^\perp \) is given by

\[
\omega = \Box^{-1/2}_{\phi,k} (d^*_\phi \alpha + d\beta),
\]

and

\[
C_{p,k} = \left\| \Box^{-1/2}_{\phi,k} \right\|_{p,p} (\left\| d \Box^{-1/2}_{\phi,k} \right\|_{q,q} + \left\| d^*_\phi \Box_{\phi,k} \right\|_{p,p}).
\]

Proof. By assumption, \( d^*_\phi \alpha \in L^p(\Lambda^k T^*M, \mu) \) and \( d\beta \in L^p(\Lambda^k T^*M, \mu) \). Since \( \Box^{-1/2}_{\phi,k} \) is bounded in \( L^p \), the Green operator \( \Box^{-1}_{\phi,k} \) exists on \( (\ker \Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^*M, \mu) \). Therefore, the Hodge system (23) has a unique solution \( \omega \in (\ker \Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^*M, \mu) \), called the canonical solution, which is given by

\[
\omega = \Box^{-1}_{\phi,k} (d^*_\phi \alpha + d\beta).
\]

Indeed, using the fact that \( d \Box^{-1}_{\phi,k} = \Box^{-1}_{\phi,k+1} d \), we have

\[
d\omega = d \Box^{-1}_{\phi,k} (d^*_\phi \alpha + d\beta)
= \Box^{-1}_{\phi,k+1} (dd^*_\phi \alpha + d^2 \beta)
= \Box^{-1}_{\phi,k+1} (dd^*_\phi \alpha + d^*_\phi d\alpha)
= \Box^{-1}_{\phi,k+1} \Box_{\phi,k+1} \alpha
= \alpha.
\]

Similarly, using the fact that \( d^*_\phi \Box^{-1}_{\phi,k} = \Box^{-1}_{\phi,k-1} d^*_\phi \), we have

\[
d^*_\phi \omega = d^*_\phi \Box^{-1}_{\phi,k} (d^*_\phi \alpha + d\beta)
= \Box^{-1}_{\phi,k-1} (d^*_\phi d^*_\phi \alpha + d^*_\phi d\beta)
= \Box^{-1}_{\phi,k-1} (dd^*_\phi \beta + d^*_\phi d\beta)\]
\[= \Box^{-1}_{\phi,k-1} \Box^{-1}_{\phi,k-1} \beta \]
\[= \beta. \]

Note that
\[\omega = \Box^{-1/2}_{\phi,k} \left( \Box^{-1/2}_{\phi,k} d_{\phi}^\ast \alpha + \Box^{-1/2}_{\phi,k} d \beta \right). \]

Since the Riesz transforms \(d\Box^{-1/2}_{\phi,k}\) and \(d_{\phi}^\ast \Box^{-1/2}_{\phi,k}\) are bounded in \(L^q(\mu)\), the duality argument shows that \(\Box^{-1/2}_{\phi,k} d_{\phi}^\ast \alpha\) and \(\Box^{-1/2}_{\phi,k} d \beta\) are bounded in \(L^p(\mu)\). Moreover,
\[\left\| \Box^{-1/2}_{\phi,k} d \right\|_{p,p} = \left\| d_{\phi}^\ast \Box^{-1/2}_{\phi,k} \right\|_{q,q}, \]
\[\left\| \Box^{-1/2}_{\phi,k} d_{\phi}^\ast \right\|_{p,p} = \left\| d \Box^{-1/2}_{\phi,k} \right\|_{q,q}. \]

Therefore
\[\|\omega\|_p \leq \left\| \Box^{-1/2}_{\phi,k} \right\|_{p,p} (\|d\Box^{-1/2}_{\phi,k}\|_{q,q}\|\alpha\|_p + \|d\Box^{-1/2}_{\phi,k}\|_{q,q}\|\beta\|_p), \]

which implies the desired estimate. The proof of Theorem 6.1 is finished. \(\square\)

