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The minimum rank problem: A counterexample

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Abstract

We provide a counterexample to a recent conjecture that the minimum rank over the reals of every sign pattern matrix can be realized by a rational matrix. We use one of the equivalences of the conjecture and some results from projective geometry. As a consequence of the counterexample we show that there is a graph for which the minimum rank of the graph over the reals is strictly smaller than the minimum rank of the graph over the rationals. We also make some comments on the minimum rank of sign pattern matrices over different subfields of $\mathbb{R}$.

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1. Introduction

The main reference for this paper is [1] in which the conjecture and its equivalences appear.

A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix*. A matrix with real entries is called a real matrix and a matrix with rational entries is called a rational matrix. For a real matrix B, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). If A is a sign pattern matrix and $\mathbb{F}$ is a subfield of $\mathbb{R}$, the *sign pattern class* of A over $\mathbb{F}$ is defined by

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$$Q_{\mathbb{F}}(A) = \{B : B \text{ is a matrix with entries in } \mathbb{F} \text{ and } \operatorname{sgn}(B) = A\}.$$

For a sign pattern matrix A and a subfield $\mathbb{F}$ of $\mathbb{R}$, the *minimum rank* of A over $\mathbb{F}$, denoted $\operatorname{mr}_{\mathbb{F}}(A)$, is defined as

$$\operatorname{mr}_{\mathbb{F}}(A) = \min_{B \in Q_{\mathbb{F}}(A)} \{\operatorname{rank} B\}.$$

In [1], the authors made the following basic conjecture:

For any $m \times n$ sign pattern matrix A, $\quad \operatorname{mr}_{\mathbb{R}}(A) = \operatorname{mr}_{\mathbb{Q}}(A)$.

They showed that the conjecture holds in certain special cases.

In [1], it was also shown that the above conjecture is equivalent to another conjecture, namely,

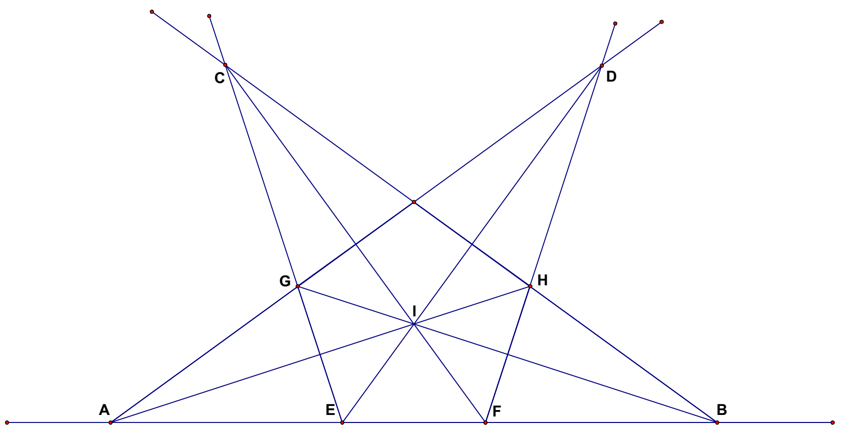
For any real matrices D, C, and E, with $DC = E$, there are rational matrices D^*, C^*, and E^* such that $\operatorname{sgn}(D^*) = \operatorname{sgn}(D)$, $\operatorname{sgn}(C^*) = \operatorname{sgn}(C)$, $\operatorname{sgn}(E^*) = \operatorname{sgn}(E)$, and $D^*C^* = E^*$.

In the following section, we shall give an example to show that this conjecture is not true. In Section 3 by making careful use of the example of Section 2, we show that there is a graph for which the minimum rank over the reals is strictly smaller than the minimum rank over the rationals. In the last section we make some comments on the minimum rank of sign pattern matrices over different subfields of $\mathbb{R}$.

2. The counterexample

Consider a configuration $\mathscr{C}$ (from [2, p. 92]) of nine points and nine lines given by A, B, C, D, E, F, G, H, I, and nine lines ABEF, ADG, AHI, BCH, BGI, CEG, CFI, DEI, DFH as drawn in Fig. 1 below starting with a regular pentagon.

Let $\ell_1, \ell_2, \ldots, \ell_9$ be the nine lines in Fig. 1 and let the equation of ℓ_i be $a_i x + b_i y + c_i = 0$. Let the nine points (with real coordinates) be (x_i, y_i), $i = 1, 2, \ldots, 9$.



The configuration consists of the nine points A, B, C, D, E, F, G, H and I, and the nine lines ABEF, ADG, AHI, BCH, BGI, CEG, CFI, DEI and DFH drawn on the plane, starting with the regular pentagon for which G, E, F, H are four of the vertices.

Fig. 1. The configuration consists of the nine points A, B, C, D, E, F, G, H and I, and the nine lines ABEF, ADG, AHI, BCH, BGI, CEG, CFI, DEI and DFH drawn on the plane, starting with the regular pentagon for which G, E, F, H are four of the vertices.