### 6.1. End of the Proof of Theorem 2.1

To prove the last part of Theorem 2.1, let \(\alpha = d\omega\) and \(\beta = d_{\phi}^\ast \omega\). Then \(\alpha\) and \(\beta\) satisfy the conditions in Theorem 6.1. Using the Weak \(L^p\)-Hodge decomposition theorem, we see that \(\omega - H_\omega\) is the canonical solution to the Hodge system
\[d(\omega - H_\omega) = \alpha \quad \text{and} \quad d_{\phi}^\ast (\omega - H_\omega) = \beta. \]

By Theorem 6.1, we can derive the \(L^p\)-Poincaré inequality (4). This ends the proof of Theorem 2.1.

### 6.2. Proof of Theorem 2.5

By Theorems 3.4 and 3.1, under the same condition as in Theorem 2.2, the Riesz potential \(\Box^{-1/2}_{\phi,k}\) is bounded in \(L^p\) for all \(p > 1\), and the Riesz transforms \(d\Box^{-1/2}_{\phi,k}\) and \(d_{\phi}^\ast \Box^{-1/2}_{\phi,k}\) are bounded in \(L^p\) for all \(p > 1\). On the other hand, under the same condition as in Theorem 2.3, Theorem 3.5 implies that the Riesz potential \(\Box^{-1/2}_{\phi,k}\) is bounded in \(L^p\) for all \(p \in (p_1, p_2)\), where
\[p_1 = 2 - \frac{2 \lambda_k}{2 \lambda_k + a_k} \quad \text{and} \quad p_2 = 2 + \frac{2 \lambda_k}{a_k}. \]

Moreover, using Theorems 3.1 and 3.3, the Riesz transforms \(d\Box^{-1/2}_{\phi,k}\) and \(d_{\phi}^\ast \Box^{-1/2}_{\phi,k}\) are bounded in \(L^p\) for all \(p \in (p_1, p_2)\). Finally, in both cases, we can derive Theorem 2.5 and the desired estimates (11) and (12) from Theorems 6.1, 3.5 as well as (14), (15) and (16).
6.3. Proof of Theorem 2.6

By Theorem 3.4, as \( W_{\phi,k} \geq \rho > 0 \), the Riesz potential \( \Box_{\phi,k}^{-1/2} \) is bounded in \( L^p \) for all \( p > 1 \) and \( \| \Box_{\phi,k}^{-1/2} \|_{p,p} \leq \rho^{-1/2} \). So the Green operator \( \Box_{\phi,k}^{-1/2} \) exists on \((\ker \Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^* M, \mu)\) for all \( p > 1 \). Therefore, the de Rham equation (3) has a unique solution in \((\ker d)^\perp \cap L^p(\Lambda^{k-1} T^* M, \mu)\), called the canonical solution, which is given by

\[
\omega = d^*_{\phi} \Box_{\phi,k}^{-1/2} \alpha = d^*_{\phi} \Box_{\phi,k}^{-1/2} \Box_{\phi,k}^{-1/2} \alpha.
\]

On the other hand, as \( W_{\phi,k-1} \geq -a \) and \( W_{\phi,k} \geq \rho \), Theorem 3.1 implies that the Riesz transform \( d^*_{\phi} (a + \Box_{\phi,k})^{-1/2} \) is bounded in \( L^p \) for all \( p > 1 \). By Theorem 3.3, the Riesz transform \( d^*_{\phi} \Box_{\phi,k}^{-1/2} \) is bounded in \( L^p \) for all \( p > 1 \). By (14), (16) and (17), there exists a constant depending only on \( a \) such that

\[
\| d^*_{\phi} \Box_{\phi,k}^{-1/2} \|_{p,p} \leq C(a + \sqrt{a}) \left( h_{\phi,k}^{-1/2} \right)_{p,p} \| d^*_{\phi} (a + \Box_{\phi,k})^{-1/2} \|_{p,p} \leq C_{k,a} (p^* - 1)^{3/2} (1 + \sqrt{a/\rho}),
\]

where \( C_{k,a} > 0 \) is a constant depending only on \( k \) and \( a \). Therefore

\[
\| \omega \|_{p,p} \leq \| d^*_{\phi} \Box_{\phi,k}^{-1/2} \|_{p,p} \| \Box_{\phi,k}^{-1/2} \|_{p,p} \| \alpha \|_{p,p} \leq C_{k,a} \rho^{-1/2} (1 + \sqrt{a/\rho}) (p^* - 1)^{3/2} \| \alpha \|_{p,p}.
\]

This proves the desired \( L^p \)-estimate (13) in Theorem 2.6.

6.4. Proof of Theorem 2.7

Based on Theorems 3.1, 3.5 and 3.3, we can prove Theorem 2.7 by the argument used in the proof of Theorem 2.6.

7. Vanishing theorems on \( L^p \)-cohomology and \( L^p \)-torsion

7.1. Two criteria

To prove Theorems 2.8 and 2.9, we need the following two criteria for the vanishing of the unreduced \( L^p \)-cohomology and the \( L^p \)-torsion on complete non-compact Riemannian manifolds, which are essentially due to Pansu [29–32].

**Theorem 7.1.** Let \( M \) be a complete Riemannian manifold, \( n = \dim M, \phi \in C^2(M) \), and \( d\mu = e^{-\phi} dv \). Let \( p > 1 \), and \( k = 0, 1, \ldots, n \). Then the unreduced \( k \)-th weighted \( L^p \)-cohomology on \( M \) vanishes, i.e.,

\[
H^{k,p}(M, \mu) = 0,
\]
if and only if for all $\alpha \in L^p(\Lambda^k T^* M, \mu) \cap \text{Ker } d$, the de Rham equation

$$d \omega = \alpha$$

has a solution in $L^p(\Lambda^{k-1} T^* M, \mu)$ and the following $L^p$-inequality holds:

$$\|\omega\|_p \leq C_{p,k} \|\alpha\|_p,$$

where $C_{p,k}$ is a constant depending only on $p$ and $k$.

**Proof.** The proof is similar to the one of Theorem 5.1 in [19]. See also [29–32]. □

**Theorem 7.2.** Let $M$ be a complete Riemannian manifold, $n = \text{dim } M$, $\phi \in C^2(M)$, and $d \mu = e^{-\phi} d\nu$. Let $p > 1$, $k = 0, 1, \ldots, n$. Then the $k$-th weighted $L^p$-cohomology on $M$ is reduced, i.e., $\overline{T}^{k,p}(M, \mu) = 0$, if and only if the following $L^p$-Poincaré inequality holds: there exists a constant $C_{p,k} > 0$ such that, for all $\omega \in \Omega_{p}^{k-1}(M, \mu)$, there exists some $\tilde{\omega} \in \Omega_{p}^{k-1}(M, \mu)$ with

$$d \tilde{\omega} = d \omega,$$

and

$$\|\tilde{\omega}\|_p \leq C_{p,k} \|d \tilde{\omega}\|_p.$$

**Proof.** The proof is similar to the one of Theorem 5.2 in [19]. See also [29–32]. □

As a corollary of Theorem 7.2, we have the following

**Theorem 7.3.** Let $p > 1$. Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$, and $d \mu = e^{-\phi} d\nu$. Suppose that the Riesz transform $d^*_\phi \square^{-1/2}_\phi \phi$ is bounded in $L^p$, and the Riesz potential $\square^{-1/2}_\phi \phi$ is bounded in $L^p$, or equivalently, there exists a constant $C_p$ such that the weighted $L^p$-Poincaré inequality holds:

$$\|\omega\|_p \leq C_p \|d \omega\|_p, \quad \forall \omega \in (\text{Ker } d)^\perp \cap \Omega^k_p(M, \mu).$$