Let D be the 9×3 matrix whose ith row is (a_i, b_i, c_i) and C be the 3×9 matrix whose jth column is the transpose of the row $(x_i, y_i, 1)$. Let $DC = E$. E is a 9×9 matrix whose (i, j)th element is 0 if the jth point is on the ith line and $\neq 0$ if the jth point is not on the ith line. The incidences of the 9 points on the 9 lines are exactly dictated by the zero and nonzero elements of E.

The result on p. 93 of [2] states that (the incidence structure) $\mathscr{C}$ cannot be realized with nine points with rational coordinates. Suppose now that there are rational matrices D^*, C^*, and E^* such that $D^*C^* = E^*$ and the zero nonzero pattern of E^* is same as the zero nonzero pattern of E. Since the third row of C^* has nonzero elements, by dividing each column of C^* and the corresponding column of E^* by a nonzero rational number we may assume that the third row of C^* has all 1's. Now, let the jth column of C^* be the transpose of $(x_j^*, y_j^*, 1)$. If D^* is the 9×3 matrix whose ith row is (a_i^*, b_i^*, c_i^*), then the jth point (x_j^*, y_j^*) will be on the line $a_i^* x + b_i^* y + c_i^* = 0$ if and only if (x_j, y_j) is on ℓ_i for $i = 1, 2, \ldots, 9$. This is because $D^*C^* = E^*$ and E^* and E have the same zero nonzero pattern. Hence (a_i^*, b_i^*) for $i = 1, 2, \ldots, 9$ will be nine points with rational coordinates with the same structure of $\mathscr{C}$.

Hence there are no rational matrices D^*, C^*, E^* such that $D^*C^* = E^*$ and E^* has the same zero nonzero pattern as E. Hence there are no rational matrices D^*, C^* and E^* such that $D^*C^* = E^*$ and $\operatorname{sgn}(D^*) = \operatorname{sgn}(D)$, $\operatorname{sgn}(C^*) = \operatorname{sgn}(C)$, $\operatorname{sgn}(E^*) = \operatorname{sgn}(E)$.

The above procedure actually gives a real 12×12 matrix $B = \begin{bmatrix} I_3 & C \\ D & E \end{bmatrix}$, such that $\operatorname{rank}(B) = 3$, for which there is no rational matrix F such that $\operatorname{rank}(F) = 3$ and F and B have the same zero nonzero pattern.

Note that in [1] it was shown that for every real matrix B of rank 2 there is a rational matrix F of rank 2 such that B and F have the same sign pattern.

3. Minimum rank of a graph – a counterexample

We shall now use the above example to construct an example of a graph for which the minimum rank over the reals is not equal to the minimum rank over the rationals.

Recall that the minimum rank over the reals for a graph G is defined to be the minimum of ranks of all real symmetric matrices whose (i, j) entry is nonzero when $i \neq j$ and i is adjacent to j in G, and whose (i, j) entry is zero when $i \neq j$ and i is not adjacent to j in G. In particular the diagonal entries of the symmetric matrices could be arbitrary. The minimum rank over the rationals of a graph G is defined analogously.

From the previous section, if we let $H = \begin{bmatrix} I_3 & C & I_3 \\ D & DC & D \\ I_3 & C & I_3 \end{bmatrix}$ where C and D are as above, the matrix H is of rank 3 and there is no rational matrix K such that K and H have the same zero nonzero pattern and $\operatorname{Rank}(K) = 3$.

If $A = \begin{bmatrix} 0 & H \\ H^T & 0 \end{bmatrix}$, then A is a 30×30 symmetric real matrix. Let G be a bipartite graph on 30 points (with 15 points on each side) whose incidence matrix has the zero nonzero pattern of A.

For this graph G, let us first show that the minimum rank of G over the reals is equal to 6. In fact, A is a symmetric real matrix that has the zero nonzero pattern of the incidence matrix of G. This A has zeros in the diagonal elements. Since rank of A is 6, the minimum rank of G over the reals is atmost 6. If M is any other real symmetric matrix that has the zero nonzero pattern of A except for the diagonal elements, one can easily see that the 13th, 14th, 15th, 16th, 17th, and 18th columns of M are independent because of the diagonal submatrices in them. Hence $\operatorname{rank}(M)$ is at least 6. This gives us that minimum rank of G over the reals is 6.

Let us now show that the minimum rank of the graph G over the rationals is strictly larger than 6. Let M be a rational symmetric matrix that has the same zero nonzero pattern of A except for the diagonal elements. M would look like

$$M = \begin{bmatrix} \diamond & \cdots & 0 & U & C^* & V \\ 0 & \cdots & 0 & D^* & E^* & D^* \\ 0 & \cdots & \diamond & W & C^* & X \\ U & D^{*\mathrm{T}} & W & \diamond & \cdots & 0 \\ C^{*\mathrm{T}} & E^{*\mathrm{T}} & C^{*\mathrm{T}} & 0 & \cdots & 0 \\ V & D^{*\mathrm{T}} & X & 0 & \cdots & \diamond \end{bmatrix}$$

where U, V, W, and X are 3×3 diagonal matrices with strictly nonzero diagonal elements, C^*, D^*, E^* are rational matrices with the same zero nonzero pattern as C, D, E respectively, and $\diamond$ may take any rational values. By dividing the 16th, 17th and 18th columns and 16th, 17th and 18th rows by appropriate numbers and making sure that the matrix is still symmetric, we can assume that $U = I_3$.