Then the $k$-th weighted $L^p$-cohomology on $(M, g)$ is reduced:

$$\overline{H}^{k,p}(M, \mu) = H^{k,p}(M, \mu).$$

Equivalently, the $k$-th weighted $L^p$-torsion on $(M, g)$ vanishes

$$T^{k,p}(M, \mu) = 0.$$

**Proof.** By the Weak $L^2$-Hodge decomposition theorem, for all $\omega \in C_0^\infty(\Lambda^{k-1} T^* M)$, we have

$$\omega = H \omega + d \square^{-1}_{\phi,k} d^*_{\phi} \omega + d^*_{\phi} \square^{-1}_{\phi,k} d \omega.$$
By the fact that the Riesz potential $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p(\Lambda^k(T^*M), \mu)$ and the Riesz transform $d^*_\phi \Box_{\phi,k}^{-1/2}$ is bounded in $L^p$, we obtain

$$
\| \omega - H\omega - d\Box_{\phi,k}^{-1/2}d^*_\phi \omega \|_p \leq \| d^*_\phi \Box_{\phi,k}^{-1/2} \|_p \| \Box_{\phi,k}^{-1/2} \|_p \| d\omega \|_p.
$$

Let $\tilde{\omega} := \omega - H\omega - d\Box_{\phi,k}^{-1/2}d^*_\phi \omega$. Then $\tilde{\omega} \in L^p(\Lambda^kT^*M, \mu)$. Moreover, we have

$$
d\tilde{\omega} = d\omega
$$

and

$$
\| \tilde{\omega} \|_p \leq C_{p,k} \| d\tilde{\omega} \|_p,
$$

where

$$
C_{p,k} = \| \Box_{\phi,k}^{-1/2} \|_p \| d^*_\phi \Box_{\phi,k}^{-1/2} \|_p.
$$

By Theorem 7.2, we have $T^{k,p}(M) = 0$, i.e., $H^{k,p}(M, \mu) = \bar{H}^{k,p}(M, \mu)$. □

### 7.2. Proof of Theorems 2.8 and 2.9

Theorem 2.8 follows from Theorems 2.6 and 7.1. By Theorem 3.5, $\Box_{\phi,k}^{-1/2}$ is bounded in $L^p$ for all $p \in (p_1, p_2)$, and $d^*_\phi \Box_{\phi,k}^{-1/2}$ is bounded in $L^p$ for all $p \in (p_1, p_2)$. Applying Theorem 7.3, we can therefore conclude that $H^{k,p}(M, \mu) = \bar{H}^{k,p}(M, \mu)$ holds for all $p \in (p_1, p_2)$. Finally, if $W_{\phi,k-1} \geq -a_k$ and $W_{\phi,k+1} \geq -a_k$, then $d\Box_{\phi,k}^{-1/2}$ and $d^*_\phi \Box_{\phi,k}^{-1/2}$ are bounded in $L^p$ for all $p \in (p_1, p_2)$. Using Theorem 7.1, we may conclude that $H^{k,p}(M, \mu) = \bar{H}^{k,p}(M, \mu) = 0$ holds for all $p \in (p_1, p_2)$. This ends the proof of Theorem 2.9.

### Acknowledgments

This paper is based on a part of the author’s Thèse d’Habilitation à Diriger des Recherches defended in December 2007 at the Université Paul Sabatier. The author is grateful to Professors D. Bakry, G. Besson, J.-M. Bismut, T. Coulhon, D. Elworthy, N. Lohoué, P. Malliavin, N. Mok, P. Pansu and Weiping Zhang for helpful discussions and constant encouragements during many years. Finally, the author would like to thank an anonymous referee for his careful reading and valuable comments.

### References


[17] X.-D. Li, Riesz transforms on forms and $L^p$-Hodge decomposition on complete Riemannian manifolds, Rev. Mat. Iberoamericana, in press.