Now, the 13th, 14th, 15th, 16th, 17th and 18th columns of M, because of the diagonal submatrices in them, are independent. If the rank of M is 6, these columns should span the columns of M. Since the 19th to 27th columns should linearly depend on the 13th, 14th, 15th, 16th, 17th and 18th columns of M, the columns of

$$\begin{bmatrix} C^* \\ E^* \end{bmatrix}$$

should depend on the columns of

$$M = \begin{bmatrix} U \\ D^* \end{bmatrix}$$

where $U = I$. Hence $D^*C^* = E^*$. Recall that C^*, D^*, E^* are rational matrices having the same zero nonzero pattern as C, D, E respectively. By section 2, this is not possible. Hence the rank of M is strictly larger than 6.

4. General results

Incidence structures with properties such as that of Fig. 1 were first constructed systematically by Maclane [3] using the "von Staudt algebra of throws". Theorem 3 of that paper states:

Theorem 1 (Maclane [3]). *Let $\mathbb{K}$ be a finite algebraic field over the field of rational numbers. Then there exists a matroid M of rank 3 which can be represented by a matrix with elements of $\mathbb{K}$, while any other representation of M by a matrix of elements in a number-field $\mathbb{K}_1$ requires $\mathbb{K}_1 \supset \mathbb{K}$.*

Using this theorem along with the argument of the previous section gives us the following general result.

Theorem 2. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}$, finite and algebraic over $\mathbb{Q}$. Then there exists a sign pattern matrix A, such that for any field $\mathbb{K}_1 \subset \mathbb{R}$ with $\mathbb{K} \not\subseteq \mathbb{K}_1$, $\mathrm{mr}_{\mathbb{K}}(A) < \mathrm{mr}_{\mathbb{K}_1}(A)$.*

In contrast, the situation completely changes for purely transcendental extensions.

Theorem 3. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, and let $\alpha \in \mathbb{R}$ be transcendental over $\mathbb{F}$. Then for any sign pattern matrix A, $\mathrm{mr}_{\mathbb{F}(\alpha)}(A) = \mathrm{mr}_{\mathbb{F}}(A)$.*

Proof. It is clear that for any sign pattern matrix A, $\mathrm{mr}_{\mathbb{F}(\alpha)}(A) \leqslant \mathrm{mr}_{\mathbb{F}}(A)$. To prove the reverse inequality, it suffices to show that for any matrix M with entries in $\mathbb{F}(\alpha)$, there exists a matrix M^* with entries in $\mathbb{F}$ such that $\mathrm{rank}(M^*) \leqslant \mathrm{rank}(M)$ and $\mathrm{sgn}(M^*) = \mathrm{sgn}(M)$.

Let M be an $m \times n$ matrix with entries in $\mathbb{F}(\alpha)$. By multiplying M by a suitable element of $\mathbb{F}[\alpha]$, it suffices to prove the theorem when M has entries in $\mathbb{F}[\alpha]$ (which is isomorphic to a polynomial ring, since α is transcendental over $\mathbb{F}$). For each $i \in [m]$, $j \in [n]$, let $M_{ij} = P_{ij}(\alpha)$, where P_{ij} is a polynomial with coefficients in $\mathbb{F}$. As α is transcendental, $P_{ij}(\alpha) = 0$ if and only if P_{ij} is the zero polynomial. Thus we may pick $\beta \in \mathbb{F}$ sufficiently close to α, so that for each i, j, $P_{ij}(\beta)$ has the same sign as $P_{ij}(\alpha)$. Now let $g : \mathbb{F}[\alpha] \to \mathbb{F}$ be the substitution homomorphism (of rings) with $g(\alpha) = \beta$. Define M^* to be the matrix whose (i, j) entry is $g(M_{ij}) = P_{ij}(\beta)$.

By construction, $\mathrm{sgn}(M^*) = \mathrm{sgn}(M)$. Let $r = \mathrm{rank}(M)$. Consider any $S \subseteq [m]$, $T \subseteq [n]$ with $|S| = |T| = r + 1$. We know that the $S \times T$ minor of M vanishes. Thus the determinant $|(\langle M_{ij}\rangle_{i\in S, j\in T})| = 0$. The corresponding minor of M^* equals the determinant $|(\langle g(M_{ij})\rangle_{i\in S, j\in T})|$ which, using the fact that g is a homomorphism, equals the determinant $g(|(\langle M_{ij}\rangle_{i\in S, j\in T})|) = g(0) = 0$. Thus we have shown that any $(r + 1) \times (r + 1)$ minor of M^* also vanishes, which gives us the result. □

Remark. The result of Section 2 that there is a sign pattern matrix for which the minimum rank over reals is not equal to the minimum rank over rationals was presented at the 14th Conference of the International Linear Algebra Society, Shanghai, 16th–20th July, 2007. There we learnt that another example was also obtained by Berman, Friedland, Hogben, Rothblum, and Shader.

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